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On generating functions on generating trees.

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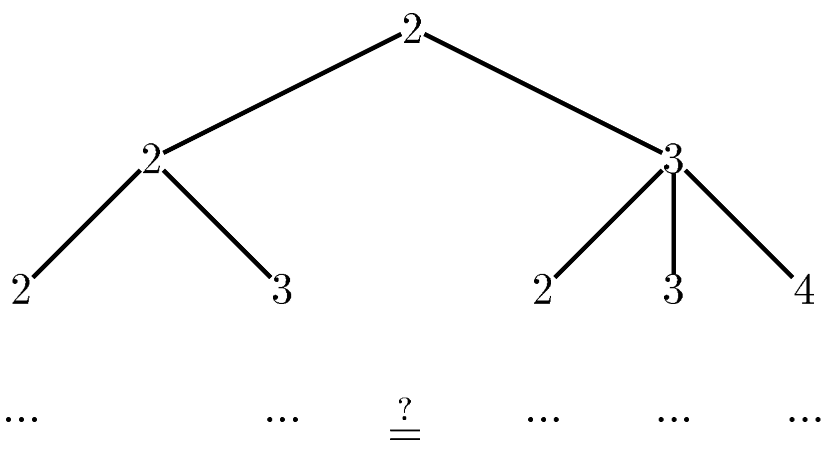
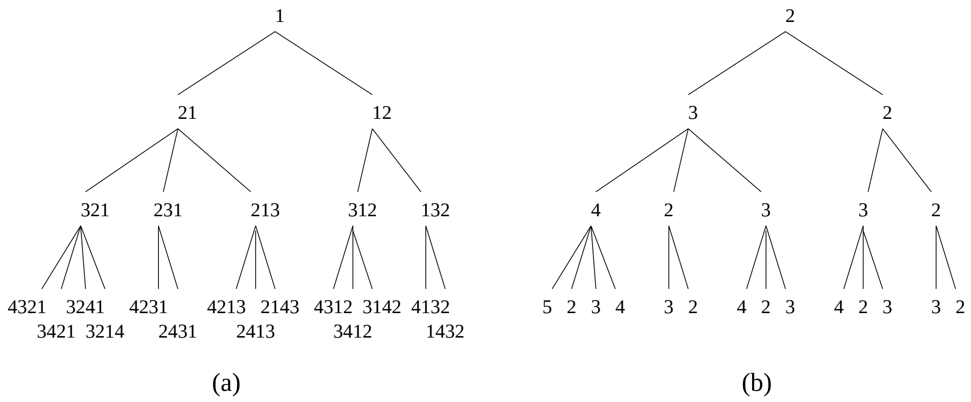


$$\frac{1-\sqrt{1-4z}}{2z}$$

Enumerating Combinatorial Objects

J. West: generating tree of 123-avoiding permutations



(a) nodes labelled by the permutations

(b) nodes labelled by the numbers of children

Morality: ECO-method of R. Pinzani & al.

$\Longleftrightarrow$ rewriting rules (ecosystem)

$(k) \to e_1(k) \ldots e_k(k)$

Usefulness: Random generation (! non trivial weights for the branches : need enumeration)

manual

123 avoiding permutations/generating tree

manual

grammar/generating tree

Generating Trees

	# of nodes	Σ labels
2	1 nodes	2
2 3	2 nodes	5
2 3 2 3 4	5 nodes	14
2 3 2 3 4 2 3 2 3 4 2 3 4 5	14 nodes	42
$\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$	f_n nodes	s_n

Generating trees: node with label $k \Rightarrow k$ children

$\exists$ associated rewriting rules e.g. (above): axiom $= (2)$ and $(k) \rightarrow (2)(3)\ldots(k+1)$.

Remark: For generating trees $f_{n+1} = s_n$

Question: How many nodes at depth n ?

Generating functions

Aim: Explicit the generating function of the generating tree

$$F(z) = \sum_{n \geq 0} f_n z^n$$

Equivalent points of view

words from an infinite "regular" grammar
$\approx$ traces in an infinite automaton
$\approx$ iterated power of an infinite matrix
$\approx$ generating trees
$\approx$ random walks on the integer

Generating function of a "walk on the integer" :

$$F(z, u) = \sum_{n \geq 0, k \geq 0} f_{n,k} u^k z^n = \sum_{n \geq 0} f_n(u) z^n$$

$n \leftrightarrow$ length $\qquad k \leftrightarrow$ height

•$f_{n,k}$: # walks with length n which start from the initial point and end at level k
$=$ # nodes with label k at depth n in the tree.

•$F(z, 1)$ counts all the walks.

•$F(z, 0)$ counts all the excursions (final height$= 0$).

manual

ecosystems

Our problem

What kind of ecosystems gives a

$$\begin{cases} rational \\ algebraic \\ transcendental \end{cases}$$

ordinary generating function ?

Three interesting families

Ecosystem of general shape:

$$(k) \to e_1(k) \dots e_k(k)$$

- Rational ecosystems:
  strong regularity conditions
  $(k) \to (2)^{k-1}(k+1)$ concentrated in $O(1)$
- Algebraic ecosystems:
  factorial form with slight modifications
  $(k) \to (2)(3) \dots (k)(k+1)$ diffused $[1..\Theta(k)]$
- Transcendental ecosystems:
  -coefficients grow too fast
  $(k) \to (k)^{k-1}(k+1)$ concentrated in $O(k)$
  -too "irregular" rules
  $$\begin{cases} (k) \to (2)^{k-2}(2)(k+1) & \text{when } k \text{ is a power of } 2 \\ (k) \to (2)^{k-2}(3)(k+1) & \text{otherwise} \end{cases}$$

Rational Systems I

$$[(s_0), (k) \rightarrow e_1(k) \ldots e_k(k)]$$

Theorem 1. *If* $e_1(k) + \cdots + e_k(k) = \alpha k + \beta$, *then*

$$F(z) \;=\; \frac{1 + (s_0 - \alpha)z}{1 - \alpha z - \beta z^2}.$$

Proof:

$$f_{n+2} = s_{n+1} = (\alpha k_1 + \beta) + \cdots + (\alpha k_{f_n} + \beta)$$

$$= \alpha s_n + \beta f_n = \alpha f_{n+1} + \beta f_n$$

$\square$

Examples: Fibonacci sequence.

$[(2),\ \{(k) \rightarrow (2)^{k-1}(k+1)\}]$ gives

$e_1(k) + \cdots + e_k(k) = 3k - 1$ thus

$F(z) = \frac{1-z}{1-3z+z^2} = 1 + 2z + 5z^2 + \cdots$

(odd entries in the Fibonacci sequence).

$[(2),\ \{(k) \rightarrow (1)^{k-1}(2k)\}]$

$[(2),\ \{(k) \rightarrow (2)^{k-2}(3 - (k \bmod 2))(k + (k \bmod 2))\}]$

$[(2),\ \{(k) \rightarrow (2)^{k-2}(3 - [k \text{ prime}])(k + [k \text{ prime}])\}]$

Same $F(z)$ but different generating trees (*idem* for bivariate functions $F(z,u)$).

Rational Ecosystems II

Theorem 2. *Let $\pi_{i,k}$:= # of "one-step transitions from i to k". If all the series $\sum_{i\geq 1} \pi_{i,k}\, t^i$ are rational then $F(z)$ is rational.*

Proof: let $F_k(z)$ the GF of #nodes with label k.

$$F(z) = \sum_{k\geq 1} F_k(z) = \sum_{i=1}^{C} F_i(z) \left[\sum_{k\geq 1} P_{i,k}(z) \right].$$

$\sum_{k\geq 1} P_{i,k}(z)$ rational as a sum of polynomials in z which can be explicited.

Remains to check that the $F_i(z)$ are rational.

But we also have a system (for $k \leq C$)

$$F_k = z \sum_{i=1}^{C} F_i(z) \left[\sum_{j\geq 1} P_{i,j}(z) \pi_{j,k} \right].$$

$\sum_{j\geq 1} P_{i,j}(z)\pi_{j,k}$ is rational (Hadamard product of two rational series).

$\Rightarrow$ linear system with rational coefficients

$\Rightarrow$ F_k rational $\Rightarrow$ F rational. $\square$

Rational Ecosystems III

Consequences of previous theorems:

The prime ecosystem paradox

$\exists$ "prime" generating tree (set of label = set of prime numbers) with rational GF. Proof: Vaughan theorem allows an affine weight. $\square$

Mixing two rules

axiom = (2) and

$$\begin{cases} (k) \rightarrow (2)^{k-2}(3)(k+1) \text{ when } k \text{ is a multiple of } 2 \\ (k) \rightarrow (2)^{k-2}(2)(k+1) \text{ otherwise} \end{cases}$$

$\Rightarrow$ rational GF.

Compatibility with arbitrary forward jumps

$[(3), (k) \rightarrow (666)^{k-1}(1999k + 2000)]$

$\Rightarrow$ rational GF.

manual

optional: The prime ecosystem paradox

Algebraic Ecosystems

Getting the functional equation.

ex.: $[(0),(k)\rightarrow(0)(1)\dots(k+1)]$

$$u^k\rightarrow u^0+\cdots+u^{k-1}+u^k+u^{k+1}=\frac{1-u^k}{1-u}+u^k+u^{k+1}$$

The basic recurrence is

$$f_{n+1}(u)=\frac{f_n(1)-f_n(u)}{1-u}+(1+u)f_n(u)$$

Multiplying by z^{n+1} and summing (for $n\geq 0$) yields

$$F(z,u)=f_0(u)+z\frac{F(z,1)-F(z,u)}{1-u}+z(1+u)F(z,u)$$

The functional equation is

$$(1-z(1+u)+z\frac{1}{1-u})F(z,u)=1+z\frac{F(z,1)}{1-u}$$

The Kernel Method

Theorem 3. *Factorial ecosystem ⇒ algebraic GF. More precisely*

$$F(z,1) = -\frac{1}{z}\prod_{i=0}^{c}(1-u_i(z)),$$

Solving the functional equation: the kernel method

$(k) \to (0)(1)\cdots(k-3)(k-1)(k+4)(k+15)$ thus $P(u) := -u^{-2} + u^4 + u^{15}$

$$F(z,u) = 1 + z\left(\frac{F(z,1)-F(z,u)}{1-u} + P(u)F(z,u) - \{u^{<0}\}\sum_{n\geq 0} z^n L[f_n](u)\right).$$

$$F(z,u)\left(1+\frac{z}{1-u} - zP(u)\right) = 1 + \frac{z}{1-u}F(z,1) - z\sum_{j=0}^{c-1} c_j(u)\partial_u^j F(z,0).$$

$$F(z,u)K(z,u) = \text{lin. comb. of } (c+1) \text{ unknown functions.}$$

Cancelling the kernel gives $c+1$ convergent branches.
linear system ⇒ the $\partial_u^j F(z,0)$ are known ⇒ F is algebraic. □

Transcendental Systems I

$$(k) \to e_1(k) \ldots e_k(k)$$

Proposition 1. *Consider a system such that:*

1. *finitely many e_i's are bounded;*
2. *for all k, there exists a forward jump from k (i.e., $e_i(k) > k$ for some i).*

Then the (ordinary) generating function $F(z)$ has radius of convergence zero.

Proof: coefficients of F grow like a factorial. □

Examples: Arrangements

$[(2),(k) \to (k)(k+1)^{k-1}]$ generates the sequence $1,2,5,16,65,326\ldots$. Triangular array $f_{n,k}$ given by the arrangement numbers $k!\binom{n}{k}$ of EGF: $e^z/(1-z)$.

Stirling polynomials

$[2,(k) \to (k)^{k-1}(k+1)]$ Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ (the number of ways one can group n objects into k nonempty subsets).

$$\widetilde{F}(z,u) = \sum_{n\geq 0} \left(\sum_{k=1}^{n} \left\{ {n \atop k} \right\} u^k \right) \frac{z^n}{n!} = \exp(u(\exp(z)-1)).$$

$u = 1 \to$ Bell

Puzzle

The previous examples are transcendental with radius of convergence 0.

¿ Are there ecosystems with a trancendental GF *and* a positive radius of convergence ?

YES ☐

NO ☐

Tick the right box and win 100 pta.

Transcendental Systems II

An example of transcendence due to irregularity but with radius of convergence > 0:

The Fredholm case axiom $= (3)$ and

$$\begin{cases} (k) \to (2)^{k-2}(2)(k+1) \text{ when } k \text{ is a power of } 2 \\ (k) \to (2)^{k-2}(3)(k+1) \text{ otherwise} \end{cases}$$

involves the "Fredholm series"

$$h(z) := \sum_{m \geq 1} z^{2^m}$$

$h(z)$ has the unit circle as a natural boundary because

$$h(z) = z^2 + h(z^2) \Rightarrow \forall n \quad h(\exp(\frac{2i\pi}{2n})) = \infty.$$

$$F(z) = \frac{z(1-z)^2 h(z)}{(1-2z)(1-z)^2 h(z) - z^4}.$$

radius of convergence $\approx 0.360102 \neq 0$. $\square$

Non Holonomic Transcendental Systems

F is D-finite (holonomic) $\Longleftrightarrow$ F satisfies a linear differential equation with polynomial coefficients in z.

Nonholonomic example:

Bessel histories.

axiom $= (2)$, $(k) \to (k-1)(k)^{k-2}(k+1)$

Classical combinatorial theory of continued fractions gives

$$F(z,0) = \frac{1}{1 - z - z^2 B(z)}$$

where $B(z)$ is the GF of "Bessel numbers".

From [FlSc90], we know that

$$1 - z^2 B(-z) \sim z \frac{J_{1/z-1}(2)}{J_1/z(2)},$$

(J_ν: Bessel J-function of order ν)

known asymptotic non compatible with holonomy [WiZe85]

$\Rightarrow F(z,u)$ and $F(z,1)$ are not holonomic. $\square$

A catalog of ecosystems of combinatorial interest

Ecosystem	Name	Id.	Generating Function
	Rational GF's		(ogf)
$(1),\ (k) \to (k)^{k-1}((k \bmod 2)+1)$	Fibonacci	M0692	$\frac{1}{1-z-z^2}$
$(2),\ (k) \to (2)^{k-1}(k+1)$	odd Fibonacci	M1439	$\frac{1-z}{1-3z+z^2}$
$(3),\ (k) \to (2)^{k-1}(k+1)$	even Fibonacci	M2741	$\frac{1}{1-3z+z^2}$
	Algebraic GF's		
$(1),\ (k) \to (1)\cdots(k-1)(k+1)$	Motzkin numbers	M1184	$\frac{1-z-\sqrt{1-2z-3z^2}}{2z}$
$(2),\ (k) \to (2)\cdots(k)(k+1)$	Catalan numbers	M1459	$\frac{1-2z-\sqrt{1-4z}}{2z}$
$(t{+}1),\ (k) \to (t{+}1)\cdots(k)(k+1)^t$	t=2: Schröder numbers	M2898	$\frac{1-2tz-z+z^t-\sqrt{(1+z-tz)^2-4z}}{2tz}$
$(3),\ (k) \to (3)\cdots(k+2)$	Ternary trees	M2926	equation: $F = 1 + zF^3$
$(t),\ (k) \to (t)\cdots(k+t-1)$	t-ary trees		equation: $F = 1 + zF^t$
	Transcendental GF's		(egf)
$(2),\ (k) \to (k-1)^{k-1}(k+1)$	Involutions	M1221	$e^{z+\frac{1}{2}z^2}$
$(2),\ (k) \to (k-1)^{k-2}(k)(k+1)$	Switchboard problem	M1461	$e^{2z+\frac{1}{2}z^2}$
$(2),\ (k) \to (k)(k+1)^{k-1}$	Arrangements	M1497	$e^z/(1-z)$
$(2),\ (k) \to (k-1)^{k-2}(k+1)^2$	Bicolored involutions	M1648	e^{2z+z^2}
$(2),\ (k) \to (k+1)^k$	Factorial numbers	M1675	$1/(1-z)$
$(2),\ (k) \to (k+1)^{k-1}(k+2)$	Increasing subsequences	M1795	$e^{z/(1-z)}/(1-z)$
	Nonholonomic GF's		
$(1),\ (k) \to (k)^{k-1}(k+1)$	Bell numbers	M1484	e^{e^z-1}
$(t),\ (k) \to (k)^{k-t}(k+1)^t$	labelled balls in t-col. urns	M1662	$e^{t(e^z-1)}$
$(2),\ (k) \to (k-1)(k)^{k-2}(k+1)$	Bessel numbers	M1462	—

Then, what ?

- obtain automatically the EGF
  (bijective approach [Corteel])
- random generation
- solve more general functional eqns
  (recurrence approach [Petkošek])
- asymptotics
  $f_n \sim \frac{A\rho^{-n}}{\sqrt{\pi n}}$ (algebraic case)

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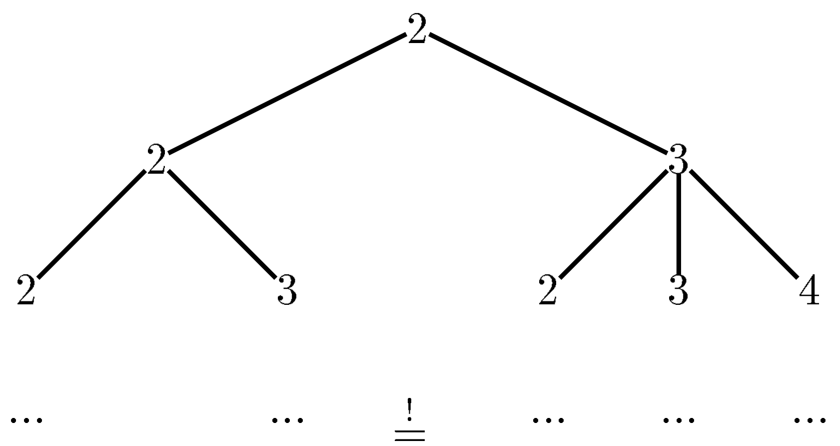

$$\frac{1-\sqrt{1-4z}}{2z}$$