Lattice Path Conference
21-25 June 2021
Presentation times for this poster:

| | |
|---|---|
| Tuesday | 1:30-2:30 pm |
| Thursday | 6-7 pm |

# Walks obeying two-step rules on the square lattice


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## Two-step rules on the square lattice

A **two-step rule** $\mathcal{R}$ is a mapping

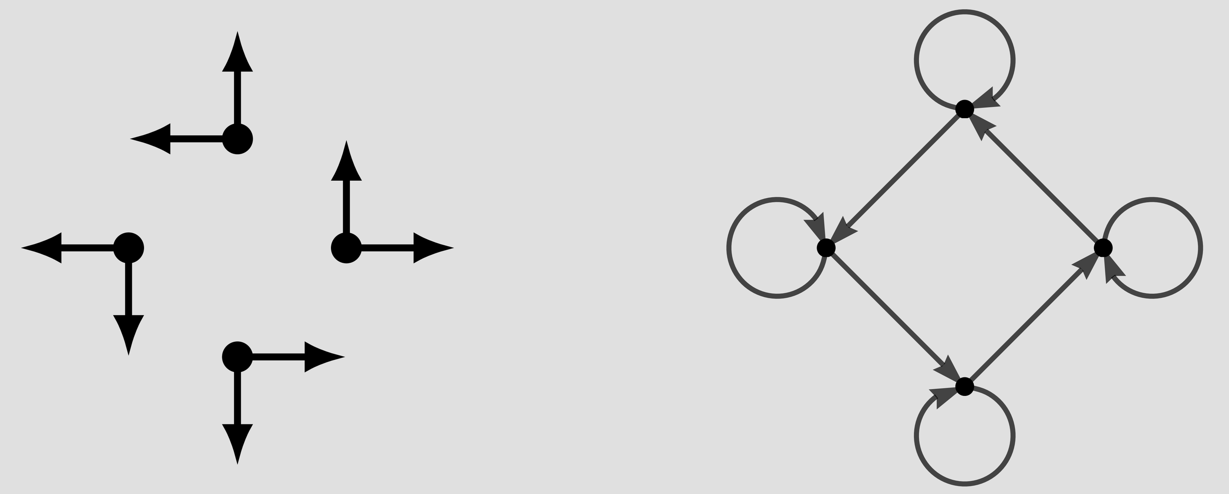
$$\{\text{east}, \text{north}, \text{west}, \text{south}\}^2 \mapsto \{0,1\}$$

where $\mathcal{R}(i,j) = 1$ if step $j$ can follow step $i$, and $\mathcal{R}(i,j) = 0$ if not.

Represent with a **transfer matrix** $T$, eg.

$$T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

or diagrammatically



A walk $w = (v_1, v_2, \ldots, v_m) \in \{\text{E}, \text{N}, \text{W}, \text{S}\}^*$ on the edges of the square lattice **obeys** $\mathcal{R}$ if $\mathcal{R}(v_i, v_{i+1}) = 1$ for $i = 1, \ldots, m-1$.

## Non-trivial rules and symmetries

There are $2^{16} = 65536$ different rules but many are trivial or isomorphic to another.

- Want rules to be **connected** and (for simplicity) **aperiodic**: there must exist $k \geq 1$ such that $(T^k)_{ij} \geq 1$ for all $i,j$. There are **25575** such rules.
- **Full plane:** any permutation of the rows/columns of $T$ gives an isomorphic rule. There are **1159** non-isomorphic aperiodic rules.
- **Upper half plane:** only allow $x \leftrightarrow -x$ symmetry. Also want rules whose walks can step arbitrarily far north (so not stuck near boundary) and arbitrarily far south (so half plane restriction is not redundant). There are **9722** such rules.
- **Quarter plane:** only allow $x \leftrightarrow y$ symmetry. Need rules whose walks do not get stuck to the boundaries and for which the quarter plane restriction is not redundant. There are **6909** such rules.

For the initial step, allow any direction (which stays in the half or quarter plane as required).

## GFs and functional equations

$F_\theta(t;x,y) \equiv F_\theta$ is the GF for walks in the **full plane** ending with step type $\theta$, counted by length and endpoint coordinate. Then

$$F_\theta = A_\theta + B_\theta F_\theta \tag{1}$$

where

- $A_\theta$ counts walks with no $\theta$ steps **except for the last step**
- $B_\theta$ counts the subset of those **which can follow a** $\theta$ **step**

$A_\theta$ and $B_\theta$ are rational, easily computable from $T$.

In the **upper half plane**,

$$H_\theta = C_\theta + B_\theta H_\theta - D_\theta H^* \tag{2}$$

where

- $C_\theta$ counts the subset of $A_\theta$ walks (ie. still full plane) which **do not start south**
- $D_\theta = A_\theta - C_\theta$ (walks that start south)
- $H^*$ counts walks (in the upper half plane) which (a) **end on the boundary** and (b) end with a step that can **precede a south step.**

$C_\theta$ and $D_\theta$ rational, easily computable.

In the **quarter plane**,

$$Q_\theta = G_\theta + B_\theta Q_\theta - D_\theta Q^{\downarrow} - J_\theta Q^{\leftarrow} \tag{3}$$

where

- $G_\theta$ counts the subset of $A_\theta$ walks which **start east or north**
- $J_\theta = A_\theta - G_\theta - D_\theta$ (walks that start west)
- $Q^{\downarrow}$ counts quarter plane walks which (a) **end on the** $x$**-axis** and (b) end with a step that can **precede a south step**. Similarly $Q^{\leftarrow}$ for walks ending on $y$-axis.

$G_\theta$ and $J_\theta$ also rational and easy to compute.

## Solutions and symmetry groups

- $F_\theta$ rational (of course)
- $H_\theta$ algebraic: kernel method (there is always a power series root $y^*$ of $1 - B_\theta$)
- $Q_\theta$ much more complicated
  - Is there a **symmetry group** of (3), like regular quarter plane lattice paths?
  - Yes! Look for values $X$ and $Y$ satisfying
  $$B_\theta(t;x,y) = B_\theta(t;X,y) = B_\theta(t;x,Y) \tag{4}$$
  - Always exactly one non-trivial (rational) solution for both $X$ and $Y$: write $X = \Psi_\theta$ and $Y = \Phi_\theta$
  - The operations $x \mapsto \Psi_\theta$ and $y \mapsto \Phi_\theta$ are involutions, generating a dihedral group
  - The actual group depends on step direction $\theta$, but the **order of the group** does not (conjecturally)

## Orbit sums and computational results

Orders of the groups:

| | |
|---|---|
| 4 : 1084 | 10 : 66 |
| 6 : 443 | 12 : 6 |
| 8 : 148 | $\infty$ : 5164 |

- For the **spiral walks** (illustrated at left) the group is $D_2$ (order 4), and the **full orbit sum** method works, giving a **D-finite** solution to each $Q_\theta$.
- This has also been observed to work for some other rules with $D_2$ groups
- What else happens?
  - Finite group but cannot cancel all terms on one side of orbit sum – series analysis says not D-finite
  - Finite group, orbit sum completely vanishes – series analysis says algebraic, but cannot get half orbit sum to work
  - Infinite group but D-finite GF (!)
  - Infinite group but algebraic GF (!!)

Computational results (500 terms each), using **Ore Algebra** package for Sage [2]:

| | alg (lower bound) | Df (?) | nonDf (upper bound) |
|---|---|---|---|
| $D_2$ | – | 659 | 425 |
| $D_3$ | 40 | 66 | 337 |
| $D_4$ | 5 | 59 | 82 |
| $D_5$ | 6 | 4 | 56 |
| $D_6$ | – | – | 6 |
| $D_\infty$ | 22 | 30 | 5112 |

Some non-D-finite will really be D-finite and some D-finite will really be algebraic.

## References


[1] Beaton, N. R.: *Walks obeying two-step rules on the square lattice: full, half and quarter planes*. Submitted. arXiv:2010.06955 (2020).

[2] Kauers, M.; Jaroschek, M.; Johansson, F.: *Ore polynomials in Sage*. github.com/mkauers/ore_algebra (2015+).