The Takagi–van der Waerden functions via elephant random walks remembering the very recent past

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Joint work with Masato Takei (Yokohama National University)
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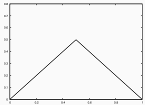
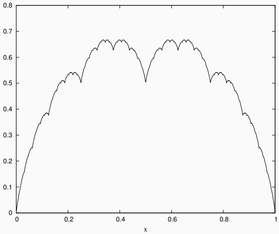
- $d(x)$: the distance from x to its nearest integer.
- $d(r^k x)$: increasing the frequency to r^k ($r \in \mathbb{Z}, r \geq 2$).



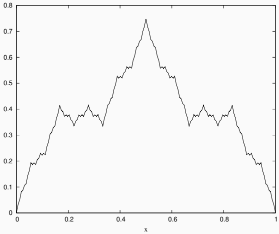
- $f_r(x) := \sum_{k=0}^{\infty} \frac{1}{r^k} d(r^k x)$: **the Takagi–van der Waerden functions.**

▶ f_r is continuous but nowhere differentiable.

[$r = 2$: Takagi (1903). $r = 10$: van der Waerden (1930).]



The graph of $f_2(x)$



The graph of $f_3(x)$

- $f_r(x) = \sum_{k=0}^{\infty} \psi_k(x)$: Takagi–van der Waerden functions.

For $h \neq 0$, we can find the unique integer $m(h)$ s.t. $|h| \in (\frac{1}{r^{m+1}}, \frac{1}{r^m}]$.

As $k \le m(h) \iff r^k|h| \le 1$,

$$|\psi_k(x+h) - \psi_k(x)| \le \begin{cases} |h| & (k \le m(h)), \\ r^{-k} & (k > m(h)). \end{cases}$$

Therefore, $|f_r(x+h) - f_r(x)| \le |h| \sum_{k=0}^{m(h)} 1 + \sum_{k=m(h)+1}^{\infty} r^{-k}$

$$= |h| \cdot m(h) + r^{-m(h)-1} \cdot \frac{r}{r-1} \le |h| \log_r(1/|h|) + \frac{r|h|}{r-1}.$$

■ For fixed $x \in \mathbb{R}$, we have

$$-1 \le \liminf_{h\to 0} \frac{f_r(x+h)-f_r(x)}{h\log_r(1/|h|)} \le \limsup_{h\to 0} \frac{f_r(x+h)-f_r(x)}{h\log_r(1/|h|)} \le 1.$$

▶ More precisely, we can obtain the following estimates.

Theorem. [N. and Takei (2026)]

$$\lim_{h\to 0} \mathrm{Leb}\left(\left\{x\in[0,1) \colon \frac{f_r(x+h)-f_r(x)}{\sigma_r h\sqrt{\log_r(1/|h|)}} \le y\right\}\right) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}dt,$$

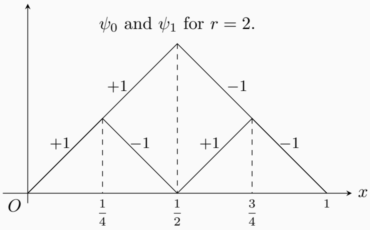
$$\limsup_{h\to 0} \frac{f_r(x+h)-f_r(x)}{\sigma_r h\sqrt{2\log_r(1/|h|)\log\log\log_r(1/|h|)}} = 1 \quad \text{Leb-a.e. } x\in[0,1).$$

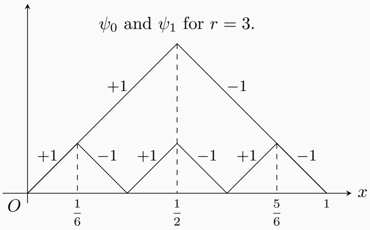
■ $\sigma_r^2 = 1$ if r is even, and $\sigma_r^2 = \dfrac{r+1}{r-1}$ if r is odd.

[$r=2$: Kôno (1987), Gamkrelidze (1990), Allaart (2009)]

$$\boxed{f_{r,N}(x) := \sum_{k=0}^{N-1} \underbrace{\frac{1}{r^k} d(r^k x)}_{=:\psi_k(x)}}$$

- $\psi_k^+(x) \in \{\pm 1\}$: the right-hand derivative
- Leb: the Lebesgue measure

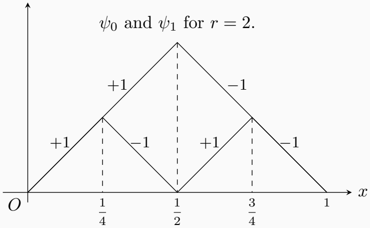


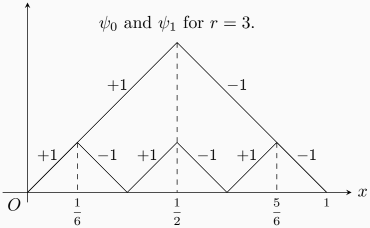


- Dividing $[0, 1)$ into $2r^k$ equal parts.

▶ If r is even, then $\mathrm{Leb}(\psi_{k+1}^+ = \pm 1 \mid \psi_k^+ = \pm 1) = 1/2$.

▶ If r is odd, then $\mathrm{Leb}(\psi_{k+1}^+ = \pm 1 \mid \psi_k^+ = \pm 1) = (r + 1)/2r$.





■ $(\Omega, \mathcal{B}, \mathrm{Leb})$: the Lebesgue prob. sp.
i.e., $\Omega = [0, 1)$, $\mathcal{B}$: Borel σ-field on Ω.

$$p_r = \begin{cases} \dfrac{1}{2} & \text{if } r \text{ is even,} \\ \dfrac{r+1}{2r} & \text{if } r \text{ is odd.} \end{cases}$$

$$\psi_{k+1}^+ = \begin{cases} \psi_k^+ & \text{w.p. } p_r, \\ -\psi_k^+ & \text{w.p. } 1 - p_r. \end{cases}$$

▶ In fact, $\{\psi_k^+\}_{k=0}^\infty$ is a stationary Markov chain.
[If r is even, $\{\psi_k^+\}$ is i.i.d. Rademacher.]

• We focus on the right-hand derivative $f_{r,N}^+(x) = \sum_{k=0}^{N-1} \psi_k^+(x)$.

■ The elephant random walk (ERW) is a discrete-time nearest neighbour stochastic process on the integers with its history, introduced by Schütz and Trimper (2004).

Def. $P(X_1 = +1) = 1 - P(X_1 = -1) = q \in [0, 1]$.

$\{\beta_n : n \in \mathbb{N}\}$: indep., β_n: $\{1, \dots, n\}$-valued r. v. (**memory variables**)

$$X_{n+1} := \begin{cases} X_{\beta_n} & \text{w.p. } p \in [0,1], \\ -X_{\beta_n} & \text{w.p. } 1-p. \end{cases}$$

■ $T_n := \sum_{k=1}^{n} X_k$: ERW with memory parameter p.

■ If $\beta_n \overset{\mathrm{d}}{=} \mathrm{Unif}\{1, \dots, n\}$, $\{T_n\}$ is original ERW.

$$\psi^+_{n+1} = \begin{cases} \psi^+_n & \text{w.p. } p_r, \\ -\psi^+_n & \text{w.p. } 1-p_r. \end{cases}$$

► $f^+_{r,N}(x) = \underline{\sum_{k=0}^{N-1} \psi^+_k(x)}$ for $x \in [0, 1)$.

ERW remembering the very recent past (ERWVRP)

■ $T_n = \sum_{k=1}^{n} X_k$: ERWVRP with parameter p.

■ $\{T_n\}$ is also known as the correlated random walk. Letting $\sigma^2 := \frac{p}{1-p}$ for $p \in (0,1)$, we have

Theorem. [Allaart (2009), Gut–Stadmüller (2021), Konno (2009)]

$$\frac{T_n}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1), \quad \limsup_{n\to\infty} \frac{T_n}{\sigma\sqrt{2n\log\log n}} = 1 \ \text{ a.s.}$$

- ▶ No matter how “memory” is strong, ERWVRP is diffusive.
- ▶ Letting $m(h) = \lfloor \log_r(1/|h|) \rfloor$,

$f_r(x+h) - f_r(x)$ can be well approximated by $h \cdot T_{m(h)}$!

Takagi–van der Waerden class functions

$$\boxed{\psi_k(x) = r^{-k} d(r^k x)}$$

- If $\{w_k r^{-k}\} \in \ell^1$,

then $T_{r,w}(x) := \sum_{k=0}^{\infty} w_k \psi_k(x)$ exists for all $x \in [0,1)$.

- $T_{r,w}$ is continuous, because

$$\sup_{x\in[0,1]} \left| \sum_{k=0}^{\infty} w_k \psi_k(x) - \sum_{k=0}^{N-1} w_k \psi_k(x) \right| \leq \sum_{k=N}^{\infty} \left| \frac{w_k}{r^k} \right|.$$

Def. Cont. func. $f(x)$ is said to be "smooth" in the sense of Zygmund if

$$f(x+h) + f(x-h) - 2f(x) = o(h) \text{ as } h \to 0 \text{ for } \forall x.$$

Note. $C^2(0,1) \subsetneq C^1(0,1) \subsetneq \{\text{"smooth" functions on } (0,1)\}$.

Theorem. [Kôno (1987)] $T_{2,w}(x)$ is "smooth" i.f.f. $T_{2,w}(x) = ax(1-x)$.

- E.g., taking $w_k = 2^{-k}$, $T_{2,w}(x) = 4x(1-x)$.

Theorem. [N. and Takei (2026)] Suppose $\{w_k r^{-k}\} \in \ell^1$.

(i) If $\{w_k\} \in \ell^2$ then $T_{r,w}$ is abs. cont. with $T'_{r,w}(x) = \sum_{k=0}^{\infty} w_k \psi_k^+(x)$ a.e.

(ii) If $\{w_k\} \notin \ell^2$ and $w_k \to 0$, then $T_{r,w}$ is not differentiable a.e., but differentiable on a set of continuum cardinality.

(iii) If $\limsup_{k\to\infty} |w_k| > 0$ then $T_{r,w}(x)$ is nowhere differentiable.

[$w_k \equiv 1$: the Takagi–van der Waerden functions]

$r = 2$: Kôno (1987),

r is even:

Ferrera–Gómez-Gil (2020), Ferrera–Gómez-Gil–Llorente (2025).

Random signs problem

- $\displaystyle\sum_{k=1}^{\infty} \frac{\boxed{+1}}{k} = \infty$ diverges, but $\displaystyle\sum_{k=1}^{\infty} \frac{\boxed{(-1)^{k-1}}}{k} = \log 2$ converges.

[Add with $+1$ signs.] [Add with alternating ± 1 signs.]

- The series with "random" signs $\displaystyle\sum_{k=1}^{\infty} \frac{\boxed{R_k}}{k}$ converges w. p. 1.

[$\{R_k\}$: i.i.d., $P(R_k = \pm 1) = 1/2$.]

Q. For a sequence $\{a_k\}_k$, does $\Sigma_n := \displaystyle\sum_{k=1}^{n} a_k R_k$ converge or diverge?

- $\{a_k\} \notin \ell^2 \implies P\left(-\infty = \varliminf_{n\to\infty} \Sigma_n < \varlimsup_{n\to\infty} \Sigma_n = +\infty\right) = 1.$
- $\{a_k\} \in \ell^2 \implies P\left(\lim_{n\to\infty} \Sigma_n \in \mathbb{R}\right) = 1.$

[Rademacher (1922), Khintchine–Kolmogorov (1925)]

- Σ_n can also be seen as a SRW with variable step length.

■ $S_n = \sum_{k=1}^{n} a_k X_k$: ERWVRP with variable step length.

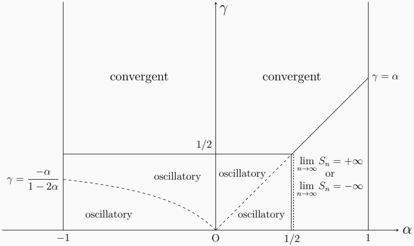
Theorem. [N. and Takei (2026)] Suppose $p \in (0,1)$.

(i) $\{a_k\} \in \ell^2 \implies \exists \lim_{n\to\infty} S_n$ a.s. and in L^2.

(ii) $\{a_k\} \notin \ell^2 \implies S_n$ diverges a.s.

▶ The convergence criterion is the same as "memoryless" case.

Taking $a_k = k^{-\gamma}$,
the original ERW may diverse
even if $\{a_k\} \in \ell^2$. [N. (2025, JSP)]



the classification of long-time behavior of original ERW taking $a_k = k^{-\gamma}$

Proof: limit theorems for ERWVRP

- $T_n = \sum_{k=1}^{n} X_k$: ERWVRP with parameter $p \in (0,1)$.

- $\{X_k\}_k \subset \{\pm 1\}$: Markov chain with its transition matrix $\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$,

satisfying $P(X_1 = +1) = P(X_1 = -1) = 1/2$.

▶ For $A \in \sigma\{X_k : k \le n\},\ B \in \sigma\{X_k : k \ge n+m\}$,

$$|P(A \cap B) - P(A)P(B)| \le P(A) \cdot \frac{|2p-1|^m}{2}.$$

$\rightsquigarrow$ $\{X_k\}_k$ is a φ-mixing sequence. [Stout (1974), Ex 3.7.2]

Applying the limit theorems for $\begin{cases} \varphi\text{-mixing} \\ \text{stationary} \end{cases}$, we complete the proof.

[E.g., Billingsley (1979) Thm 27.5, Stout (1974) Thm 5.4.4]

Proof: convergence condition for weighted ERWVRP

$$\boxed{S_n = \sum_{k=1}^{n} a_k X_k.}$$

$$\boxed{A_n = \sum_{k=1}^{n} (a_k)^2.}$$

Theorem. (recalled)

(i) $\{a_k\} \in \ell^2 \implies \exists \lim_{n\to\infty} S_n$ a.s. and in L^2.

(ii) $\{a_k\} \notin \ell^2 \implies S_n$ diverges a.s.

Proof sketch.

(i) Using the Yoshihara (1978) inequality,

$$E[(S_n - S_m)^4] \le C_1(A_n - A_m)^2.$$

[Using the φ-mixing condition.]

By the Móricz (1976) inequality, for $n > m \ge 1$,

$$E\Big[\Big(\max_{m<l\le n} |S_l - S_m|\Big)^4\Big] \le C_1 C_2 (A_n - A_m)^2.$$

$\rightsquigarrow$ L^2-convergence implies a.s.-convergence.

Proof sketch. (ii) Using the following lemma in order to prove (ii).

Lemma. For any $\{a_k\}$ and $n > m \geq 1$, there exists $K(p) \geq 1$ s.t.

$$\frac{1}{K(p)} \sum_{m<k\leq n} a_k^2 \leq E\left[(S_n - S_m)^2\right] \leq K(p) \sum_{m<k\leq n} a_k^2.$$

- Supposing $\{S_n\}$ converges w.p. 1, $P(|S_n - S_m| < {}^{\exists}M)$ is close to 1.

$\rightsquigarrow$ $\{a_k\} \in \ell^2$. I.e., $\{a_k\} \notin \ell^2$ implies $P(\{S_n\}$ converges$) < 1$.

- By the tail triviality for a φ-mixing seq., $P(\{S_n\}$ converges$) = 0$. ∎

Lemma. For $h \in (0, 1/r)$, letting $m(h) = \lfloor \log_r(1/h) \rfloor$,

$$f_r(x+h) - f_r(x) = h \cdot s_{m(h)}(x) + o(h\sqrt{m(h)}) \text{ as } h \downarrow 0$$

for Leb-a.e. $x \in [0,1)$, where $s_n(x) := \sum_{k=0}^{n-1} \psi_k^+(x)$.

Proof of the modulus of continuity results.

Since $s_{m(h)}(x)$ is the ERWVRP with parameter p_r,

$$\frac{s_{m(h)}}{\sigma_r \sqrt{m(h)}} \xrightarrow{d} N(0,1), \quad \varlimsup_{h \downarrow 0} \frac{s_{m(h)}(x)}{\sigma_r \sqrt{2m(h) \log\log m(h)}} = 1 \text{ for a.e. } x.$$

- Noting $m(h) \sim \log_r(1/h)$. ■

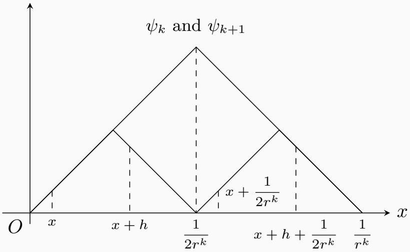
Proof of Lemma.

- Focusing on $f_r(x+h) - f_r(x) = \sum_{k=0}^{\infty} (\psi_k(x+h) - \psi_k(x))$.
- Letting $N_0(x,h) := \{k \geq 0 \colon \psi_k(x+h) - \psi_k(x) = h \cdot \psi_k^+(x)\}$, we have

$$f_r(x+h) - f_r(x) = h \cdot \sum_{k \in N_0(x,h)} \psi_k^+(x) + \text{“remainder terms”}$$

We shall show that $N_0(x,h)$ is sufficiently large as $h \to 0$.

▶ $k \in N_0(x,h) \iff x$ and $x+h$ coincide up to the k-th fractional digit & $x + 1/(2r^k)$ and $x+h+1/(2r^k)$ coincide up to the k-th fractional digit.



▶ When r is even, a sufficient condition for both conditions is that

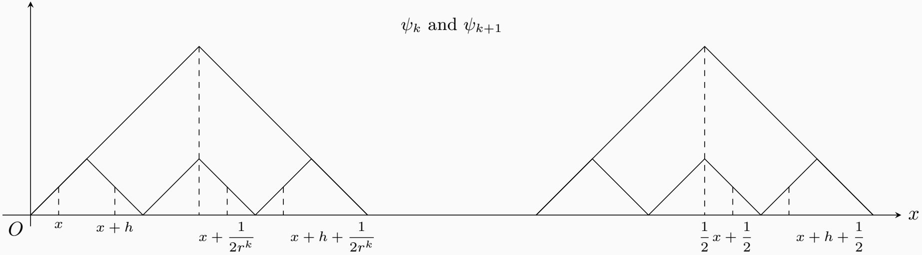
$$x \text{ and } x+h \text{ coincide up to the } (k+1)\text{-th fractional digit.}$$

For the base-r expansions $x = \varepsilon_0 + \sum_{k=1}^{\infty} \frac{\varepsilon_k}{r^k}$ and $x+h = \varepsilon_0' + \sum_{k=1}^{\infty} \frac{\varepsilon_k'}{r^k}$, letting $k_0(x,h) := \max\{k\colon \varepsilon_0 = \varepsilon_0', \dots, \varepsilon_k = \varepsilon_k'\}$, [$\max\varnothing := -1$.]

$$\text{if } 0 \le k < k_0(x,h), \text{ then } k \in N_0(x,h).$$

🤔 What if r is odd?

▶ $k \in N_0(x,h) \Leftrightarrow x$ and $x+h$ coincide up to the k-th fractional digit. & $x+1/(2r^k)$ and $x+h+1/(2r^k)$ coincide up to the k-th fractional digit.



■ When r is odd, the latter condition is equivalent to "$x+1/2$ and $x+h+1/2$ coincide up to k-th fractional digit". I.e.,

$$\text{If } 0 \le k < \min\left\{k_0(x,h), k_0\left(x+\frac{1}{2}, h\right)\right\}, \text{ then } k \in N_0(x,h).$$

■ need to show that $k_0(x,h) \to \infty$ as $h \downarrow 0$. Letting $m(h) = \lfloor \log_r(1/h) \rfloor$,

Lemma. $k_0(x,h) = m(h) + O\big(\log_r m(h)\big)$ Leb-a.e. $x \in [0,1)$.

Thus, when r is even,

$$
\begin{aligned}
f_r(x+h) - f_r(x) = h \cdot \sum_{k=0}^{m(h)} \psi_k^+(x) \\
&+ \sum_{k=k_0(x,h)}^{m(h)} \left(\psi_k(x+h) - \psi_k(x) - h \cdot \psi_k^+(x)\right) \\
&+ \sum_{k>m(h)} \left(\psi_k(x+h) - \psi_k(x)\right).
\end{aligned}
$$

- The 1st term: ERWVRP at time $m(h) \sim \log_r(1/h)$.
- The 2nd and 3rd terms: As $h \downarrow 0$, $o(h \log_r(1/h))$.

• When r is odd, replace $k_0(x,h)$ with $\min\Big\{k_0(x,h), k_0\big(x+\frac{1}{2}, h\big)\Big\}$. ■

Future Works

- the modulus of continuity of $f_{r,a}(x)$?
- almost sure invariance for $f_r(x)$?

$$f_r(x+h) - f_r(x) = h \cdot \sigma_r B(m(h)) + O(h \cdot m(h)^{1/4+\delta}) \text{ as } h \to 0, \text{ a.s.},$$

where $m(h) = \lfloor \log_r(1/|h|) \rfloor$, $B(t)$: standard BM.

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