Continued Fractions, Comparisons Algorithms and Fine Structure Constants

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[summary by Cyril Banderier]

Abstract

A simple problem of comparing fractions (Gosper's algorithms for continued fractions in the "Hacker's Memorandum") and of deciding the orientation of triangles in computational geometry leads to an interesting complexity analysis with an incursion into a surprising variety of domains: dynamical systems, number theory (continued fractions), special functions (multiple zeta values), functional analysis (transfer operators), numerical analysis (series acceleration) and complex analysis (Riemann hypothesis). These domains all eventually contribute to a detailed characterization of the complexity of comparison and sorting algorithms, either on average or in probability. (Joint work with Brigitte Vallée, Caen.)

1. Introduction

To compare two rational numbers (or similarly, to determine the sign of a 2×2 determinant, a relevant problem in computational geometry) is a delicate problem when you have to work with a numerical calculator limited to a given number of digits. For example, since $\frac{312689}{99532} - \frac{833719}{265381} \approx 3 \times 10^{-11}$, a computer with 10-digits arithmetic cannot compare "naively" the two rational numbers.

In the "Hacker's Memorandum" [2], it is showed that it is always possible to solve this comparison problem without exceeding the calculator's accuracy. The algorithm consists in expanding the two rational numbers into continued fractions, and stopping as soon as one gets two different coefficients. This algorithm is easily generalized to any number representation system (binary, d-ary, centered or classical continued fractions, etc.) and also to the comparison of n rational numbers.

2. Results

The functions $\bar{U}(x) = \{1/x\}$ and $\hat{U}(x) = \{\{1/x\}\}$ (where $\{\{y\}\}$ stands for the distance to the nearest integer from y) are the functions whose iteration produces classical continued fractions and centered continued fractions respectively. Under a uniform probabilistic model (over the set of legal inputs, that is, an interval of the shape $[\,0, \alpha\,]$), the number L of iterations needed to compare two numbers satisfies $\Pr(L \geq k+1) = \sum_{|h|=k} \left| \frac{h(0)-h(\alpha)}{\alpha} \right|^2$ and the moment sums of order l satisfy $\rho^{(l)} = \sum_h \left| \frac{h(0)-h(\alpha)}{\alpha} \right|^l$. These sums are indexed by the inverse branches of U, which appear to be linear fractional transformations of a specific shape [6] in the case of continued fractions. Thus:

Theorem 1. *The expected cost of the classical ($\bar{\rho}$) and centered ($\hat{\rho}$) comparison algorithms are expressible as sums over lattice points in* $\mathbb{N}^2$

$$\bar{\rho}^{(l)} = 1 + \frac{1}{2^l} + \frac{2}{\zeta(2l)} \sum_{d<c<2d} \frac{1}{c^l d^l} \qquad \textit{and} \qquad \hat{\rho}^{(l)} = \frac{2^l}{\zeta(2l)} \sum_{d\phi<c<d\phi^2} \frac{1}{c^l d^l} \qquad \left(\phi = \left(1+\sqrt{5}\,\right)/2\right)$$

With the help of double zeta values (also known as Euler-Zagier sums), defined as

$$\zeta^{++}(s,t) = \sum_{n=1}^{\infty}\sum_{q=1}^{n-1} \frac{1}{n^s q^t} \qquad \text{and} \qquad \zeta^{-+}(s,t) = \sum_{n=1}^{\infty}\sum_{q=1}^{n-1} \frac{(-1)^n}{n^s q^t},$$

it is possible to rewrite $\bar{\rho}^{(2)}$ as a peculiar combination of multiple zeta values and a polylogarithm. This implies that $\bar{\rho}^{(2)}$ can be computed to any precision in polynomial time:

Theorem 2. *The mean number $\bar{\rho}^{(2)}$ of digit comparisons in the comparison algorithm based on classical continued fraction is expressed as*

$$\bar{\rho}^{(2)} = \frac{3}{4} + \frac{360}{\pi^4}\zeta^{-+}(2,2) = 17 - \frac{60}{\pi^4}\left(24\,\mathrm{Li}_4\,(1/2) - \pi^2(\ln 2)^2 + 21\zeta(3)\ln 2 + (\ln 2)^4\right) = 1.35113157\ldots.$$

There exist several alternative expressions, due to intriguing relations between multiple zeta values. This is otherwise a topic of active research [1, 5, 7], found relevant to knot invariants, Feynman diagrams and even the theory of perverse sheaves. For higher moment sums, polynomial time computations are also possible, via some nice series/integral representations [6].

For the comparison of n real numbers, the cost of sorting n numbers depends on the position of the nontrivial zeroes of the Riemann zeta function (see [3] for an approach by Dirichlet depoissonization and Mellin transform, using the Vallée secant operator or [6] for an approach by Nörlund–Rice integrals, via a complex lifting of the moment sums):

Theorem 3. *The expected cost of sorting n uniform real numbers given by their classical continued fraction representations satisfies*

$$\bar{P}(n) = n\sum_{l-1}^{n-1}(-1)^{l-1}\binom{n-1}{l}\bar{\rho}^{(l+1)} = K_0 n\ln n + K_1 n + Q(\ln n) + O(1),$$

where K_0 is Lévy's entropy constant and K_1 is a Porter-like constant (see [4]*):*

$$K_0 = \frac{6\ln 2}{\pi^2}, \qquad K_1 = 18\frac{\gamma\ln 2}{\pi^2} + 9\frac{(\ln 2)^2}{\pi^2} - 72\frac{\ln 2\,\zeta'(2)}{\pi^4} - \frac{1}{2}.$$

The function $Q(u)$ is an oscillating function with mean value 0 that satisfies $Q(n) = O(u^{\delta/2})$, where δ is any number such that $\delta > \sup\{\,\Re(s) \mid \zeta(s) = 0\,\}$.

For more details, we refer to Flajolet and Vallée's articles, available on their web pages:

`http://algo.inria.fr/flajolet` and `http://www.info.unicaen.fr/~brigitte`.

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