Remarks on the Elephant Random Walk

Ulrich Stadtmüller, *Ulm University*

joint work with Allan Gut, *University of Uppsala*

Webinar, 5.05.2026

Outline

1. 0. Introduction of myself
2. 1.The classical random walk

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   - 2.b Remembering only the recent past
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   - Bernoulli ERW's
   - Behavior of S_n

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   - Behavior of S_n
7. Some Literature, related to the talk

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Position: Retired Professor from Ulm University

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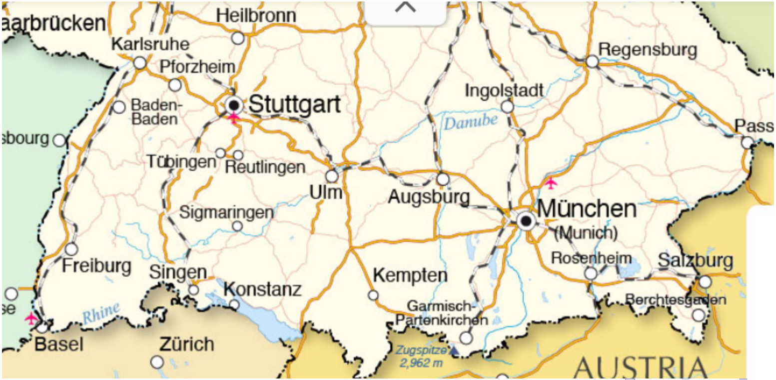

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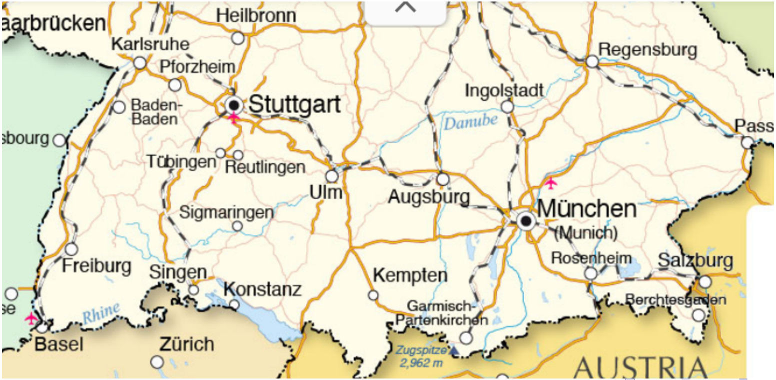
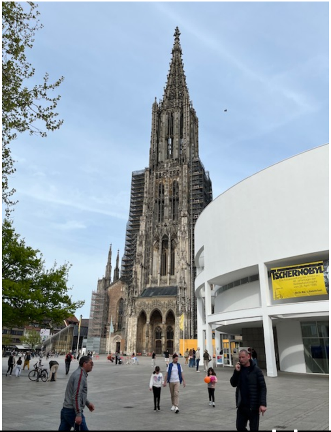
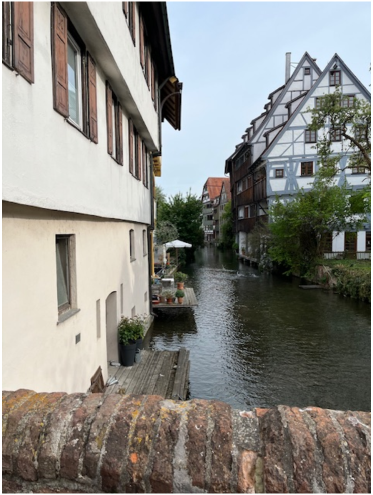
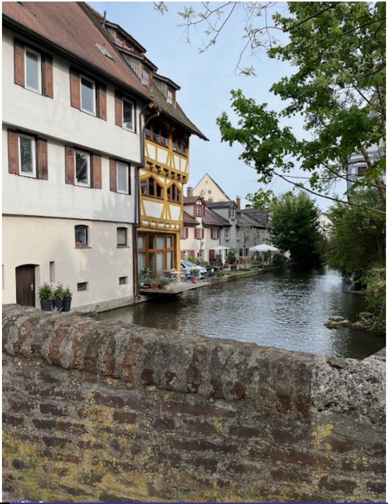


Ulm is a midsize city lying at the Danube with a gothic cathedral with the highest church spire in the world



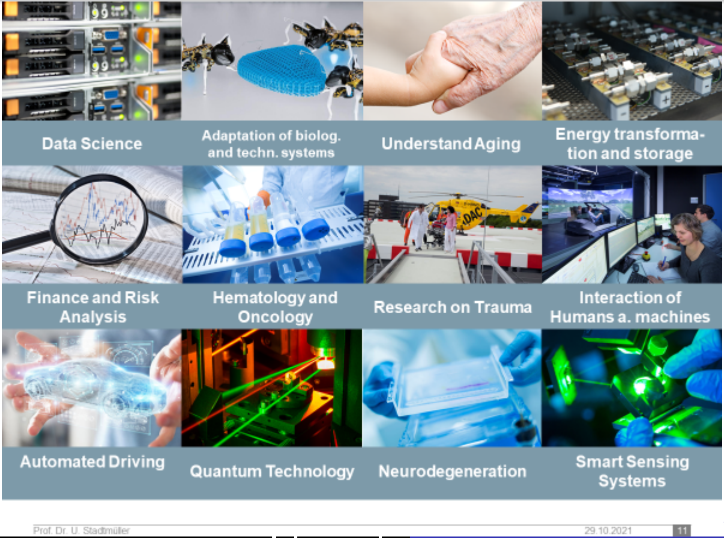




Ulm University, founded in 1967 with

Departments: Medicine, Natural Sciences, Electrical Engineering, Computer Science and Psychology, Mathematics & Management Sciences

Excellency Cluster in Battery Research


Data Science
Adaptation of biolog. and techn. systems
Understand Aging
Energy transforma-tion and storage
Finance and Risk Analysis
Hematology and Oncology
Research on Trauma
Interaction of Humans a. machines
Automated Driving
Quantum Technology
Neurodegeneration
Smart Sensing Systems
Prof. Dr. U. Stadtmüller
29.10.2021
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One of the starting points of Probability was the classical random walk where

$X_k = \pm 1$ with Prob. p and $1-p$ resp. for $k \geq 1$ (with $p \in (0,1)$) are independent random variables
Their partial sums are denoted by $S_n = \sum_{k=1}^{n} X_k$ for $n \geq 1$.

The classical limit theorems where proven

WLN & SLN

$$\frac{S_n}{n} \to E(X_1) = 2p-1 \quad \text{as } n \to \infty \quad \text{in Prob. or a.s.}$$

CLT

$$\frac{S_n - n(2p-1)}{\sqrt{n}} \xrightarrow{d} \mathcal{N}_{0,4p(1-p)} \quad \text{as } n \to \infty.$$

LIL

$$\limsup_{n\to\infty}(\liminf_{n\to\infty})\frac{S_n - n(2p-1)}{\sqrt{8p(2p-1)n\log\log n}} \overset{a.s.}{=} 1(-1).$$

Furthermore Invavriance Principles, Level Crossings etc.

There are many different models to generalize the simple situation of an ordinary random walk, e.g.:

i) Random walks in a random enviroment (RWRE)

ii) Random walks with certain dependence structures:

e.g., The Elephant Random Walk (ERW)

An elephant is known to have a long memory, so for an ERW the step at time n depends on the whole past.

Belongs to the family of reinforced random processes.

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For the elephant random walk $X_1 \sim 2\mathcal{B}_{1,p} - 1\,,\ \ (p \in (0,1))$ as before and then for $n \geq 1$ inductively

$$X_{n+1} = \begin{cases} +X_K, & \text{with probability } p, \\ -X_K, & \text{with probability } 1-p, \end{cases}$$

where K is again a random variable being uniformly distributed on $\{1, \ldots, n\}$ and being independent of $\mathcal{G}_n = \sigma(X_1, \ldots, X_n)\,.$
S_n denote the partial sums again.

Rem.:

0)

Name, because the memory of an Elephant is known to be long ranging.

i) Not a Markovian process.

ii) Can as well be formulated as an urn model

iii)

$$E(X_{n+1} \mid \mathcal{G}_n) = \frac{1}{n}\left(p\sum_{k=1}^{n} X_k + (1-p)\sum_{k=1}^{n}(-X_k)\right) = (2p-1)\frac{S_n}{n}$$

and hence we get

$$E(S_{n+1}\,|\,\mathcal{G}_n) = S_n + E(X_{n+1}\,|\,\mathcal{G}_n) = \frac{n+2p-1}{n}\,S_n =: \gamma_n S_n\,.$$

Then with

$$a_n = \prod_{k=1}^{n-1} \gamma_k^{-1} = \frac{\Gamma(2p)\cdot\Gamma(n)}{\Gamma(n+2p-1)}, \quad n \geq 1,$$

we find that $(M_n) = (a_n\,S_n)$ is a martingale.
(Coletti et al. (2017), Bercu (2018), see also C.Heyde (2004))
and we can use martingale theory.
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We may write

$$M_n = \sum_{k=1}^{n} a_k \varepsilon_k \text{ with } \varepsilon_k = (S_k - \gamma_{k-1} S_{k-1})$$

and we find

$$E(\varepsilon_{n+1}^2 | \mathcal{G}_n) = 1 - (2p-1)^2 (S_n/n)^2 \leq 1$$

implying $< M_n >= \sum_{k=1}^{n} a_k^2 E(\varepsilon_k^2 | \mathcal{G}_{k-1}) \approx \sum_{k=1}^{n} a_k^2$.

For the asymptotic behavior of (M_n) the quantity $\nu_n = \sum_{k=1}^n a_k^2$ is essential and

$$\nu_n \begin{cases} \sim (\Gamma(2p))^2 \frac{n^{1-2(2p-1)}}{3-4p}, & \text{for} p < 3/4 \\ \sim \frac{\pi}{4} \log n, & \text{for } p = 3/4 \\ \to C, & \text{for } p > 3/4 \end{cases}$$

one obtains by well-known results for martingales for M_n and then for S_n

Theorem (Bercu (2018))

(a) *For $p < 3/4$ (diffusive scheme)),*

$$\frac{S_n}{n} \stackrel{a.s.}{\to} 0 \quad \textit{and} \quad \frac{S_n}{\sqrt{n}} \stackrel{d}{\to} \mathcal{N}_{0,\frac{1}{3-4p}} \,, \quad \textit{as } n \to \infty.$$

(b) *For $p = 3/4$ (critical scheme),*

$$\frac{S_n}{\sqrt{n}\log n} \stackrel{a.s.}{\to} 0 \quad \textit{and} \quad \frac{S_n}{\sqrt{n\log n}} \stackrel{d}{\to} \mathcal{N}_{0,1} \,, \quad \textit{as } n \to \infty.$$

(c) *For $p > 3/4$ (superdiffusive scheme),*

$$\frac{S_n}{n^{2p-1}} \stackrel{a.s.}{\to} L \,, \quad \textit{as } n \to \infty,$$

where L is a non-degenerate random variable.

and

Theorem

(LIL) For $p < 3/4$

$$\limsup_{n\to\infty}(\liminf_{n\to\infty})\frac{S_n - n(2p-1)}{\sqrt{8p(2p-1)n\log\log n}} \overset{a.s.}{=} \frac{1}{\sqrt{3-4p}}\left(\frac{-1}{\sqrt{3-4p}}\right).$$

and more.

There were simulation papers dealing with partial memory of the elephant. People in Physics asked what happens for different memories? When is there such a dichotomy?

We now deal with a partial memory of the elephant described by $\mathcal{F}_n = \sigma(X_j\,,\ j \in I_n)$ for some memory subset $I_n \subset \{1, \ldots, n\}$ of the elephant. Then,

$$\begin{aligned} E(X_{n+1} \,|\, \mathcal{F}_n) &= p \cdot \sum_{i \in I_n} \frac{1}{|I_n|} X_i + (1-p) \cdot \sum_{i \in I_n} \frac{1}{|I_n|}(-X_i) \\ &= (2p-1) \cdot \frac{\sum_{i \in I_n} X_i}{|I_n|}, \end{aligned}$$

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Remember: I_n is the set of indices describing the memory

Now, assume $m \in N$ is fixed and

$$I_n = \{1, ..., n\} \text{ for } n = 1, \ldots m \quad \text{and} \quad I_n = \{1, \ldots, m\} \text{ for } m < n$$

The process (S_n) has neither a martingale nor a Markovian structure, but one can prove the following (here $m = 2$).

Theorem

1a) As $n \to \infty$

(a) $$\frac{S_n}{n} \xrightarrow{d} \begin{cases} 2p-1, & \text{with probability } p^2, \\ 0, & \text{with probability } 1-p, \\ -(2p-1), & \text{with probability } p(1-p), \end{cases} ;$$

(b) $E(S_n/n) \to p(2p-1)^2$ *and*

$$Var(S_n/n) \to p(1-p)(2p-1)^2(4p^2+1).$$

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Or using random centering:

Theorem

1b) As $n \to \infty$

(a)

$$\frac{S_n - n(2p-1)\,(X_1+X_2)/2}{\sqrt{n}} \xrightarrow{d} p \cdot \mathcal{N}_{0,4p(1-p)} + (1-p) \cdot \mathcal{N}_{0,1};$$

(b) $$\frac{S_n - n(2p-1)\,(X_1+X_2)/2}{n} \xrightarrow{a.s.} 0;$$

(c) $$\limsup_{n\to\infty}(\liminf_{n\to\infty}) \frac{S_n - n(2p-1)\,(X_1+X_2)/2}{\sqrt{2n\log\log n}} \overset{a.s.}{=} (-)\sigma(X_1,X_2)$$

where

$$\sigma(X_1,X_2) = \begin{cases} 4p(1-p)\,, & \text{for } \omega \in \{\omega \in \Omega : X_1(\omega)\cdot X_2(\omega) = 1\}, \\ 0\,, & \text{otherwise.} \end{cases}$$

For general $m \in \mathbb{N}$

Theorem

1c) Let $q_k = P(S_m = m - 2k)$, $r_k = \big((m-k)p + k(1-p)\big)/m$, *and* $p_k = (m-2k)(2p-1)/m$, *where* $0 \le k \le m$ *and* $m \in \mathbb{N}$, *then*

$$\frac{S_n}{n} \xrightarrow{d} \sum_{k=0}^{m} q_k \delta_{p_k} \ \text{as } n \to \infty,$$

and

$$\frac{S_n - n(2p-1)S_m/m}{\sqrt{n}} \xrightarrow{d} \sum_{k=0}^{m} q_k \mathcal{N}_{0,4r_k(1-r_k)} \quad n \to \infty.$$

Question: What about $m = m_n \to \infty$ s.t. $m_n/n \to 0$?

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Question: What about $m = m_n \to \infty$ s.t. $m_n/n \to 0$?

Theorem

(a) If $0 < p < 3/4$, *then*

$$\frac{S_n\sqrt{m_n}}{n} \xrightarrow{d} \mathcal{N}_{0,(2p-1)^2/(3-4p)}\,, \ \ as\ n \to \infty\,.$$

(b) If $p = 3/4$, *then*

$$\frac{S_n\sqrt{m_n/\log m_n}}{n} \xrightarrow{d} \mathcal{N}_{0,1/4}\,, \ \ as\ n \to \infty\,.$$

(c) If $3/4 < p < 1$, *then*

$$\frac{S_n m_n^{2(1-p)}}{n} \xrightarrow{d} (2p-1)\,L\,, \ \ as\ n \to \infty\,,$$

where the random variable L is the same as in Theorem above.

Remark *If* $p = 1/2$, *we recall that the ERW reduces to coin-tossing, hence,* $S_n/\sqrt{n} \xrightarrow{d} \mathcal{N}_{0,1}$ *as* $n \to \infty$.*Note also that Theorem part (a) reduces to* $S_n\sqrt{m_n}/n \to 0$ *as* $n \to \infty$ *in that case.*

What happens if $m_n/n \to c \in [0, 1]$?

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What happens if $m_n/n \to c \in [0, 1]$?

Rafik Aguech and El Machkouri (2024) showed that there is asymptotic normality with variance

$$\sigma_c^2 = \frac{\rho^2}{3-4p} + c(1-c)\,,$$

where $\rho = c + (1-c)(2p-1)$.

Also the other cases are treated. in $p > 3/4$

$$\frac{m_n^{2(1-p)}\, S_n}{n} \stackrel{a.s.}{\to} (c + (1-c)(2p-1)\, L\,.$$

Again Rafik in 2024 considered the more general case $\tilde{S}_n = \sum_{k=1}^{n} X_k Z_k$, where (Z_k) are iid random variables independent from the rest and showed asymptotic normality.

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For the case of nonuniform distribution of K, see Laulin (2022), Majumdar & Maulik (2025).

E.g., Laulin uses $P(K=k) = \frac{\Gamma(n+\beta)}{\Gamma(n)\Gamma(\beta)}$ with $\beta \geq 0$ and shows among other results in the diffusive case $p < \frac{4\beta+3}{4(\beta+1)}$

$$\frac{S_n}{\sqrt{n}} \xrightarrow{d} \mathcal{N}_{0,\sigma_\beta^2} \text{ as } n \to \infty$$

where $\sigma_\beta^2 = \frac{\beta+1-p}{(1-p)(1+2\beta-2(2p-1)(\beta+1))}$.

More realistic, the elephant remembers the last m steps, e.g., $m = 2$. Now assume that for $n \geq 3$ we have $I_n = \{n-1, n\}$.

Now we have a recurrent Markov chain (X_n) with finite state space of order 2 and we can apply results for Markov chains.

Theorem

for $p \in (0,1)$ *we have as* $n \to \infty$

$$\frac{S_n}{n} \xrightarrow{a.s.} 0 \text{ and } \frac{S_n}{\sigma_2\sqrt{n}} \xrightarrow{d} \mathcal{N}_{0,1}.$$

where $\sigma_2^2 = 1 + \frac{(2p-1)(5-2p)}{2(1-p)(3-2p)}$.

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Rem.:

If the elephant remembers the last m steps then we have again asymptotic normality with variance

$$\sigma_m^2 = \frac{m-1+2p}{2(1-p)\big(2(1-p)m+2p-1\big)}$$

see Ben Ari et al. (2021)

As $m \to \infty$ we have $\sigma_v^2 \to \frac{1}{4(1-p)^2}$ (being different from the classical ERW).

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How to proof limit results if $m = m_n \to \infty$?

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How to proof limit results if $m = m_n \to \infty$?

Here we suppose for $n \geq 2$ that $I_n = \{1, n\}$.

Theorem

We have as $n \to \infty$

$$\frac{S_n}{n} \xrightarrow{d} S = \begin{cases} \frac{2p-1}{3-2p}, & \text{with probability } p, \\ -\frac{2p-1}{3-2p}, & \text{with probability } 1-p, \end{cases}.$$

Moreover, $E(S_n/n)^r \to E(S^r)$ *for all* $r > 0$.

And with random centering:

Theorem

We have as $n \to \infty$

$$\frac{S_n - \frac{(2p-1)X_1}{3-2p}n}{\sigma_2\sqrt{n}} \xrightarrow{d} \mathcal{N}_{0,1} \, .$$

The larger the memory, the more tedious are calculations of e.g., moments, typically one can use difference equations, as in the most simple case. Let be given $X_1 = 1$

$$\begin{aligned} E(X_{n+1}) &= E(E(X_{n+1} \,|\sigma\{X_1, X_n\}) = (2p-1)\left(\frac{E(X_1)+E(X_n)}{2}\right) \\ &= \frac{(2p-1)}{2}\Big(E(X_n)+1\Big) \quad \text{and thus} \end{aligned}$$

$$E(X_n) = \frac{2p-1}{3-2p} + (p-1/2)^{n-1}\frac{4(1-p)}{3-2p}$$

and

$$E(S_n) = n \cdot \frac{2p-1}{3-2p} + \frac{8(1-p)}{(3-2p)^2} + o(1)\,.$$

etc . for higher moments

The order of the difference equations increase with moments and the size of the memory.

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The order of the difference equations increase with moments and the size of the memory.

Assume $p < 3/4$ How long do we have to wait till the first passage through zero $\tau_0 = \inf\{j \geq 1 \ : \ S_j = 0\}$?

What about the number of zeros $Z(n) = \left|\{1 \leq j \leq n \ : \ S_j = 0\}\right|$?

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Theorem (Bertoin (2022))

i) $P(\tau_0 > n) \sim c_p n^{2p-3/2}$ *and* $E(\tau_0) < \infty$ *iff* $p < 1/4$,

ii) $Z(n)/\sqrt{n} \xrightarrow{d} V$ *with a nondegenerate rv* V.

Remark i) The distribution of τ_0 is heavy tailed.

ii) Compare to Markov chain with finite state space.

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ii) $Z(n)/\sqrt{n} \xrightarrow{d} V$ *with a nondegenerate rv* V.

Remark i) The distribution of τ_0 is heavy tailed.

ii) Compare to Markov chain with finite state space.

The basis for the proof is a Brownian embedding:

Notations

$$M_k(n) = a_{n+k}S_{n+k} \text{ is a binary splitting martingale given } S_k = 0$$

Then one can construct an increasing sequence $(T_{k,n}), n \geq 1$ such that for fixed k

the sequences $(M_k(n))$ and $(B(T_{k,n}))$ have the same law.

Finally we can relate the zeros as

$$\{n \geq 0: \ M_k(n) = 0\} = \{n \geq 0: \ B(T_{k,n} = 0)\}.$$

Consider possible asymptotics for the first passage times

$$\tau_n = \min\{k \in \mathbb{N} : \; S_k \geq n\} \; = \min\{k : \; S_k = n\}, \quad n \geq 1.$$

A first observation is that the stopping times are not proper random variables in the superdiffusive case.

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If τ_n would be finite a.s. for a subsequence $n_k \nearrow \infty$, then, as $\tau_{n_k} \nearrow \infty$,

$$\frac{S_{\tau_{n_k}}}{\tau_{n_k}^{2p-1}} = \frac{n_k}{\tau_{n_k}^{2p-1}} \stackrel{a.s.}{\to} L \text{ as } k \to \infty.$$

However, since $E(S_n/n^{2p-1} \mid X_1 = -1) \sim -\frac{2p}{\Gamma(2p)}$ as $n \to \infty$, L cannot be a positive random variable, thus yielding a contradiction. Hence, $\tau_{n_k} = \infty$ for all $k \geq k_0$ on a set of positive measure, which implies that $\tau_n = \infty$ for all $n \geq k_0$ there.

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In the diffusive case we obtain strong laws and almost sure bounds.

Theorem

$$(i)\quad \textit{For } 0 < p < 3/4, \quad \liminf_{n\to\infty} \frac{\tau_n \log\log n}{n^2} \geq \frac{3-4p}{2} \ \textit{a.s.}$$

$$(ii)\quad \textit{For } p = 3/4, \quad \liminf_{n\to\infty} \frac{\tau_n \log n \, \log\log\log n}{n^2} \geq \frac{1}{2} \ \textit{a.s.}$$

Rem.:

(i) For $0 < p \leq 3/4$ it follows, in particular, that $\frac{\tau_n}{n} \stackrel{a.s.}{\to} \infty$ and, hence, that $\frac{E(\tau_n)}{n} \to \infty$ as $n \to \infty$. This might be interpreted as a negative renewal theorem, in that the almost sure limit equals $1/\mu < \infty$ for random walks with positive mean μ.

(ii) For $3/4 < p < 1$ the LLN for S_n does not imply an LLN for τ_n. However, for

$$\tau_n^* = \min\{k : \ |S_k| = n\} \quad \text{we have}$$

$$\frac{\tau_n^*}{n^{1/(2p-1)}} \stackrel{a.s.}{\to} |L|^{-1/(2p-1)} n \to \infty.$$

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Distributional convergence

Theorem

(i) For $0 < p < 3/4$, $P\left(\frac{\mathcal{T}_n}{n^2} \le x\right) \xrightarrow{d} G_p(x)$ *as* $n \to \infty$, *where*

$$G_p(x) = P\left(\frac{1}{\sqrt{3-4p}} \max_{0<t\le x}\{t^{2p-1}W(t^{3-4p})\} \ge 1\right)$$
$$= P\left(\max_{0<t\le 1}\left\{t^{(2p-1)/(3-4p)}W(t)\right\} \ge \sqrt{\frac{3-4p}{x}}\right).$$

(ii) For $p = 3/4$,
$P\left(\frac{\mathcal{T}_n \log n}{n^2} \le x\right) \xrightarrow{d} 2\left(1 - \mathcal{N}_{0,1}(1/\sqrt{x})\right)$ *as* $n \to \infty$.

Rem.: $E(\tau_n) = \infty$ for $p \in [1/4, 3/4)$.

What about different memories, e.g., $I_n = \{n\}$

Theorem

Here we have

$$\frac{p\,\tau_n}{(1-p)n^2} \xrightarrow{d} S(1/2, 1/2) \quad n \to \infty$$

where $S(1/2, 1/2)$ *is the distribution of* $1/\chi_1^2$.

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Now stops are permitted, that is, we assume that our basic random variable follows

$$X_1 = \begin{cases} +1, & \text{with probability } p, \\ -1, & \text{with probability } q, \\ \ 0\ , & \text{with probability } r = 1 - p - q \end{cases}$$

and

$$X_{n+1} = \begin{cases} +X_K, & \text{with probability } p, \\ -X_K, & \text{with probability } q \\ \ 0\ , & \text{with probability } r = 1 - p - q \end{cases}$$

That means the random walk stops from time to time and then goes on but with a slightly different dynamic.

Then we may ask: How many zeros do occur, i.e,
$N_n = \sum_{k=1}^{n} \mathbb{1}_{\{X_k=0\}} =?$ or $N_n^* = n - N_n =?$

Theorem

Here we have

$$\frac{N_n^*}{n^{1-r}} = \frac{n - N_n}{n^{1-r}} \overset{a.s.}{\rightarrow} Y$$

with some positive random variable Y having moments

$$E(Y) = \frac{1}{\Gamma(1-r)} \quad and \quad Var(Y) = d_r - \frac{1}{\Gamma^2(1-r)} \, .$$

Rem.: Related results we obtain for different memories.

Now, the limit distributions for S_n are quite different.
We have to normalize by the number of the nonzero contributions in the walk: N_n^*
For that we can consider a so-called Bernoulli ERW $T_n = \sum^n Y_k$, where with K as before

$$Y_{n+1} = \begin{cases} +Y_K, & \text{with probability } 1-r, \\ 0, & \text{with probability } r, \end{cases}$$

Now we need to define the Mittag–Leffler distributions ML_α which for $0 < \alpha < 1$ is the distribution with moment generating function $M_\alpha(t) = \sum_{n=0}^{\infty} t^n / \Gamma(1 + n\,\alpha)$

Theorem

For $0 < r < 1$ *we have as* $n \to \infty$

$$P\left(\left(\frac{T_n}{n^{1-r}} \leq x\right) \to (1-r)ML_{1-r}(x) + r\delta_0(x)\,.\right.$$

Wen can also deal with different memories, e.g., $I_n = \{1, n\}$ then

Theorem

As $n \to \infty$ *we have for* $x \in R$

$$P\left(\frac{S_n - \frac{np}{2-p}\mathbb{1}\{X_1 = 1\}}{\sqrt{n}} \leq x\right) \to p\,\mathcal{N}_{0,\sigma(p)^2}(x) + (1-p)\,\delta_0(x)\,.$$

Now $S_n/\sqrt{N_n^*}$ is asymptotically normal and by a result of Heyde (2004) we come to

$$F_{r,\sigma^2}(x) = \int_0^\infty \int_{-\infty}^{x/\sqrt{v}} e^{-w^2/(2\sigma^2)} \frac{dw}{\sqrt{2\pi\sigma^2}} ML_{1-r}(dv),$$

as multipicative convolution.

Then we get

Theorem

(a) For $p < \frac{3}{4}(1-r)$ and $\sigma^2 = 1/(3(1-r)-4p)$,

$$P\left(\frac{S_n}{\sqrt{n^{1-r}}} \leq x\right) \xrightarrow{d} (1-r)F_{r,\sigma^2}(x) + r\,\delta_0(x) \text{ as } n \to \infty.$$

Furthermore

$$\frac{S_n}{n^{1-r}} \xrightarrow{a.s.} 0 \text{ as } n \to \infty.$$

(b) For $\frac{3}{4}(1-r) < p < 1$ we have

$$\frac{S_n}{n^{2p-1+r}} \xrightarrow{a.s.} L(\mathbb{1}_{\{X_1=1\}} - \mathbb{1}_{\{X_1=-1\}}).$$

Let us come back to a restricted memory. What happens if we remember only $I_n = \{n-\ell, n-\ell+1, \dots, n\}$ $(\ell = 0, 1, 2, \dots)$ with the process?

Then the process stops with $\ell + 1$ zeros, after time τ_ℓ with the total sum S ..

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If $\ell = 0$ then τ, S are geometric with mean $1/r$ and $(1-r)/r$ resp.

If $\ell = 1$ then $E(\tau) = \frac{3+6r-r^2}{4r}$ and $E(S) = \frac{(1-r)(3+5r^2)}{4r(1+r)}$,

If $\ell = 2$ then $E(\tau) = \frac{62+303r-36r^2-5r^4}{108r}$ ...??

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Thank very much you for your attention!

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