A two-step rule $R$ is a mapping
{east, north, west, south}^2 \to \{0,1\}
where $R(i,j) = 1$ if step $j$ can follow step $i$, and $R(i,j) = 0$ if not.
Represent with a transfer matrix $T$, eg.
\[
T = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\]
or diagrammatically

A walk $w = (v_1, v_2, \ldots, v_m) \in \{E, N, W, S\}^*$ on the edges of the square lattice obeys $R$ if $R(v_i, v_{i+1}) = 1$ for $i = 1, \ldots, m - 1$.

Non-trivial rules and symmetries

There are $2^{16} = 65536$ different rules but many are trivial or isomorphic to another.

- Want rules to be connected and (for simplicity) aperiodic: there must exist $k \geq 1$ such that $(T^k)_{ij} \geq 1$ for all $i,j$. There are 25575 such rules.
- Full plane: any permutation of the rows/columns of $T$ gives an isomorphic rule. There are 1159 non-isomorphic aperiodic rules.
- Upper half plane: only allow $x \leftrightarrow -x$ symmetry. Also want rules whose walks can step arbitrarily far north (so not stuck near boundary) and arbitrarily far south (so half plane restriction is not redundant). There are 9722 such rules.
- Quarter plane: only allow $x \leftrightarrow y$ symmetry. Need rules whose walks do not get stuck to the boundaries and for which the quarter plane restriction is not redundant. There are 6909 such rules.

For the initial step, allow any direction (which stays in the half or quarter plane as required).

GFs and functional equations

$F_\theta(t; x, y) \equiv F_\theta$ is the GF for walks in the full plane ending with step type $\theta$, counted by length and endpoint coordinate. Then
\[
F_\theta = A_\theta + B_\theta F_\theta
\]
where
- $A_\theta$ counts walks with no $\theta$ steps except for the last step
- $B_\theta$ counts the subset of those which can follow a $\theta$ step $A_\theta$ and $B_\theta$ are rational, easily computable from $T$.

In the upper half plane,
\[
H_\theta = C_\theta + D_\theta H_\theta - D_\theta H^*
\]
where
- $C_\theta$ counts the subset of $A_\theta$ walks which do not start south
- $D_\theta = A_\theta - C_\theta$ (walks that start south)
- $H^*$ counts walks (in the upper half plane) which (a) end on the boundary and (b) end with a step that can precede a south step.
- $C_\theta$ and $D_\theta$ rational, easily computable.

In the quarter plane,
\[
Q_\theta = G_\theta + B_\theta Q_\theta - D_\theta Q_\theta - J_\theta Q^* - J_\theta Q^{**}
\]
where
- $G_\theta$ counts the subset of $A_\theta$ walks which start east or north
- $J_\theta = J_\theta = G_\theta - D_\theta$ (walks that start west)
- $Q^*$ counts quarter plane walks which (a) end on the $x$-axis and (b) end with a step that can precede a south step. Similarly $Q^{**}$ for walks ending on $y$-axis.
- $G_\theta$ and $J_\theta$ also rational and easy to compute.

Solutions and symmetry groups

- $F_\theta$ rational (of course)
- $H_\theta$ algebraic: kernel method (there is always a power series root $y'$ of $1 - B_\theta$)
- $Q_\theta$ much more complicated
  - Is there a symmetry group of (3), like regular quarter plane lattice paths?
  - Yes! Look for values $X$ and $Y$ satisfying $B_\theta(x, y) = B_\theta(x, Y) = B_\theta(t, x, Y)$
  - Always exactly one non-trivial (rational) solution for both $X$ and $Y$: write $X = \Psi_y$ and $Y = \Phi_y$
  - The operations $x \mapsto \Psi_x$ and $y \mapsto \Phi_y$ are involutions, generating a dihedral group
  - The actual group depends on step direction $\theta$, but the order of the group does not (conjecturally)

Orbit sums and computational results

Orders of the groups:
\[
\begin{array}{lll}
4 & 1084 & 66 \\
6 & 443 & 126 \\
8 & 148 & \infty
\end{array}
\]
- For the spiral walks (illustrated at left) the group is $D_2$ (order 4), and the full orbit sum method works, giving a D-finite solution to each $Q_\theta$.
- This has also been observed to work for some other rules with $D_2$ groups.
- What else happens?
  - Finite group but cannot cancel all terms on one side of orbit sum – series analysis says not D-finite
  - Finite group, orbit sum completely vanishes – series analysis says algebraic, but cannot get half orbit sum to work
  - Infinite group but D-finite GF (1)
  - Infinite group but algebraic GF (2)

Computational results (500 terms each), using Ore Algebra package for Sage [2]:

<table>
<thead>
<tr>
<th>\theta</th>
<th>alg (lower bound)</th>
<th>Df (?)</th>
<th>nonDf (upper bound)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_2$</td>
<td>-</td>
<td>-</td>
<td>659</td>
</tr>
<tr>
<td>$D_3$</td>
<td>40</td>
<td>66</td>
<td>12</td>
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<tr>
<td>$D_4$</td>
<td>5</td>
<td>59</td>
<td>82</td>
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<td>$D_5$</td>
<td>6</td>
<td>4</td>
<td>56</td>
</tr>
<tr>
<td>$D_6$</td>
<td>-</td>
<td>-</td>
<td>6</td>
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<tr>
<td>$D_7$</td>
<td>22</td>
<td>30</td>
<td>5112</td>
</tr>
</tbody>
</table>

Some non-D-finite will really be D-finite and some D-finite will really be algebraic.

References