Lattice Path Conference
21-25 June 2021
Presentation times for this poster:
Monday  1:30-2:30 pm
Tuesday  1:30-2:30 pm

Lattice Pathology and symmetric functions
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Lattice paths
- **Step set**: \( S = \{ s_1, s_2, \ldots, s_n \} \subset \mathbb{Z} \)
- largest left and right steps: \( c := \min S \) and \( d := \max S \)
- **n-step lattice path**: sequence of steps \( (v_1, \ldots, v_n) \in S^n \)
  \( \sim \) can be seen as a directed lattice path in \( \mathbb{N} \times \mathbb{Z} \)

New tool for lattice path surgery: prime walks
- Set \( A_k \) of arches = walks starting at 0, ending at altitude \( k \), and staying always strictly above altitude \( k \) except for its first and final position.
- The set \( P \) of prime walks is defined as the following sets of arches
  \[ P = \bigcup_{k=0}^{d} A_k. \]

Theorem (Universal context-free grammar)
Meanders and excursions are generated by the following grammar:

- \( M \rightarrow \varepsilon + PM \) (meanders),
- \( E \rightarrow \varepsilon + AE \) (excursions),

i.e., "meanders are sequences of prime walks": \( M = \text{Seq} \left( \sum_{k=0}^{d} A_k \right) \)
and "excursions are sequences of arches": \( E = \text{Seq}(A_0) \),
where the arches \( A_k \) from 0 to \( k \) are generated by

\[ A_k \rightarrow k + \sum_{j=k+1}^{d} A_j E A_{k-j} \] (arcs for \( k \geq 0 \)),

\[ A_k \rightarrow k + \sum_{j=k+1}^{d} A_{k-j} E A_j \] (arcs for \( k < 0 \)),

with the convention that the part \( A_k \rightarrow k \) is omitted whenever \( k \notin S \).

Prime walk decomposition

Theorem (Bivariate Spitzer/Sparre Andersen's identities)
The GF \( W^+(z, u) = \sum w^+_n(u) z^n \) of walks ending at a positive altitude and the GF \( M(z, u) = \sum m_n(u) z^n \) of meanders (where \( u \) encodes the final altitude and \( z \) encodes the length) are related by the formula

\[ M(z, u) = \exp \left( \int_0^z W^+(t, u) - 1 \right) dt = \exp \left( \sum_{k=1}^{\infty} w^+_k(u) z^k \right). \]

Proof (Spitzer/Sparre Andersen-like decomposition)

A non-empty walk \( W^+(z, u) \) consisting of a maximal meander \( M(z, u) \) starting at the minimum and a pointed prime walk \( \phi_2^{-1}\phi_1^{-1} \),

\[ W^+(z, u) - 1 = M(z, u) z \frac{\partial \phi_1}{\partial z} \left( 1 - \frac{1}{M(z, u)} \right). \]

Theorem (Bivariate version of Wiener–Hopf formula)
The GFs \( W_i(z, u) \) and \( W_s(z, u) \) of walks (\( u \) marks the positive and negative height; not the altitude!) are related to the GFs \( M^+(z, u) \) of positive and \( M^-(z, u) \) of negative meanders (\( u \) marks the final altitude):

\[ W_i(z, u) = M^+(z) E(z) M^-(z, u) = \frac{1}{n^d} \prod_{j=1}^{d} \left( 1 - u_j(z) \right) \prod_{i=1}^{\infty} \left( 1 - u_i(z) \right) \]

\[ W_s(z, u) = M^+(z, u) E(z) M^-(z) = \frac{1}{n^d} \prod_{j=1}^{d} \left( 1 - u_j(z) \right) \prod_{i=1}^{\infty} \left( 1 - u_i(z) \right) \]

Symmetric polynomials of degree \( k \) in \( d \) variables
- Complete hom. sym. pol.
  \( h_k(x_1, \ldots, x_d) = \sum_{1 \leq i_1 < \cdots < i_k \leq d} x_{i_1} \cdots x_{i_k} \)
- Elementary sym. pol.
  \( e_k(x_1, \ldots, x_d) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq d} x_{i_1} \cdots x_{i_k} \)
- Power sum sym. pol.
  \( p_k(x_1, \ld. \ld, x_d) = \sum_{i=1}^{d} i^k \)

Symmetric polynomials and new types of lattice paths
- From \( k = 0 \) to \( k \)
- \( M^+(z) \)
- \( h_k \)
- \( e_k \)

Theorem (Asymptotics: explicit multiplicative constants)
The radius of convergence is \( r := 1/S(\tau) \), s.t. \( \tau > 0 \) given by \( S(\tau) = 0 \).

\[ [z^n] M_{\alpha}(z) = \frac{\alpha S(\tau)^n}{2\sqrt{\pi}} \left( 1 + \frac{1}{n^2} \right), \quad \alpha = \frac{\partial h_k}{\partial x_k}(\tau, u_1, \ldots, u_d). \]

References