Popular Assignments and Extensions

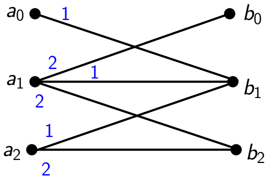
Kavitha Telikepalli

(Tata Institute of Fundamental Research, Mumbai)

Laboratoire d'Informatique de Paris-Nord (LIPN)
Université Sorbonne Paris Nord
May 21, 2024

The original model

Popular matching algorithms were first studied in the model of *one-sided* preferences.



Vertices on the left are agents and those on the right are items.

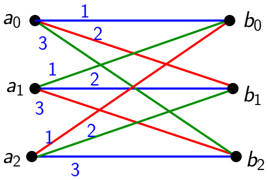
- agents have preferences (ties are allowed) over their neighbors;
- items have no preferences.

This is also called a *house allocation* instance.

Popular matchings in this model

We say $M \succ N$, i.e., M is *more popular* than N, if

$$\underbrace{|\{a \in A : M \succ_a N\}|}_{\#\text{ of agents that prefer } M} > \underbrace{|\{a \in A : N \succ_a M\}|}_{\#\text{ of agents that prefer } N}.$$



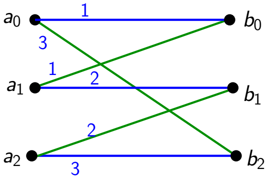
Let us hold elections between some pairs of matchings here.

- Say, between the green matching and the blue matching.

Popular matchings in this model

We say $M \succ N$, i.e., M is *more popular* than N, if

$$\underbrace{\left|\{a \in A : M \succ_a N\}\right|}_{\#\text{ of agents that prefer } M} > \underbrace{\left|\{a \in A : N \succ_a M\}\right|}_{\#\text{ of agents that prefer } N}.$$



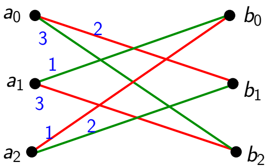
The green matching is more popular than the blue matching.

- In the green vs blue election: green gets 2 votes and blue gets only 1.

Popular matchings in this model

We say $M \succ N$, i.e., M is *more popular* than N, if

$$\underbrace{|\{a \in A : M \succ_a N\}|}_{\#\text{ of agents that prefer } M} > \underbrace{|\{a \in A : N \succ_a M\}|}_{\#\text{ of agents that prefer } N}.$$



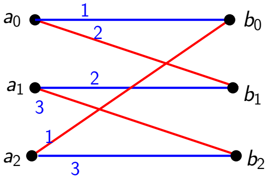
The red matching is more popular than the green matching.

- In the red vs green election: red gets 2 votes and green gets only 1.

Popular matchings in this model

We say $M \succ N$, i.e., M is *more popular* than N, if

$$\underbrace{\left|\{a \in A : M \succ_a N\}\right|}_{\#\text{ of agents that prefer } M} > \underbrace{\left|\{a \in A : N \succ_a M\}\right|}_{\#\text{ of agents that prefer } N}.$$



The blue matching is more popular than the red matching.

- In the blue vs red election: blue gets 2 votes and red gets only 1.

Popular matchings

So we have blue $\succ$ red $\succ$ green $\succ$ blue.

- For every matching here, there is a *more popular* matching.
- So this instance has no popular matching.

The popular matching problem

- *Given an instance* $G = (A \cup B, E)$*, does* G *admit a popular matching?*

Is there a simple characterization of popular matchings?

- Such a characterization is known.
- This leads to an efficient algorithm for the popular matching problem. [Abraham, Irving, K, and Mehlhorn, 2007]

Structure of popular matchings for strict rankings

For every $a \in A$, let us add a dummy item $d(a)$ as a's *worst* item.

- Henceforth, only A-perfect matchings are interesting.

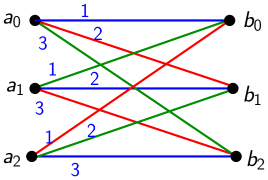
For any $a \in A$:

- let $f(a) = a$'s top choice item;
- let $s(a) = a$'s favorite item that is nobody's top item.

CLAIM. For any $a \in A$ and any popular matching M:

- $M(a)$ is either $f(a)$ or $s(a)$.

Structure of popular matchings for strict rankings



- Here $f(a_0) = f(a_1) = f(a_2) = b_0$.
- And $s(a_0) = s(a_1) = s(a_2) = b_1$.

M is popular $\Rightarrow M(a) \in \{f(a) \cup s(a)\} = \{b_0, b_1\}$ for all $a \in A$.

- There is no such A-perfect matching.
- Hence there is no popular matching.

Structure of popular matchings for strict rankings

1. Suppose a is matched in M to an item worse than $s(a)$.
   - Match $a' = M(s(a))$ to $f(a')$ [note that $s(a) \neq f(a')$].
   - Match a to $s(a)$.
   - Leave $M(f(a'))$ unmatched.

2. Suppose a is matched to an item strictly sandwiched between $f(a)$ and $s(a)$.
   - Observe that $M(a) = f(a')$ for some $a' \neq a$ [since $M(a) \notin \{f(a), s(a)\}$].
   - Match a' to $M(a)$.
   - Match a to $f(a)$.
   - Leave $M(f(a))$ unmatched.

In both cases, the resulting matching is more popular than M.

Structure of popular matchings with ties in rankings

Let $E_1 = \{$top edges in $G\}$, i.e., $ab \in E_1 \iff b$ is a top item for a.

- Matching M is popular $\Rightarrow M \cap E_1$ is a maximum matching in the top subgraph $G_1 = (A \cup B, E_1)$.
- What are the other edges in a popular matching M?

Call an item b non-critical if:

- b is left unmatched in some maximum matching in G_1.

For each $a \in A$:

- let $s(a) = \{a$'s favorite non-critical items$\}$;
- let $f(a) = \{a$'s top items$\}$.

M is popular $\Rightarrow M(a) \in f(a) \cup s(a)$ for all $a \in A$.

An efficient algorithm

The popular matching algorithm

- Let $E' = \{ab : a \in A \text{ and } b \in f(a) \cup s(a)\}$.
- Find a maximum matching M in the graph $G' = (A \cup B, E')$.
  - If M is not A-perfect then return "no popular matching".
  - Else return an A-perfect matching M^* in G' that maximizes $|M^* \cap E_1|$.

The algorithm finds a maximum matching M in the subgraph G' that has

- all edges ab s.t. $b \in f(a) \cup s(a)$.
  - M is A-perfect $\Rightarrow M^*$ is popular.
  - M is not A-perfect $\Rightarrow G$ has no popular matching.

An interesting example

The popular matching algorithm works when ties are allowed in preferences.

- However it does not work when preferences are partial orders.
- For partial order preferences, *indifference* is not necessarily transitive.

Consider the following instance on the complete bipartite graph with $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$.

a_1	$b_1 \succ b_3$; $b_2 \succ b_3$.
a_2	$b_1 \succ b_3$.
a_3	$b_2 \succ b_1$; $b_2 \succ b_3$.

In $M = \{a_1 b_1, a_2 b_2, a_3 b_3\}$, we have $M(a) \in f(a) \cup s(a)\ \forall a \in A$.

- But $N \succ M$ where $N = \{a_1 b_1, a_2 b_3, a_3 b_2\}$.
- a_3 prefers N to M while a_1 and a_2 are indifferent between M and N.

Random popular matchings

Consider a "random" instance $G = (A \cup B, E)$.

- Every a picks its ranking independently and uniformly at random from the set of all permutations of B.
- Thus every $a \in A$ has a complete and strict ranking.

If $|B| > (1.42 \cdot |A|) \Rightarrow$ popular matchings almost surely exist [Mahdian, 2006].

- In fact, there is a *phase transition* at 1.42.
- So $|B| < (1.42 - \delta) \cdot |A|$ where $\delta > 0$ is some constant
  $\Rightarrow$ almost surely the instance has no popular matching.

Mixed matchings

A mixed matching is a probability distribution over matchings, i.e.,

$$\Pi = \{(M_0, p_0), \ldots, (M_k, p_k)\},$$

where $M_0, \ldots, M_k$ are matchings in G and $\sum_i p_i = 1$ and $p_i \geq 0 \ \forall i$.

- A mixed matching is a lottery over matchings.

For any two matchings M and N:

$$\text{let } \Delta(M, N) \ = \ \# \text{ of votes for } M - \# \text{ of votes for } N.$$

- Define $\Delta(\Pi, N) = \sum_i p_i \cdot \Delta(M_i, N)$.

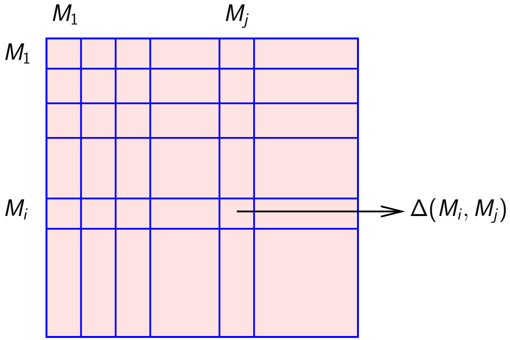
Definition. A mixed matching Π is popular if $\Delta(\Pi, N) \geq 0 \ \forall$ matchings N.

Popular mixed matchings

Do popular mixed matching always exist?

- Yes [K, Mestre, and Nasre, 2011].

We model this as a 2-player game.

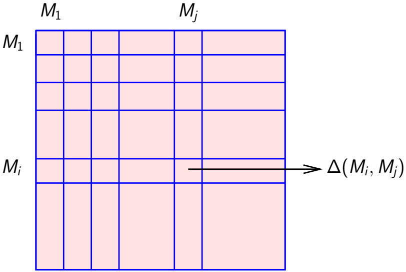


We need to show $\exists\, \Pi$ such that $\Delta(\Pi, N) \geq 0$ for all matchings N.

Popular mixed matchings

Consider the following game where the row player chooses a probability distribution $\langle p_1, \ldots, p_k \rangle$ over the rows.

- The column player chooses a column N.



- Value of the game is $\Delta(\Pi, N) = \sum_i p_i \cdot \Delta(M_i, N)$.

Popular mixed matchings

Row player is the *max-player* and column player is the *min-player*.

CLAIM. $\max_\Pi \min_N \Delta(\Pi, N) \leq 0$.

- Observe that $\Delta(\Pi, \Pi) = \sum_i \sum_j p_i p_j \cdot \Delta(M_i, M_j) = 0$.
  (since $\Delta(M_i, M_j) = -\Delta(M_j, M_i) \; \forall i, j$)
- Thus there exists a matching N such that $\Delta(\Pi, N) \leq 0$.

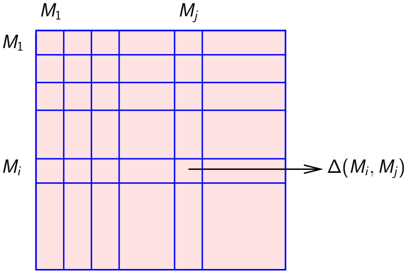
Hence for every Π there exists some N such that $\Delta(\Pi, N) \leq 0$.

- So $\max_\Pi \min_N \Delta(\Pi, N) \leq 0$.

Popular mixed matchings

Consider the *dual* game where the column player chooses a probability distribution $\langle p'_1, \ldots, p'_k \rangle$ over the columns first.

- The row player chooses a row N'.



- Value of the dual game is $\Delta(N', \Pi') = \sum_i p'_i \cdot \Delta(N', M_i)$.

Popular mixed matchings

Recall that the column player is the *min-player* and the row player is the *max-player*.

CLAIM. $\min_{\Pi'} \max_{N'} \Delta(N', \Pi') \geq 0$.

- Since $\Delta(\Pi', \Pi') = \sum_i \sum_j p'_i p'_j \cdot \Delta(M_i, M_j) = 0$:
  - there exists a matching N' such that $\Delta(N', \Pi') \geq 0$.

Hence for any Π' there exists an N' such that $\Delta(N', \Pi') \geq 0$.

- So $\min_{\Pi'} \max_{N'} \Delta(N', \Pi') \geq 0$.

Popular mixed matchings

We know from von Neumann's minimax theorem that

$$\max_{\Pi} \min_{N} \Delta(\Pi, N) = \min_{\Pi'} \max_{N'} \Delta(N', \Pi').$$

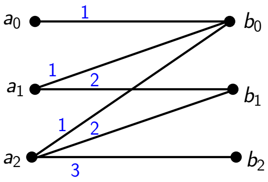
- Thus $0 \geq$ the left side $=$ the right side ≥ 0.
- Hence $\max_{\pi} \min_{N} \Delta(\Pi, N) = 0$, i.e., $\exists \Pi$ s.t. $\Delta(\Pi, N) \geq 0$ $\forall$ matchings N.

Thus popular mixed matchings always exist.

- Such a mixed matching can be computed efficiently as a popular fractional matching.

When cardinality is more important than popularity

Suppose the most important attribute of a matching is its cardinality.



So it is only maximum matchings that are admissible solutions.

What we seek is a maximum matching M such that:

- *no* maximum matching defeats M in their head-to-head election.
- a smaller matching may defeat M.

When cardinality is more important than popularity

The cardinality of the matching is important in many applications:

- assigning staff to hospitals in emergencies such as a pandemic;
- allocation problems for humanitarian organizations;
- assigning medical students to residencies.

We seek a maximum matching in these applications.

- Among maximum matchings, we want a "best" one.
- Thus we seek popularity *within* the set of maximum matchings.

Popular assignments

OUR PROBLEM. Find a popular maximum matching in G, if one exists.

By adding appropriate dummy agents and dummy items to G:

- we can assume wlog that G has a perfect matching, i.e., an assignment.

The popular assignment problem

- *Given an instance $G = (A \cup B, E)$, does G admit a popular assignment?*

- This generalizes the popular matching problem.

Popular assignments

- Add $|A|$ dummy items (one per agent as its last choice).
- Add $|B|$ dummy agents that are adjacent to all the $|A \cup B|$ items.
  - All neighbors are tied for any dummy agent.

Then any matching $M \rightsquigarrow$ a perfect matching $\tilde{M}$.

- $\Delta(M, N) = \Delta(\tilde{M}, \tilde{N})$ for any pair of matchings M and N.

Thus the popular assignment problem generalizes the popular matching problem.

Popular assignments

For 2-sided preferences:

- our algorithm in the red/blue graph

  $\longrightarrow$ the popular maximum matching algorithm in the colorful graph.

For 1-sided preferences:

- it is not clear how to generalize the popular matching algorithm to the popular assignment algorithm.

No combinatorial characterization of popular assignments is known.

The LP-method for popular assignments

Given an assignment M, define edge weights in G as follows. For any edge ab:

$$\mathrm{wt}_M(ab) \;=\; \begin{cases} 1 & \text{if } a \text{ prefers } b \text{ to its partner;} \\ -1 & \text{if } a \text{ prefers its partner to } b\text{;} \\ 0 & \text{otherwise.} \end{cases}$$

OBSERVATION. For any assignment N:

$$\mathrm{wt}_M(N) \;=\; \sum_{e \in N} \mathrm{wt}_M(e) \;=\; \#\text{ of votes for } N \;-\; \#\text{ of votes for } M.$$

- M is popular $\iff$ $\mathrm{wt}_M(N) \le 0$ for all assignments N in G.

Since $\mathrm{wt}_M(M) = 0$:

- M is popular $\iff$ M is a max-weight assignment under wt_M.

The LP-method

LP for max-weight assignment:

$$\max \sum_{e \in E} \mathsf{wt}_M(e) \cdot x_e$$

$$\sum_{e \in \delta(v)} x_e \;=\; 1 \;\; \forall v \in A \cup B \quad \text{and} \quad x_e \;\geq\; 0 \;\; \forall e \in E.$$

M is popular $\iff$ the optimal value of this LP is 0.

The dual LP:

$$\min \sum_{v} \alpha_v$$

$$\alpha_a + \alpha_b \;\geq\; \mathsf{wt}_M(ab) \;\; \forall ab \in E$$

M is popular $\iff$ the optimal value of the dual LP is 0.

Dual certificate

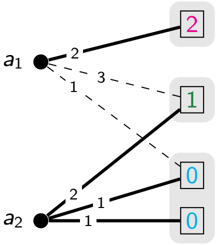
CLAIM. M is popular $\iff$ $\exists$ dual feasible solution $\vec{\alpha}$ such that $\sum_v \alpha_v = 0$ and

- $\alpha_a \in \{0, 1, 2, \ldots, n-1\}$ for all $a \in A$;
- $\alpha_b \in \{0, -1, -2, \ldots, -(n-1)\}$ for all $b \in B$.

Such a solution $\vec{\alpha}$ to the dual LP is a *dual certificate* for M.

- Let $c : B \to \{0, 1, 2, \ldots, n-1\}$.
  - For each $a \in A$: let $c^*(a) = \underbrace{\max\{c(b) : b \in \text{Nbr}(a)\}}_{\text{highest color among } a\text{'s neighbors}}$.
  - We can define a subgraph $G_c = (A \cup B, E_c)$ of G as follows.

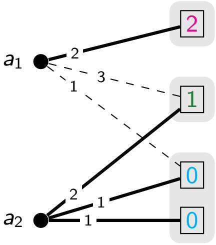
The subgraph G_c



Here $c^*(a_1) = 2$ and $c^*(a_2) = 1$.

- Each a keeps edges to its most preferred neighbors in color $c^*(a)$.
- Furthermore, a keeps edges to its most preferred neighbors in color $c^*(a) - 1$ if they are preferred to all neighbors in color $c^*(a)$.
- The bold edges are in E_c and the dashed edges are not.

The right function $c \iff$ there is a popular assignment



G has a popular assignment if and only if

- $\exists c : B \to \{0, 1, 2, \ldots, n-1\}$ s.t. G_c admits a perfect matching M;
  - $\alpha_b = -c(b)$ for $b \in B$ and $\alpha_a = c(M(a))$ for $a \in A$ is M's dual certificate.

PROBLEM: Find a right function $c : B \to \{0, 1, 2, \ldots, n-1\}$, if there is one.

The popular assignment algorithm

Input: $G = (A \cup B, E)$ where $|A| = |B| = n$.

1. Initialize $c(b) = 0$ for every $b \in B$.
2. Compute a maximum matching M in the subgraph G_c.
3. If M is perfect then return M.
4. For every unmatched $b \in B$ do: $c(b) = c(b) + 1$.
5. If $c(b) \leq n - 1$ for all $b \in B$ then go back to Step 2; else return "no".

The above algorithm solves the popular assignment problem [K, Király, Matuschke, Schlotter, and Schmidt-Kraepelin, 2022].

Analysing the popular assignment algorithm

Eventually, either a perfect matching M in G_c is returned
or $c(b) = n$ for some $b \in B$.

- If M is returned: $\alpha_b = -c(b)$ for $b \in B$ and $\alpha_a = c(M(a))$ for $a \in A$ is $\underline{M\text{'s dual certificate}}$.
- Suppose $c(b) = n$ for some $b \in B$.

Let $\vec{\beta}$ be a dual certificate for some popular assignment in G.

- We show $c(b) \le |\beta(b)|$ for all $b \in B$ where $c(b)$ is b's c-value at the end. This means:

$$n \;=\; c(b) \;\le\; |\beta(b)| \;\le\; n-1, \text{ a contradiction.}$$

- Our algorithm says "no" $\Rightarrow$ there is indeed no popular assignment in G.

A popular matching algorithm

Input: $G = (A \cup B, E)$ where $|A| = |B| = n$.

1. Initialize $c(b) = 0$ for every $b \in B$.
2. Compute a maximum matching M in the subgraph G_c.
3. If M is perfect then return M.
4. For every unmatched $b \in B$ do: $c(b) = c(b) + 1$.
5. If $c(b) \leq n - 1$ for all $b \in B$ then go back to Step 2; else return "no".

REMARK. Suppose "$c(b) \leq n - 1$" in step 5 is replaced with "$c(b) \leq 1$".

- Then the resulting algorithm solves the popular matching problem.
- This algorithm works for partial order preferences as well.

Popularity with forced edges

Given a set $\{e_1, \ldots, e_k\}$ in G:

- Is there a popular assignment in G that contains all these k edges?

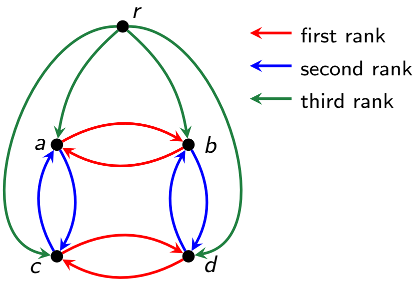
Our algorithm can be easily updated to solve the above problem.

- Suppose there is no such popular assignment.
- Find a popular assignment that contains as many of these k edges as possible.

This problem is NP-hard.

- Thus it is NP-hard to find a min-cost popular assignment when there is a function cost : $E \to \{0, 1\}$.
- This hardness holds even when all agents have strict rankings.

Liquid democracy



- There are n voters.
- Every voter considers its in-neighbors to be better informed than itself.
- It seeks to delegate its vote to an in-neighbor.
- It has preferences over its in-neighbors.
- Delegation cycles are forbidden.

E.g., a considers b as her best in-neighbor and c as her second best in-neighbor.

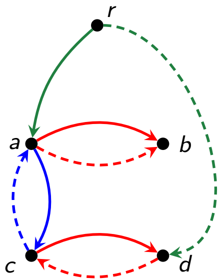
- For convenience, a dummy vertex r has been added as the root.

PROBLEM. Find an optimal arborescence as per vertex preferences.

Comparing two arborescences

A vertex prefers the arborescence where it has a more preferred in-neighbor.

- Let us compare the solid arborescence A with the dashed arborescence A'.



- a prefers A' to A since it prefers c to r;
- b is indifferent between A and A';
- c prefers A' to A since it prefers d to a;
- d prefers A to A' since it prefers c to r;
- so A' gets 2 votes and A gets 1 vote, thus $A' \succ A$.

Arborescence A is <u>popular</u> if there is no arborescence A' such that $A' \succ A$.

- A popular arborescence represents a stable way of delegating votes.

Popular arborescences

QUESTION. Does an instance $G = (V \cup \{r\}, E)$ have a popular arborescence? If so, find one.

- Our popular assignment algorithm can be extended to solve this problem.

Matroids [Whitney, 1935]

Combinatorial structures that generalize the notion of linear independence in matrices.

- Assignments are common bases in the intersection of two *partition* matroids.
- Arborescences are common bases in the intersection of a partition matroid with a *graphic* matroid.

The LP-method for popular arborescences

For any arborescence A and $v \in V$, let $A(v)$ be the unique edge in $A \cap \delta(v)$.

(here $\delta(v)$ is the set of v's incoming edges)

Given an arborescence A, define edge weights in G as follows. For any $e \in \delta(v)$:

$$\mathsf{wt}_A(e) \;=\; \begin{cases} 1 & \text{if } v \text{ prefers } e \text{ to } A(v); \\ -1 & \text{if } a \text{ prefers } A(v) \text{ to } e; \\ 0 & \text{otherwise.} \end{cases}$$

OBSERVATION. For any arborescence A':

$$\mathsf{wt}_A(A') \;=\; \sum_{e \in A'} \mathsf{wt}_A(e) \;=\; \#\text{ of votes for } A' \;-\; \#\text{ of votes for } A.$$

- A is popular $\iff$ $\mathsf{wt}_A(A') \le 0$ for all arborescences A' in G.

The LP-method for popular arborescences

Since $\text{wt}_A(A) = 0$:

- A is popular $\iff$ A is a max-weight arborescence under wt_A.

LP for max-weight arborescence:

$$\max \sum_{e \in E} \text{wt}_A(e) \cdot x_e$$

$$\sum_{e \in S} x_e \quad \leq \quad \text{rank}(S) \;\; \forall S \subseteq E$$

$$\sum_{e \in \delta(v)} x_e \quad = \quad 1 \;\; \forall v \in V \quad \text{and} \quad x_e \;\geq\; 0 \;\; \forall e \in E.$$

For any $S \subseteq E$: $\text{rank}(S)$ is the maximum size of an acyclic subset of S in G.

- A is a popular arborescence $\iff$ the optimal value of this LP is 0.

The dual LP

$$\min \left(\sum_{v \in V} \alpha_v + \sum_{S \subseteq V} \text{rank}(S) \cdot y_S \right)$$

$$\begin{aligned} \sum_{S : e \in S} y_S \; + \; \alpha_v \quad & \geq \quad \text{wt}_A(e) \quad \forall e \in \delta(v), \; \forall v \in V \\ y_S \quad & \geq \quad 0 \quad \forall S \subseteq E. \end{aligned}$$

A is popular $\iff$ the optimal value of the dual LP is 0.

- $\exists$ an integral optimal solution $(\vec{y}, \vec{\alpha})$ s.t. $\{S : y_S > 0\}$ is a *chain*.
- A chain $\mathcal{C} = \{C_0, C_1, \ldots, C_k\}$ has the form $C_0 \subset C_1 \subset \cdots \subset C_k$.

Moreover, we will have a chain $\emptyset \subset C_0 \subset \cdots \subset C_k = E$.

Dual certificates

Our chain $\mathcal{C}$ induces a coloring $c : E \to \{0, 1, 2, \ldots, k\}$ where

- $c(e) =$ the index i such that $e \in C_i \setminus C_{i-1}$.
- For each $v \in V$: let $c^*(v) = \underbrace{\max\{c(e) : e \in \delta(v)\}}_{\text{highest color among } v\text{'s incoming edges}}$.

We define $E_{\mathcal{C}} \subseteq E$: for any $v \in V$, edge $e \in \delta(v)$ is in $E_{\mathcal{C}}$ if:

- either $c(e) = c^*(v)$ and $e \succeq_v e'$ for all $e' \in \delta(v)$ with color $c^*(v)$
- or $c(e) = c^*(v) - 1$ and (i) $e \succeq_v e'$ for all $e' \in \delta(v)$ with color $c^*(v) - 1$ and (ii) $e \succ_v e'$ for all $e' \in \delta(v)$ with color $c^*(v)$.

Dual certificates

Arborescence A is popular $\iff \exists\, \mathcal{C} = \{C_0, \ldots, C_k\}$ such that

- $\emptyset \subset C_0 \subset \cdots \subset C_k = E$;
- $A \subseteq E_{\mathcal{C}}$;
- $\text{span}(A \cap C_i) = C_i$ for all i where

  for any $S \subseteq E$: $\text{span}(S) = \{e : \text{rank}(S \cup \{e\}) = \text{rank}(S)\}$.

The dual certificate $(\vec{y}, \vec{\alpha})$ for A will be:

- Let $y_S = 1 \ \forall\, S \in \mathcal{C}$ and $y_S = 0$ for all other S.
- Let $\alpha_v = -(\#$ of sets in $\mathcal{C}$ that $A(v)$ belongs to$)$.

PROBLEM. Find an arborescence A and chain $\mathcal{C}$ if there exist such an A and $\mathcal{C}$.

The popular arborescence algorithm

Input: $G = (V \cup \{r\}, E)$ where $|V| = n$.

1. Initialize $k = 0$ and $C_0 = E$.
2. Compute a branching $I \subseteq E_{\mathcal{C}}$ that lex-maximizes $(|I \cap C_0|, \ldots, |I \cap C_k|)$.
3. If $|I \cap C_i| = \text{rank}(C_i)\ \forall i$ then return I.
4. Let j be the minimum index such that $|I \cap C_j| < \text{rank}(C_j)$.
5. Update $C_j = \text{span}(I \cap C_j)$.
6. If $j = k$ then
   - If $k \leq n - 1$ then $k = k + 1$, $C_k = E$, and $\mathcal{C} = \mathcal{C} \cup \{E\}$; go back to step 2.
   - Else return "no".

The above algorithm solves the popular arborescence problem [K, Makino, Schlotter, and Yokoi, 2024].

References

1. M. Mahdian. *Random popular matchings.* In EC 2006.

2. D. J. Abraham, R. W. Irving, T. Kavitha, and K. Mehlhorn. *Popular matchings.* SIAM Journal on Computing, 2007.

3. T. Kavitha, J. Mestre, and M. Nasre. *Popular mixed matchings.* Theoretical Computer Science, 2011.

4. T. Kavitha, T. Király, J. Matuschke, I. Schlotter, U. Schmidt-Kraepelin. *The popular assignment problem: when cardinality is more important than popularity.* In SODA 2022.

5. T. Kavitha, K. Makino, I. Schlotter, Y. Yokoi. *Arborescences, colorful forests, and popularity.* In SODA 2024.

Thank you! Any questions?