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NOTE

ON SIGNED DIGRAPHS WITH ALL CYCLES NEGATIVE

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In "On signed digraphs with all cycles negative", *Discrete Appl. Math.* 12 (1985) 155–164, F. Harary, J.R. Lundgren and J.S. Maybee, identify certain families of such digraphs: the class of strong and upper digraphs and the class $\bar{U}$. We give here a characterization of the latter class and new proofs of two results concerning these classes, by using the c-minimal strongly connected digraphs. This note answers some questions of the authors.

The definitions not given here can be found in [1].

Let $D(V,E)$ be a digraph. V is the vertex set, E is the arc set. $|V|=n$, $|E|=m$.

A *chain* is a digraph with vertex set $\{v_1,\dots,v_k\}$ and arc set $\{e_1,\dots,e_{k-1}\}$ where $e_i=(v_i,v_{i+1})$ or (v_{i+1},v_i), $1\le i\le k-1$. If $v_1=v_k$, the chain is called a *cycle*. A cycle can be considered as a vector of $\mathbb{Z}^m$. Several cycles are *independent* if the corresponding vectors are independent.

A *path* is a digraph with vertex set $\{v_1,\dots,v_k\}$ and arc set $\{(v_i,v_{i+1})\mid 1\le i\le k-1\}$. If $v_1=v_k$, the path is called a *circuit* (or *directed cycle*).

An *elementary cycle* contains no vertex twice. (In [5], for digraphs, 'cycle' means 'elementary directed cycle'.)

A *signed digraph* is a digraph whose arcs have been signed positive or negative by a sign function $\sigma: E\to\{1,-1\}$.

Theorem 1 [2–4]. *For a strongly connected digraph D the following are equivalent:*

(i) *D has a minimal number of elementary circuits (i.e., $m-n+1$).*

(ii) *All the elementary circuits are independent.*

(iii) *D has a* **circulation tree** (*spanning tree such that each elementary cycle associated with is a circuit*).

(iv) *For every pair of vertices $\{x,y\}$ on the same elementary circuit, there is exactly one path from x to y or there is exactly one path from y to x.*

If a strongly connected digraph verifies any of these propositions, it is called a c-minimal strongly connected (*cmsc*) digraph.

Property 1 [2–4]. *Let D be a cmsc digraph and T a circulation tree of D.*

(1) *Each arc of $\bar{T}$ $(D-T)$ is in one and only one elementary circuit of D.*

(2) *Each elementary circuit of D contains one and only one arc of $\bar{T}$.*

(3) *Each elementary circuit of D is formed with one path in T and one arc in $\bar{T}$.*

The explicit construction of circulation trees is completed in [6].

Denote by $\mathscr{C}$ the class of c-minimal digraphs: digraphs whose strongly connected components are cmsc.

For signed digraphs, we keep the notation of [5]: $\mathscr{N}$ is the class of all signed digraphs with all (directed) cycles negative. $\mathscr{M}$ is the set of all digraphs D for which there exists a sign function σ such that $\sigma D \in \mathscr{N}$. A digraph is *upper* if there is a labelling of V such that the resulting adjacency matrix $A=[a_{ij}]$ verifies $a_{ij}=0$ whenever $i-j>1$. $\bar{U}$ is the class of *free cyclic* digraphs D: every cycle of D contains at least one arc which is not in any other cycle of D.

Theorem 2. $\mathscr{C} \subset \mathscr{M}$.

Proof. Let $D(V,E)$ be a cmcs digraph and T a circulation tree of D. We define the sign function $\sigma : \sigma(e)=-1$ if $e \in \bar{T}$, $\sigma(e)=+1$ if $e \in T$. By Property 1, $D \in \mathscr{M}$. □

Theorem 3. (1) *Each strong and upper digraph is cmsc.*

(2) $\mathscr{C}=\bar{U}$.

Proof. (1) Let D be a strong and upper digraph, $H=(v_p, \dots, v_1)$ a Hamiltonian path of D. It is clear that H is a circulation tree of D.

(2) If $D \in \mathscr{C}$, then each elementary circuit of D contains at least one arc which is not on any other circuit of D; it follows that $D \in \bar{U}$.

If $D \in \bar{U}$, then all the elementary circuits of a strong component are independent; it follows that $D \in \mathscr{C}$. □

Corollary 1 [5, Theorem 2]. *If D is strong and upper, then $D \in \mathscr{M}$.*

Corollary 2 [5, Theorem 5]. *If $D \in \bar{U}$, then $D \in \mathscr{M}$.*

From Theorem 1(iv) and Theorem 3(2), it follows that the class $\bar{U}$ is indeed a generalization of *unipathic* digraphs. (A digraph is unipathic if whenever v is reachable from u, there is exactly one path from u to v [5].)

Furthermore, the digraphs of $\bar{U}$, called *free cyclic* digraphs in [5], are characterized, in particular by using trees. That answers some questions of [5].

References

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