The optimal switching problem with signed switching costs

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Alkhwarizmi Applied Mathematics Webinar, KSU, May 19, 2026.

Outlines

- Motivation and setting of the switching problem.
- The framework of signed switching costs
- Verification Theorem. Optimal strategy.
- The Markovian case and HJB system of PDEs.

1. Motivation and setting of the switching problem

- $B := (B_t)_{t \leq T}$ a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ whose completed natural filtration is $(F_t)_{t \leq T}$.
- **C** is a power plant.

1.1. Features of the power plant

- The power plant **C** has $m \geq 3$ modes of production (if e.g. $m = 3$, "1=no production", "2=normal mode" and "3=intensive one"). The case of $m = 2$ is dual.
- Electricity cannot be stored, when produced it should be sold and consumed. The manager of **C** will put it dynamically in the most profitable mode.
- If **C** is in mode $i \in \mathcal{J} := \{1, ..., m\}$, the yield per dt is $\psi_i(t, \omega)dt$.
- Switching **C** from mode i to mode $j \neq i$ at t induces a payoff which equals to

$$\ell_{ij}(t, \omega)$$

an adapted continuous stochastic process.

1.2. Switching strategies

A management strategy of **C** has two components δ and ξ

- (i) $\delta = (\tau_n)_{n\geq 0}$ a sequence of stopping times such that $\tau_n \leq \tau_{n+1}$ and $\tau_n \to T$. At τ_n the manager switches the production from the current mode to another one.

- (ii) $\xi = (\xi_n)_{n\geq 0}$ a sequence of $r.v.'s$ such that:

$$\xi_0 = 1 \text{ and } \forall n \geq 1, \xi_n \in \mathcal{J} \text{ and } \xi_n \; is \; F_{\tau_n} - meas..$$

  ξ_n is the new working mode chosen at time τ_n.

- The pair (δ, ξ) is called a switching strategy of management strategy of the power plant. $\square$

1.3. The payoff

Let $(u_t)_{t\leq T}$ be the process indicator of the production mode at t of **C**:

$$u_0 = 1 \text{ and } u_t = \xi_n \text{ if } t \in]\tau_n, \tau_{n+1}] \ (n \geq 0).$$

When a strategy (δ, ξ) is implemented the yield is given by:

$$J(\delta,\xi) := \mathbb{E}[\int_0^T \psi_{u_s}(s)ds - \sum_{n\geq 1} \ell_{\xi_{n-1},\xi_n}(\tau_n) 1\!\!1_{[\tau_n < T]} + G_{u_T}].$$

G_{u_T} is the terminal payoff at time T.

Remark : (i) The process $u = (u_t)_{t\leq T}$ is in a one-to-one correspondence with (δ, ξ).
(ii) When $\ell_{ij}(\tau) > 0$ (resp. < 0), the switching from i to j at τ incures a cost (rep. a subsidy or profit) for the decision maker. □

1.4. Problems

1) The wellposedness of the problem if we do not have $\ell_{ij} \geq 0$.

2) Existence of an optimal strategy (δ^*, ξ^*), i.e.,

$$J(\delta^*, \xi^*) = \sup_{(\delta,\xi)} J(\delta, \xi).$$

3) What can be said about

$$\sup_{(\delta,\xi)} J(\delta, \xi)$$

in terms of characterization, properties, simulation, etc. ?

Remark : The quantity

$$\Xi := \sup_{(\delta,\xi)} J(\delta, \xi)$$

is the price of the power plant in the energy market. □

The model fits also for:

(i) a change of technology of a corporate;
(ii) the management of a portfolio (allocation) of life insurances (pensions, etc.) in a financial market;
(iii) the management of a cluster in cyber-security;
(iv) crude oil fields, etc.

$\square$

Remark : There are several papers on this subject. Mainly in the case when the switching payoffs are non-negative + appropriate assumptions.

2. The framework of signed switching payoffs

Assumptions:

i) For any $i, j \in \mathcal{J}$, the stochastic process $t \in [0, T] \mapsto \ell_{ij}(t)$ is increasing (resp. decreasing).

ii) $\ell_{ij}(.)$ satisfy the triangle inequality, i.e., for any $i, j, k \in \mathcal{J}$, $\forall t \leq T$,

$$\ell_{ij}(t) < \ell_{ik}(t) + \ell_{kj}(t).$$

$\ell_{ii} = 0$; $\ell_{ij} \in \mathcal{S}^2$ (cont. uniformly square integrable processes).

iii) Consistency condition: The terminal payoffs G_i are F_T-r.v. and verify: For any $i \in \mathcal{J}$,

$$\mathbb{E}[(G_i)^2] < \infty \text{ and } G_i \geq \max_{i \in \mathcal{J} - \{i\}} \{G_j - \ell_{ij}(T)\}.$$

Remark

- *The triangle inequality implies the non free loop property , i.e., for any sequence of indices $i_1, \ldots, i_k \in \mathcal{J}$ such that $i_1 = i_k$ and card$\{i_1, ..., i_k\} = k - 1$ we have:*

$$\ell_{i_1 i_2}(t) + \ell_{i_2 i_3}(t) + \ldots + \ell_{i_{k-1} i_k}(t) + \ell_{i_k i_1}(t) > 0$$

because

$$\begin{aligned} &\ell_{i_1 i_2}(t) + \ell_{i_2 i_3}(t) + \ldots + \ell_{i_{k-1} i_k}(t) + \ell_{i_k i_1}(t) \\ &\quad > \ell_{i_1 i_3}(t) + \ell_{i_3 i_4}(t) + \ldots + \ell_{i_{k-1} i_k}(t) + \ell_{i_k i_1}(t) \\ &\quad \ldots \\ &\quad > \ell_{i_1 i_1}(t) = 0. \end{aligned}$$

- **Example**: $m = 2$, $\ell_{12} = -1$ *and* $\ell_{21} = 2$.

Admissiblility: A switching strategy $(\delta, \xi) := (\tau_n, \xi_n)_{n \geq 0}$ is called admissible if:

(i) $\tau_n \leq \tau_{n+1}$ and

$$\mathbb{P}[\tau_n < T, \forall n \geq 0] = 0.$$

(ii) ξ_n is a $\mathcal{J}$-valued $\mathcal{F}_{\tau_n}$-r.v. such that for any n:

- $\mathbb{P}[\xi_n = \xi_{n+1}, \tau_n < T] = 0.$
- the n-partial payoff (payoff after n switchs):

$$C_n^{\delta,\xi} := \sum_{m=1}^{n} \ell_{\xi_{m-1}\xi_m}(\tau_m) \mathbf{1}_{[\tau_m < T]}$$

verifies:

$$\mathbb{E}[\sup_{n \geq 1} |C_n^{\delta,\xi}|^2] < \infty.$$

In this case, the sequence $(C_n^{\delta,\xi})_{n \geq 1}$ converges in $L^2(d\mathbb{P})$ to

$$C^{\delta,\xi} := \sum_{m \geq 1} \ell_{\xi_{m-1}\xi_m}(\tau_m) \mathbf{1}_{[\tau_m < T]}.$$

Proposition: Assume that the triangle inequality holds. Then for all $N \geq 1$ and any $(\tau_n, \xi_n)_{n \leq N}$ (an N-truncated strategy) we have:

i) If the switching costs are increasing in time, then :

$$\underbrace{\sum_{j=1}^{N} -\ell_{\xi_{j-1}\xi_j}(\tau_j)\mathbf{1}_{[\tau_j < T]}}_{-C_N^{\delta,\xi}} \leq \max_{k \in \mathcal{J}} |\ell_{\xi_0 k}(\tau_1)|. \qquad (1)$$

(ii) If the switching costs are decreasing in time, then:

$$-C_N^{\delta,\xi} = \sum_{j=1}^{N} -\ell_{\xi_{j-1}\xi_j}(\tau_j)\mathbf{1}_{[\tau_j < T]} \leq \max_{k \in \mathcal{J}}(|\ell_{\xi_0 k}(T)|. \qquad (2)$$

Proof: By induction. We focus on (i).
(1) obviously holds for N= 1. Suppose it holds up to $N-1$. Let

$$\mathbb{D} = \{\omega \in \Omega \text{ such that } \ell_{\xi_{N-1}\xi_N}(\tau_N) < 0\ \},$$

then:

$$\begin{aligned} -\ C_N^{\delta,\xi} &= -\sum_{j=1}^{N} \ell_{\xi_{j-1}\xi_j}(\tau_j)\mathbf{1}_{[\tau_j<T]} \\ &= -\sum_{j=1}^{N} \ell_{\xi_{j-1}\xi_j}(\tau_j)\mathbf{1}_{[\tau_j<T]}\mathbf{1}_{\mathbb{D}^c} - \sum_{j=1}^{N} \ell_{\xi_{j-1}\xi_j}(\tau_j)\mathbf{1}_{[\tau_j<T]}\mathbf{1}_{\mathbb{D}}. \end{aligned} \tag{3}$$

For the first term,

$$\begin{aligned} -\sum_{j=1}^{N} \ell_{\xi_{j-1}\xi_j}(\tau_j)\mathbf{1}_{[\tau_j<T]}\mathbf{1}_{\mathbb{D}^c} &\leq -\sum_{j=1}^{N-1} \ell_{\xi_{j-1}\xi_j}(\tau_j)\mathbf{1}_{[\tau_j<T]}\mathbf{1}_{\mathbb{D}^c} \\ &\leq \max_{k\in\mathcal{J}} |\ell_{\xi_0 k}(\tau_1|\mathbf{1}_{\mathbb{D}^c} \end{aligned}$$

For the other term in (3), since $[\tau_N < T] \subset [\tau_{N-1} < T]$ and $\ell_{\xi_{N-1}\xi_N}(\tau_{N-1}) \leq \ell_{\xi_{N-1}\xi_N}(\tau_N)$, then (on $\mathbb{D}$)

$$
\begin{aligned}
& -\sum_{n=1}^{N} \ell_{\xi_{j-1}\xi_j}(\tau_j)\mathbf{1}_{[\tau_j<T]}\mathbf{1}_{\mathbb{D}} \\
&= \Bigg[-\sum_{j=1}^{N-2} \ell_{\xi_{j-1},\xi_j}(\tau_j)\mathbf{1}_{[\tau_j<T]} \\
&\qquad -\ell_{\xi_{N-2}\xi_{N-1}}(\tau_{N-1})\mathbf{1}_{[\tau_{N-1}<T]} - \ell_{\xi_{N-1}\xi_N}(\tau_N)\mathbf{1}_{[\tau_N<T]}\Bigg]\mathbf{1}_{\mathbb{D}}. \\
&\leq \Bigg[-\sum_{j=1}^{N-2} \ell_{\xi_{j-1},\xi_j}(\tau_j)\mathbf{1}_{[\tau_j<T]} \\
&\qquad -(\ell_{\xi_{N-2}\xi_{N-1}}(\tau_{N-1}) + \ell_{\xi_{N-1}\xi_N}(\tau_{N-1}))\mathbf{1}_{[\tau_{N-1}<T]}\Bigg]\mathbf{1}_{\mathbb{D}}.
\end{aligned}
$$

By the triangle inequality we get,

$$
\begin{aligned}
&-\sum_{j=1}^{N} \ell_{\xi_{j-1}\xi_j}(\tau_j)\mathbf{1}_{[\tau_j<T]}\mathbf{1}_{\mathbb{D}} \\
&\leq \Big[-\textstyle\sum_{j=1}^{N-2} \ell_{\xi_{j-1}\xi_j}(\tau_j)\mathbf{1}_{[\tau_j<T]} - \ell_{\xi_{N-2}\xi_N}(\tau_{N-1})\mathbf{1}_{[\tau_{N-1}<T]}\Big]\mathbf{1}_{\mathbb{D}} \\
&= \Big[\sum_{l\in\mathcal{J}} \big[-\sum_{j=1}^{N-2} \ell_{\xi_{j-1}\xi_j}(\tau_j)\mathbf{1}_{[\tau_j<T]} - \ell_{\xi_{N-2}l}(\tau_{N-1})\mathbf{1}_{[\tau_{N-1}<T]}\big]\mathbf{1}_{[\xi_N=l]}\Big]\mathbf{1}_{\mathbb{D}} \\
&= \Big[\sum_{l\in\mathcal{J}} \big[-\sum_{j=1}^{N-1} \ell_{\tilde{\xi}_{j-1}\tilde{\xi}_j}(\tau_j)\mathbf{1}_{[\tau_j<T]}\big]\mathbf{1}_{[\xi_N=l]}\Big]\mathbf{1}_{\mathbb{D}} \leq \max_{k\in\mathcal{J}} |\ell_{\xi_0 k}(\tau_1)|\mathbf{1}_{\mathbb{D}},
\end{aligned}
$$

where $\tilde{\xi}_j = \xi_j$ for $j = 1, ..., N-2$, and $\tilde{\xi}_{N-1} = l$. Thus the desired result.

Point (ii) is obtained in the same way. First

$$-\sum_{j=1}^{N} \ell_{\xi_{j-1}\xi_j}(\tau_j)\mathbf{1}_{[\tau_j<T]} \;\leq\; -\sum_{j=1}^{N} \ell_{\xi_{j-1}\xi_j}(T)\mathbf{1}_{[\tau_j<T]}. \quad P-a.s.$$

Then by considering the set

$$\mathbb{D}_1 = \{\omega \in \Omega \text{ such that } \ell_{\xi_{N-1}\xi_N}(T) < 0\},$$

we show by induction that for any N and $(\tau_n, \xi_n)_{n\leq N}$,

$$-\sum_{j=1}^{N} \ell_{\xi_{j-1}\xi_j}(T)\mathbf{1}_{[\tau_j<T]} \;\leq\; \max_{k\in\mathcal{J}} |\ell_{\xi_0 k}(T)| \tag{4}$$

since $\ell_{ij}(T)$, $i,j \in \mathcal{J}$, verify the triangle inequality. □

3. Verification Theorem and its solution

We assume that $m = 3$; $\mathcal{J} = \{1, 2, 3\}$ and $\mathcal{J}^{-i} = \mathcal{J} - \{i\}$.

Theorem: There exist continuous processes $(Y^i, Z^i, K^i)_{i=1,2,3}$ $(Y^i \in \mathcal{S}^2)$ such that: for $i = 1, 2, 3$ and $t \leq T$,

$$\begin{cases} Y_t^i = G_i + \int_t^T \psi_i(u)du - \int_t^T Z_u^i dB_u + K_T^i - K_t^i; \\ Y_t^i \geq \max_{j \in \mathcal{J}^{-i}} \{-\ell_{ij}(t) + Y_t^j\}; \\ \int_0^T (Y_u^i - \max_{j \in \mathcal{J}^{-i}} \{-\ell_{ij}(u) + Y_u^j\}) dK_u^i = 0 \end{cases} \tag{5}$$

Idea of the proof: recursive approximations.
For $i \in \mathcal{J} := \{1, 2, 3\}$

$$Y_t^{i,0} = E[\int_t^T \psi_i(s)ds + G_i | F_t]$$

and, for $n \geq 1$, $(Y^{i,n}, Z^{i,n}, K^{i,n})$ verifies: for $i = 1, 2, 3$ and $t \leq T$,

$$\begin{cases} Y_t^{i,n} = G_i + \int_t^T \psi_i(u)du - \int_t^T Z_u^{i,n} dB_u + K_T^{i,n} - K_t^{i,n}; \\ Y_t^{i,n} \geq \max_{j \in \mathcal{J}^{-i}} \{-\ell_{ij}(t) + Y_t^{j,n-1}\}; \\ \int_0^T (Y_u^{i,n} - \max_{j \in \mathcal{J}^{-i}} \{-\ell_{ij}(u) + Y_u^{j,n-1}\}) dK_u^{i,n} = 0 \end{cases} \qquad (6)$$

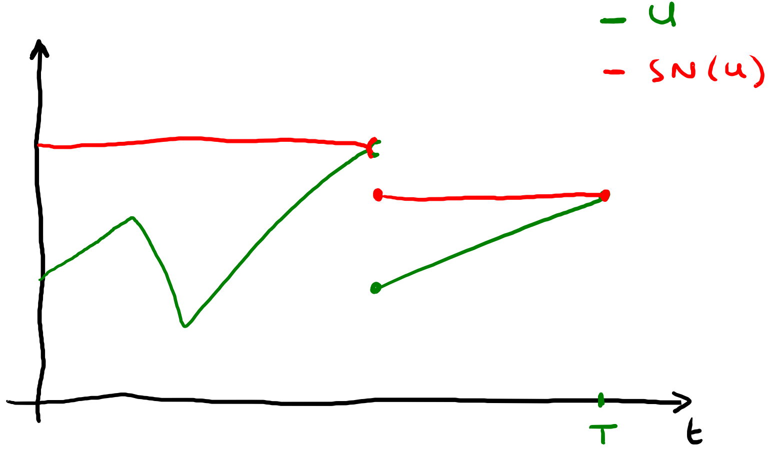
□

(i) Note that

$$Y_t^{i,n} + \int_0^t \psi_i(s)ds = \text{ess sup}_{\tau \geq t}\mathbb{E}[\int_0^\tau \psi_i(s)ds + \max_{k \in \mathcal{J}^{-i}}\{-\ell_{ik}(\tau) + Y_\tau^{k,n-1}\}1\!\!1_{[\tau < T]} + G_i 1_{\{\tau = T\}}|F_t].$$

The process $(Y_t^{i,n} + \int_0^t \psi_i(s)ds)_{t \leq T}$ is a Snell envelope of the process

$$(\int_0^t \psi_i(s)ds + \max_{k \in \mathcal{J}^{-i}}\{-\ell_{ik}(t) + Y_t^{k,n-1}\}1\!\!1_{[t < T]} + G_i 1_{\{t = T\}})_{t \leq T}.$$


u
SN(u)
T
t

(ii) Let

$$\mathcal{D}_t^{i,n} = \{u = (\tau_n, \xi_n)_{n\geq 1} \text{ admissible}, u_0 = i,\ \tau_1 \geq t \quad \text{and} \quad \tau_{n+1} = T\}.$$

Then

$$\begin{aligned} Y_t^{i,n} &= esssup_{u\in\mathcal{D}_t^{i,n}} \mathbb{E}[\int_t^T \psi_{u_s}(s)ds - \sum_{j\geq 1} \ell_{\xi_{j-1}\xi_j}(\tau_j)\mathbf{1}_{[\tau_j<T]} + G_{u_T}|\mathcal{F}_t] \\ &= esssup_{u\in\mathcal{D}_t^{i,n}} \mathbb{E}[\int_t^T \psi_{u_s}(s)ds - \sum_{j=1}^{n} \ell_{\xi_{j-1}\xi_j}(\tau_j)\mathbf{1}_{[\tau_j<T]} + G_{u_T}|\mathcal{F}_t]. \end{aligned}$$

Since $\mathcal{D}_t^{i,n} \subset \mathcal{D}_t^{i,n+1}$, we then have:

i) $Y_t^{i,n} \leq Y_t^{i,n+1}$.
ii)

$$Y_t^{i,n} \leq \mathbb{E}\left[\int_t^T \sum_{k\in\mathcal{J}} |\psi_k(s)|ds + \sum_{k\in\mathcal{J}} |G_k| + \sum_{k\in\mathcal{J}} \sup_{s\leq T} |\ell_{ik}(s)| |\mathcal{F}_t\right].$$

Thus the processes $Y^{i,n}$ are uniformly bounded. We set:

$$Y^i = \lim_n Y^{i,n}.$$

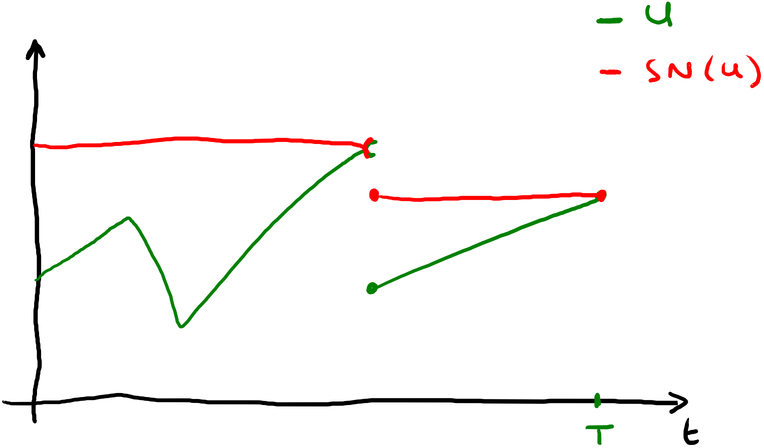
(i) Y^i is càdlàg.

$$Y_t^{i,n} = \text{ess sup}_{\tau \geq t} \mathbb{E}[\int_t^\tau \psi_i(s)ds + \max_{k \in \mathcal{J}^{-i}} (-\ell_{ik}(\tau) + Y_\tau^{k,n-1}) \mathbb{1}_{[\tau < T]} 1_{\{t<T\}} + G_i 1_{\{\tau = T\}} | F_t].$$

Then $(Y_t^{i,n} + \int_0^t \psi_i(s)ds)_{t \leq T}$ is a supermartingale and Y^i is càdlàg as a limit of the increasing sequence.

(ii) Y^i verifies:

$$Y^i_t = \text{ess sup}_{\tau \geq t} \mathbb{E}[\int_t^\tau \psi_i(s)ds + \max_{k \in \mathcal{J}^{-i}} (-\ell_{ik}(\tau) + Y^k_\tau) 1\!\!1_{[\tau < T]} 1_{\{t < T\}} + G_i 1_{\{\tau = T\}} | F_t].$$



u
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t

(ii) Y^i is continuous: Thanks to the non free loop property.

If for some t_0, $\triangle_{t_0} Y^i := Y^i_{t_0} - Y^i_{t_0-} < 0$ then there exists $j \neq i$ such that

$$\Delta_{t_0} Y^j < 0 \text{ and } Y^i_{t_0-} = -g_{ij}(t_0) + Y^j_{t_0-}.$$

Repeat the procedure to obtain a loop $j_1, .., j_{p-1}$ such that

$$g_{j_1 j_2}(t_0) + \cdots + g_{j_{p-1} j_1}(t_0) = 0.$$

This is contradictory and then Y^i is continuous. By Dini's theorem

$$Y^{i,n} \to_n Y^i \text{ in } \mathcal{S}^2.$$

We set:

$$Z^i = \lim_n Z^{i,n}, K^i = \lim_n K^{i,n}$$

we obtain that $(Y^i, Z^i, K^i)_{i=1,2,3}$ verify the above system of reflected BSDEs with inter-connected obstacles (6).

3.1 Existence of an optimal strategy

Theorem: There is an optimal strategy $u^* = (\tau_n^*, \xi_n^*)_{n \geq 1}$.

Hint for the proof:

a) Definition of u^*:

- τ_1^*=the first time that Y^i reaches the obstacle $(\max_{j \in \mathcal{J}^{-i}}\{-\ell_{ij}(t) + Y_t^j\})_{t \leq T}$.

- ξ_1^* is the optimal index at τ_1^*.

- τ_2^*=the first time that $Y^{\xi_1^*}$ reaches the obstacle after τ_1^*.

- ξ_2^* is the optimal index at τ_2^*, etc.

b) Part i) of admissibility stems from the NFLP implied by the triangle inequality.

c) Part ii) is due to:

$$Y_0^1 = \sum_{k=1}^{n} G_{u_{\tau_{k-1}^*}} \mathbf{1}_{[\tau_k^* = T]} \mathbf{1}_{[\tau_{k-1}^* < T]} - \sum_{k=1}^{n} \ell_{u_{\tau_{k-1}^*} u_{\tau_k^*}}(\tau_k^*) \mathbf{1}_{[\tau_k^* < T]} + Y_{\tau_n^*}^{u_{\tau_n^*}} \mathbf{1}_{[\tau_n^* < T]} + \sum_{k=1}^{n} \int_{\tau_{k-1}^*}^{\tau_k^*} \psi_{u_{\tau_{k-1}^*}}(r)dr - \underbrace{\sum_{k=1}^{n} \int_{\tau_{k-1}^*}^{\tau_k^*} Z_r^{u_{\tau_{k-1}^*}} dB_r}_{=\int_0^{\tau_n^*} Z_r^* dB_r}.$$

Thus

$$\mathbb{E}[\sup_{n \geq 1} |C_n^{u^*}|^2] < \infty$$

and the strategy u^* is admissible. Take the limit wrt n to obtain

$$Y_0^1 = J(u^*).$$

d) For any other admissible strategy u we have:

$$Y_0^1 \geq J(u). \quad \square$$

4.The Markov framework

Let $X^{t,x}$ be the solution of the following SDE:

$dX_s^{t,x} = b(s, X_s^{t,x})\, ds + \sigma(s, X_s^{t,x})\, dB_s$ for $t \leq s \leq T$ and $X_s^{tx} = x$ for $s \leq t$.

Next assume that :

a) $\psi_i(s) = \psi_i(s, X_s^{t,x})$

b) $\ell_{ij}(s)$ does not depend on x

c) $G_i = G_i(X_T^{t,x})$

where $\psi_i(t,x)$, $\ell_{ij}(t)$ and $G_i(x)$ are continuous and of polynomial growth.

Then there exist deterministic functions v^i, $i = 1, ..., m$, of polynomial growth such that

$$Y_s^{i,t,x} = v^i(s, X_s^{t,x}), \quad s \in [t, T].$$

Moreover:

Theorem:

The functions $(v^i)_{i=1,m}$ are continuous of polynomial growth and unique viscosity solution of: $\forall i = 1, ..., m,$

$$\begin{cases} \min\left\{ v_i(t,x) - \max_{j \in \mathcal{J}^{-i}} \{-\ell_{ij}(t) + v_j(t,x)\}, -\partial_t v_i(t,x) - \mathcal{A} v_i(t,x) - \psi_i(t,x) \right\} \\ \qquad = 0, \\ \\ v_i(T,x) = G_i(x). \end{cases} \tag{7}$$

Example

- $m = 3,\ T = 1,\ \sigma = 1.5,\ b = 1,\ \mathcal{I} = \{1, 2, 3\}.$
- $\psi_1 = (-x^2 + 5x) \div 10,\ \psi_2 = 0,\ \psi_3 = (x^2 - 5x) \div 10,$
- $\ell_{12}(t) = t + 1.5,\ \ell_{13}(t) = t + 2,\ \ell_{21}(t) = t - 0.4,\ \ell_{23}(t) = t + 1.5,\ \ell_{31}(t) = t - 1,\ \ell_{32}(t) = t - 0.1\,.$
- $G_i = 0.$

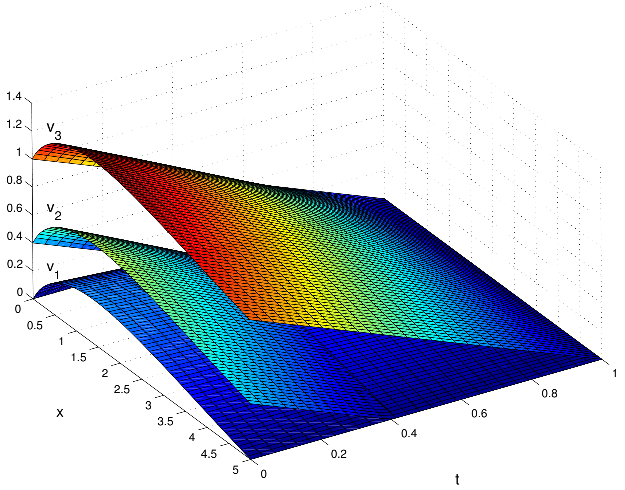

Figure: Value functions.

5. Extension

- The switching payoffs can be monotonic on $[0, t_1]$ and on $[t_1, T]$ with different monotonicities.
- We can do the same if similar phenomenon happens on $[0, t_1]$, $[t_1, t_2]$, ...,$[t_k, T]$. □

Some references

- $T = \infty$ (Dixit-Pindyck '94, Duckworth-Zervos '01, Pham-Vath '07, Pham-Vath-Zhou '09).
- two modes (Ham.-Jeanblanc MOR '07)
- several modes (Djehiche-H.-Popier '09, H.-Zhang '10, Hu-Tang '10)
- exponential utilities (Porchet-Touzi-Warin, '06; H.-Wang '09)
- randomization (Fuhrman-Morlais '20, Benezet-Chassagneux-Richou '22)
- ergodicity (Bayrakhtar-Cosso-Pham '18)
- numerical treatment within several modes (Carmona-Ludkovski, '06, Chassagneux-Elie-Kharroubi '12, Chassagneux-Richou '19, etc.).

The list is far from exhaustive.

Thanks for your attention.