Luigi Accardi

Probabilistic Quantization: Why Applied Mathematics Interested Should Check It Out

Alkhwarizmi Applied Mathematics webinar: Special session

March 31, 2026

Email: accardi@volterra.uniroma2.it
Email: andreasboukas@yahoo.com
WEB page: http://volterra.uniroma2.it

Simple question:
Where quantization comes from ?

2025 has been declared
the year of the quantum because it is the centenary of the famous 1925 Heisenberg's article where the mathematical formalism of quantization was first introduced.
The basic innovation is expressed by Heisenberg commutation relations:

$$[q, p] = i\hbar \quad \hbar > 0 \text{ is Planck's constant}$$

Heisenberg did not deduce this formula (in fact, in his original manuscript, he did not write it in the form used today).

Throughout his life, he insisted that the only thing we can say is that it works admirably and that the only thing we can do is to utilize this formalism, even if we do not understand where it comes from.

The goal of my talk is to explain how a non-trivial generalization of this formula, and in fact of the whole formalism of quantum theory, naturally emerges from the combination of classical probability with the theory of orthogonal polynomials.

Mathematics can prove that $\hbar > 0$ (in non-trivial cases), but cannot distinguish among all possible strictly positive numbers because in probabilistic quantization, $\hbar > 0$ is the variance of a Gaussian.

Plan of my talk.

(I) Illustration of the main idea of probabilistic quantization in the simplest possible case.

(II) Mentioning of some of the research lines, in pure and applied directions, arising from the basic idea.

(III) Mentioning of some important achievements of quantum probability before probabilistic quantization.

Probabilistic quantization.
(I) Notations form classical probability

– Start from a **classical real valued random variable** X

random variable $\equiv$ random variable with all moments.

– Form the (complex) **polynomial algebra** of X:

$$\mathcal{P} := \{ \sum_{j=0}^{n} a_j X^j : n \in \mathbb{N} \ , \ a_j \in \mathbb{C} \} \ (\equiv \mathcal{P}_X)$$

$$= \{ P \circ X : P \in \mathbb{C}[x] := \text{polynomial functions} : \mathbb{R} \to \mathbb{C} \}$$

$\mathcal{P}$ is a $*$-algebra with

$$\begin{cases} \text{point-wise addition and multiplication} \\ \text{point-wise complex conjugation} \\ \text{identity - the constant function} = 1 \ =: 1_{\mathcal{P}} \end{cases}$$

$P_X \in \mathrm{Prob}(\mathbb{R})$ the probability distribution of X,
one can identify

$\mathcal{P} \subseteq L^2_{\mathbb{C}}(\mathbb{R}, P_X)$ (polynomials are square-integrable)

This allows to consider
the usual scalar product on $L^2(\mathbb{R}, P_X)$.

$$\langle Q, R \rangle := P_X(Q^* R) = \int_{\mathbb{R}} \overline{Q}(x) R(x) P_X(dx) \quad (1)$$

(II) Notations from orthogonal polynomials

– Use the $L^2(\mathbb{R}, P_X)$-scalar product to **orthogonalize the monomials**, that is: $\forall n$,

$$X^n \mapsto \Phi_n :=$$

$$X^n - \left\{ \begin{array}{l} \text{its orthogonal components} \\ \text{on polynomials of degree } < n \end{array} \right.$$

monic orthogonal polynomial of degree n

monic polynomial $\iff$
term of **highest degree has coefficient** 1 (in particular, the Φ_n are not normalized to 1).
From now on:

$X \equiv$ multiplication operator by X in $L^2(\mathbb{R}, P_X)$

when we want to interpret X as a **vector** in $L^2(\mathbb{R}, P_X)$, we write

$$X \cdot \Phi_0$$

where, to make connection with quantum physics, we use the notations:

$$\Phi_0 := \text{ constant function } = 1$$

and we call Φ_0 the **vacuum vector**.
Recall that, when the same constant function is understood as the identity of the algebra $\mathcal{P}$, we write $1_{\mathcal{P}}$ or simply 1.

Finally, we us the notation

$$\mathcal{P} \cdot \Phi_0 := \{P \cdot \Phi_0 \colon P \in \mathcal{P}\} =$$

polynomials in X considered as vectors in $L^2(\mathbb{R}, P_X)$

From now on, we only consider classical random variables such that

$$\|\Phi_0\|_{L^2(\mathbb{R},P_X)} \neq 0 \quad , \ \forall n \in \mathbb{N}$$

– Fact (Jacobi $\sim$ 150 years ago):
the monic orthogonal polynomial
satisfy the Jacobi 3–diagonal relation:

$$X\Phi_n(X)$$

$$= \Phi_{n+1}(X) + \alpha_n\Phi_n(X) + \omega_n\Phi_{n-1}(X) \quad (2)$$

(ω_n) := principal Jacobi sequence;
(α_n) := secondary Jacobi sequence.
– operator interpretation of (2):
By orthogonality, the following linear operators are well defined

$$\begin{cases} a^{+}\Phi_n := \Phi_{n+1}, & \text{creator} \\ a\Phi_n := \omega_n\Phi_{n-1}, & \text{annihilator} \\ a^{0}\Phi_n := \alpha_n\Phi_n, & \text{preservator} \end{cases}$$

so we can write (2) in vector form

$$X\Phi_n := (a^{+} + a^{0} + a)\Phi_n \quad , \quad \forall n \qquad (3)$$

Again using the fact that the Φ_n are an orthogonal basis of

$$\left(\mathcal{P} \cdot \Phi_0 \ , \ \langle \cdot \ \cdot \rangle_{L^2(\mathbb{R}, P_X)} \Big|_{\mathcal{P} \cdot \Phi_0} \right)$$

we conclude that the vector identity (3) is equivalent to the following operator identity:

$$X = a^+ + a^0 + a^- \qquad (4)$$

The operator identity (4) is called: the canonical quantum decomposition of the classical random variable X
and the operators a^+, a^-, a^0 are called the CAP operators canonically associated to the classical random variable X
(**Creation, Annihilation, Preservation**)

Warning

There exist infinitely many quantum decompositions of a classical real valued random variable.
The canonical quantum decomposition is special because of the following result.

Theorem 1 The canonical quantum decomposition is unique up to unitary isomorphisms intertwining the CAP operators.

wave–particle aspects of classical random variables (fields)

The canonical quantum decomposition holds also for stochastic processes.
In the case of a stochastic process (X_t) $(t \in \mathbb{R})$ one has

$$X_t = \begin{cases} a_t^+ \ + \ a_t^- , & \text{diffusion part (wave)} \\ + \ a_t^0 , & \text{jump part (particle)} \end{cases} \tag{5}$$

So, the wave–particle duality appears as the physical manifestation of the universal probabilistic phenomenon, that any stochastic process (or random field) has a continuous part and a jump part.

Theorem 2

$$(a^+)^* = a \equiv a^- \tag{6}$$

$$a^-\Phi_0 = 0 \quad \text{Fock property} \tag{7}$$

i.e., the annihilator annihilates the vacuum.

This justifies the name
$\Phi_0 \iff$ vacuum vector.

Given the **quantum decomposition of the classical random variable** X

$$X = a^+ + a^0 + a^- \tag{8}$$

one can define:

– The orthogonal gradation

$$\mathcal{P} \cdot \Phi_0 = \bigoplus_{n \in \mathbb{N}} \mathcal{P}_n = \bigoplus_{n \in \mathbb{N}} \mathbb{C} \cdot \Phi_n \tag{9}$$

Note that, by definition of the creator

$$a^+ \Phi_n := \Phi_{n+1} \ , \ \forall n \iff \Phi_n = a^{+n} \Phi_0 \ , \ \forall n$$

Therefore we can write the orthogonal gradation in the form

$$\mathcal{P} \cdot \Phi_0 = \bigoplus_{n \in \mathbb{N}} \mathbb{C} \cdot a^{+n} \Phi_0 = \bigoplus_{n \in \mathbb{N}} \mathcal{P}_n \tag{10}$$

and this gives the translation code:

n–th orthogonal polynomial Φ_n

$\iff$ n–particle vector $|n\rangle$.

1) Any classical random variable X is canonically associated to a quantum probability space.

Definition 1 An algebraic probability space is a pair

$$(\mathcal{A}, \varphi) \quad ; \quad \mathcal{A} \ *\text{–algebra} \ ; \ \varphi \text{ state on } \mathcal{A}$$

– classical algebraic probability space (CPS), if $\mathcal{A}$ is commutative (example: $L^{\infty}(\mathbb{R}, gauss)$);
– quantum probability space (QPS), if $\mathcal{A}$ is non–commutative (example: $(\mathcal{B}(\mathcal{H}), \mathrm{Tr}(W\,\cdot\,))$ where W is a density matrix).

We started from the classical algebraic probability space

$$(\mathcal{P}, P_X)$$

But we know that X has a quantum decomposition

$$X = a^+ + a^0 + a^-$$

So we can construct

$$\mathcal{P}(a^+, a^-, a^0)$$

$$:= \text{algebraic span of } \{a^+, a^-, a^0, 1\} \qquad (11)$$

$$= *\text{–algebra generated by } \{a^+, a^-, a^0, 1\}$$

this is a non–commutative $*$–algebra.
A natural state on this algebra is
the quantum extension of the probability distribution of the classical random variable X

$$\varphi_{vac}(z) := \langle \Phi_0, z\Phi_0 \rangle \ , \ \ z \in \mathcal{P}(a^+, a^-, a^0) \quad (12)$$

Summing up:

$$X \mapsto (\mathcal{P}(a^+, a^-, a^0), \varphi_{vac}) \text{ QPS}$$

Since

$$\mathcal{P} \subseteq \mathcal{P}(a^+, a^-, a^0) \quad ; \quad \varphi_{vac}\Big|_{\mathcal{P}} = \varphi = P_X \quad (13)$$

the QPS of X is a natural extension of the CPS of X.

Recall the orthogonal gradation

$$\mathcal{P} \cdot \Phi_0 = \bigoplus_{n\in\mathbb{N}} \mathcal{P}_n = \bigoplus_{n\in\mathbb{N}} \mathbb{C} \cdot \Phi_n \tag{14}$$

Definition 2 A linear operator

$$A \colon \mathcal{P} \cdot \Phi_0 \ \rightarrow \ \mathcal{P} \cdot \Phi_0$$

is called gradation preserving if

$$A\mathcal{P}_n \subseteq \mathcal{P}_n \quad , \quad \forall n$$

Theorem 3 Define the **number operator** by:

$$\Lambda\Phi_n := n\Phi_n \quad , \quad \forall n$$

An operator A is gradation preserving iff, there exists a function

$$F \colon n \in \mathbb{N} \rightarrow F_n \in \mathbb{C}$$

satisfying

$$A\Phi_n = F_\Lambda \Phi_n := F_n \Phi_n \quad , \quad \forall n \in \mathbb{N} \tag{15}$$

Multiplication table for the CAP operators.

The definition of CAP operators:

$$a^+\Phi_n = \Phi_{n+1} \quad , \quad a\Phi_n = \omega_n\Phi_{n-1}$$

implies the **multiplication table**

$$a^-a^+\Phi_n = \omega_{n+1}\Phi_n \iff a^-a^+ = \omega_{\Lambda+1}$$

$$a^+a^-\Phi_n = \omega_n\Phi_n \iff a^+a = \omega_\Lambda$$

From the multiplication table, we deduce the probabilistic commutation relations:

$$\iff [a^-, a^+]\Phi_n = (\omega_{n+1} - \omega_n)\Phi_n \quad , \quad \forall n$$

and their operator valued formulation:

$$\iff [a^-, a^+] = \omega_{\Lambda+1} - \omega_\Lambda =: \partial\omega_\Lambda \qquad (16)$$

Equation (16) includes all the known *deformations* of the usual Heisenberg CR, and in fact many more.

However in this approach they are they are not artificially put by hands, but deduced and framed within a 150–year old branch of classical analysis.

The existence of the
probabilistic commutation relations
naturally raises the following question:

are there classical random variables
whose canonically associated CR are the
original Heisenberg ones?

Theorem.
For every **strictly positive real number** σ, the **Gaussian measure** $\gamma_{0,\sigma}$, with zero mean and variance σ, is the **unique symmetric probability measure on** $\mathbb{R}$ whose CAP operators satisfy

$$[a^-, a^+] = \sigma \cdot 1 \tag{17}$$

Proof. We know from equation (16) that

$$[a^-, a^+]\Phi_n = (\omega_{n+1} - \omega_n)\Phi_n \quad , \quad \forall n \in \mathbb{N}$$

From the theory of orthogonal polynomials, we know that the gaussian with covariance σ, is the unique symmetric probability measure on $\mathbb{R}$ satisfying

$$\omega_n = \sigma n \quad , \quad \forall n \in \mathbb{N} \tag{18}$$

Since $\omega_0 = 0$, (18) is equivalent to

$$(\omega_{n+1} - \omega_n) = \sigma = \text{ constant } , \ \forall n \in \mathbb{N} \tag{19}$$

For $\sigma = \hbar$, we find the Heisenberg CR. $\square$

For the gaussian,
$\Phi_n = n$-th Hermite polynomial.

Remark.

There exists an infinite dimensional extension of the above theorem showing that the CR associated to the free Fock Boson field coincide with the probabilistic CR canonically associated to the classical Gaussian random field with covariance given by the identity operator times $\hbar$.

Probabilistic meaning of the preservation operator

Definition.
A classical real valued random variable X, with probability distribution P_X, is called **(moment–)symmetric** if its odd moments vanish

$$\int_{\mathbb{R}} X^{2n+1} dP_X = 0 \qquad , \qquad \forall n \in \mathbb{N}$$

Theorem. A classical real valued random variable X with canonical quantum decomposition

$$X = a^+ + a^0 + a^- \tag{20}$$

is **moment–symmetric if and only if**

$$a^0 = 0$$

Usual definition of position operator in physics

$$X = a^+ + a^- \tag{21}$$

Now we know that what is considered as an innocuous definition in physics is in fact
a strong probabilistic constraint:
it only allows position operators (identified to a real valued classical random variable in the standard probabilistic way) whose vacuum distribution is moment symmetric.

Momentum operators in classical probability

Up to now, we have formulated the Heisenberg commutation relations in terms of creation and annihilation operators.

However **Heisenberg original formulation of his commutation relations was in terms of** p **(momentum) and** q **(position), not in terms of** a **and** a^+.

But, up to now, **only position operator appeared** identified to the multiplication operator by the classical random variable X which we know to correspond to the operator q.

Natural question: is there an analogue of p in probabilistic quantization?

Generalized momentum operators.

Definition 3 Let $X = a^{+} + a^{-}$ $(v \in \mathbb{R})$ be a **symmetric** real valued classical random variable. The hermitean operator

$$P_X := i(a^{+} - a^{-}) \tag{22}$$

is called the **(probabilistic) momentum operator** associated to the classical random variable X.

Only symmetric classical real valued random variables have a conjugate momentum (non–symmetric random variables have not). Note that

$$P_X^* := (i(a^{+} - a^{-}))^* = -i(a^{-} - a^{+}) = P_X$$

So, the probabilistic momentum operator associated to the classical random variable X, is an **observable** in the quantum sense.

Remark. One can prove that, when X is the standard Gaussian, (22) reduces (up to a scalar multiple) to the **usual definition of momentum in quantum mechanics**.

Theorem 4 For a symmetric random variable

$$[X, P_X] = 2i\partial\omega_\Lambda \tag{23}$$

In the Gaussian case one can choose

$$\partial\omega_\Lambda = 1/2$$

and wit this choice (23) reduces to

$$[X, P_X] = i \tag{24}$$

which is the original formulation Heisenberg position–momentum commutation relations.

Therefore (23) is the probabilistic extension, **to an arbitrary (symmetric) classical random variable** X of Heisenberg position–momentum commutation relations.

Open problem
In the gaussian case (usual QM) the conjugate momentum of the classical random variable X has a natural physical interpretation.

Which is the **probabilistic interpretation** of the conjugate momentum of **an arbitrary classical random variable** X?

Part (II) of the talk.
Here we only list some research lines with lapidary comments.
In Part (III), for some of these lines, some more comments are added.

In my talk up to now, I have explained the new idea of probabilistic quantization in its simplest case: a single real valued random variable.

Even in this simplest case there is a large quantity of new results which I had no time to describe.

Below I list some examples.

1) The conjugate momentum of the classical semi-circle random variable

This shows the natural emergence, in probabilistic quantization, of the *Hilbert transform with respect to the semi-circle measure*.
This transform has important applications, for example to aerodynamics via the airfoil equation or to image reconstruction.
See the references in the paper:

L. Accardi, T. Hamdi, Y.G. Lu:
The quantum mechanics canonically associated to free probability I:
Free momentum and associated kinetic energy.
Open Syst. Inf. Dyn., 29 (2022) 2250017

2) The conjugate momentum of the classical arc–sine random variable

Here the momentum operator turns out to be the same as the semi-circle law plus a rank-one operator.

L. Accardi, T. Hamdi and Y.G. Lu:
Quantum Mechanics of Arc-Sine and Semi-Circle Distributions: A Unified Approach, Complex Analysis and Operator Theory (2025) 19:195

3) Another consequence of the quantum decomposition of X:

quantum Gaussianization of any classical random variable

This is strictly related to the notion of *quantum moments of a classical random variable*.

4) The multi–dimensional case

– Type I commutation relations:
they generalize the usual boson ones in a similar way as in the one–dimensional case.

– Type II commutation relations a new type of CR typical of the multi–dimensional case.

– Product states on $\mathcal{P}$.

Up to now, I have only discussed the latest, and in my opinion the most important, development of QP, which covers the last 20 years.

However, in the previous 25-30 years, many other important developments took place in quite different directions.

Some of these developments are currently widely used nowadays in physics, engineering (control theory, reliability theory), machine learning, numerical analysis, ... ,

In addition to
(I) probabilistic quantization.
Let me mention

(II) Quantum Markov Chains (QMC) and Quantum Hidden Markov Processes (QHMP).

(III) Quantum conditioning.

(IV) Notions of stochastic independence and CLT.

(V) Application to physics (the stochastic limit of quantum theory)

(VI) Recent developments of the debate on the foundations of quantum theory.

Comments on some of the research lines mentioned above.

(II) Quantum Markov Chains (QMC) and Quantum Hidden Markov Processes (QHMP).

Abdessatar Souissi, Abdessatar Barhoumi
Causal Architecture in Hidden Quantum Markov Models
February 26, 2026

The paper:
Luigi Accardi, Abdessatar Souissi, El Gheteb Soueidi, Mohamed Rhaima
Quantum Viterbi Algorithm
Preprint (2026)

proposes a quantum extension of the Viterbi Algorithm for Hidden Quantum Markov Models (HQMMs).

Given a finite observation sequence, the algorithm identifies hidden trajectories that maximize a joint score functional.

The main result is the proof of a quantum advantage:

coherent hidden trajectories attain decoding scores that surpass those achievable by any classical path restricted to diagonal effects.

The conjugate momentum of the classical semi-circle random variable

We have seen that the usual QM $\iff$ centered Gaussian measure with variance ω, characterized by

$$\omega_{n+1} - \omega_n = \omega \iff \omega_n = n\omega > 0 \quad , \quad \forall n \geq 0$$

where ω is a **strictly positive** constant.
We have also seen that the condition $\omega = 0$

$$\omega_{n+1} - \omega_n = 0 \quad , \quad \forall n \geq 0$$

characterizes the δ–**measures at single points**.

The **semi-circle measure** μ_{sc}, supported on the interval $[-\omega, \omega]$ (with variance $\omega > 0$) is given by the density function

$$x \longmapsto \frac{1}{2\pi\omega}\sqrt{4\omega - x^2}\chi_{(-2\sqrt{\omega},2\sqrt{\omega})}(x) \qquad (25)$$

So, it is symmetric and it is known from classical probability that its principal Jacobi sequence is **characterized by**

$$\omega_{n+1} - \omega_n = 0 \text{ **for** } n \geq 1$$

$$\iff \omega_n = \omega \geq 0 \quad ; \quad \forall n \in \mathbb{N}^* \tag{26}$$

Recall the definition of Hilbert transform on $\mathbb{R}$:

$$Hf(x) := 2\text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy \tag{27}$$

To simplify computations it is convenient to normalize the **semi-circle measure** μ_{sc}, choosing its support as the interval $[-2, 2] \subset \mathbb{R}$. So:

$$\mu_{sc}(dy) := \chi_{[-2,2]}(y) f(y) \frac{1}{\pi} \sqrt{4 - y^2} \tag{28}$$

With these notations, the μ_{sc}–Hilbert transform on $\mathbb{R}$ is defined by:

$$H_{\mu_{sc}} f(x) := 2\text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x-y} \mu(dy) \tag{29}$$

$$= 2\text{p.v.} \int_{-2}^{2} \frac{f(y)}{x-y} \frac{1}{\pi} \sqrt{4 - y^2} dy$$

One can prove that
(L.A., Y.G. Lu, Tarek Hamdi 2021):

$$P_{X_{sc}} = H_{\mu_{sc}} f(x) \tag{30}$$

$$= 2\text{p.v.} \int_{-2}^{2} \frac{f(y)}{x-y} \frac{1}{\pi} \sqrt{4-y^2} dy$$

Extensive research on the operator $H_{\mu_{sc}}$ has been carried out both for its own interest and for its important application to aerodynamics via the **airfoil equation**:

F. G. Tricomi:
Integral Equations,
Interscience Publisher Inc (1957)

F. King:
Hilbert Transforms,
Encyclopedia of Mathematics and its applications,
Vol.1, 124, Cambridge (2009)

Another interesting application of this operator includes **tomography** and problems arising in **image reconstruction**:

A. Katsevich, A. Tovbis:
Finite Hilbert transform with incomplete data: null-space and singular values,
Inverse Probl. 28 (2012) 105006

The conjugate momentum of the classical arc–sine random variable

The **arc–sine** distribution on the interval $[-2, 2] \subset \mathbb{R}$ is given by the density function

$$x \longmapsto \frac{1}{2\pi\omega}\sqrt{4\omega - x^2}\chi_{(-2\sqrt{\omega},2\sqrt{\omega})}(x) \qquad (31)$$

$$\omega_n = \begin{cases} \omega_1, & \text{if } n = 1 \\ \frac{1}{2}\omega_1, & \text{if } n \geq 2 \end{cases}$$

We normalize the arc–sine distribution by taking $\sqrt{\omega} = \omega = 1$: for all $x \in [-2, 2]$

$$\mu_{as}(dx) = \frac{1}{\pi\sqrt{4 - x^2}}\chi_{(-2,2)}(x)\, dx$$

Let $(T_n)_{n\in\mathbb{N}}$ denote the monic Chebyshev polynomials of first kind, associated to μ_{as}

$$T_n(x) = \begin{cases} 2\cos\Big(n \arccos(x/2)\Big), & \text{if } n \geq 1 \\ 1, & \text{if } n = 0 \end{cases} \tag{32}$$

and $(\Phi_n)_{n\in\mathbb{N}}$ the monic Chebyshev polynomials of second kind associated to μ_{sc}

$$\Phi_n(x) = \frac{\sin\Big((n+1)\arccos(x/2)\Big)}{\sin\Big(\arccos(x/2)\Big)}$$

Proposition 1 *The identity*

$$(4-x^2)H_{as}f(x)+T_{1,0}f(x) = H_{sc}f(x)-2T_{0,1}f(x) \quad (33)$$

holds everywhere on $(-2,2)$ *for every* $f \in L^2([-2,2],\mu_{as})$*, where for any* $\{i,j\} \in \{0,1\}$*,* $T_{i,j}$ *denotes the operator in* $L^2([-2,2],\mu_{as})$ *given by*

$$T_{i,j}(f)(x) := \frac{1}{||T_j||_2^2}\langle T_j, f\rangle_{L^2([-2,2],\mu_{as})}T_i(x)$$

$$= \left(\frac{1}{||T_j||_2^2}T_iT_j^*f\right)(x)$$

One has

$$L^2([-2,2],\mu_{as}) = \{T_0\}^{\perp} \oplus \{T_0\}$$

With these notations:

$$Pf(x) = iH_{sc}f(x) - 2iT_{0,1}f(x) \tag{34}$$

the arc–sine CAP operators $a^{\varepsilon}_{\mu_{as}}$ $(\varepsilon \in \{+,-\})$ are mapped into the following CAP operators

$$A^{+} = \frac{1}{2}\Big(Q + (4 - x^2)H_{as} + T_{1,0}\Big)$$
$$A^{-} = \frac{1}{2}\Big(Q - (4 - x^2)H_{as} - T_{1,0}\Big)$$

The arc–sine kinetic energy operator E_{as} is mapped into the operator $E{:=}\,\frac{1}{2}P^2$ in

$$L^2([-2,2],\mu_{as})$$

given by:

$$Ef(x) = \frac{1}{2}(4 - x^2)f(x) + T_{1,1}f(x) \tag{35}$$

The considerations above imply the following **Translation code** between mathematics and physics:
theory of **classical random variables** $\Longleftrightarrow$ **extensions of quantum mechanics**

$$1 := \text{constant function} \equiv \Phi_0 \ \textbf{vacuum} \text{ vector}$$

$$(a^+)^n\Phi_0 = \Phi_n$$

(this **follows from** $a^+\Phi_n = \Phi_{n+1}$)

$=$ **monic orthogonal polynomials** of degree n

$$\equiv |n\rangle \ n\text{–}\textbf{particle vector}$$

$$\mathcal{H}_0 := \mathbb{C} \cdot \Phi_0$$

$$\mathcal{H}_n := a^+\mathcal{H}_{n-1} = (a^+)^n\mathcal{H}_0 = \mathbb{C} \cdot \Phi_n$$

$$= \text{orthogonal polynomials of degree } n \ \equiv$$

$$\equiv n\text{–}\textbf{particle space}$$

The μ–orthogonal gradation

$$\mathcal{P}_X \cdot \Phi_0 = \bigoplus_{n \in \mathbb{N}} \mathbb{C} \cdot \Phi_n =: \bigoplus_{n \in \mathbb{N}} \mathcal{P}_n$$

$$=: \Gamma_X(\mathbb{C})$$

When X is the standard Gaussian, $\Gamma_X(\mathbb{C})$ is the usual **boson Fock space** over the complex Hilbert space $\mathbb{C}$.

Another consequence of the quantum decomposition of X:
quantum Gaussianization of any classical random variable

There are many characterizations of the classical standard Gaussian measure on $\mathbb{R}$ (which, having mean zero, is symmetric).
One of them is that all moments are expressed explicitly in terms of the covariance.

It can be proved that all moments of any classical symmetric random variable X can be expressed in terms of its vacuum quantum variance.

In this sense, from the quantum point of view, any random variable is a (generalized) Gaussian.
The theorem has recently been extended to

the non symmetric case, where it has a different formulation.

Let a^+, a be the CAP operators of a real valued **symmetric** random variable
X and $\Phi_0 \in \mathcal{H}$ the associated vacuum vector. Then the quantum decomposition of X is

$$X = a^+ + a$$

Therefore the **classical moments** of X are, for $n \in \mathbb{N}$

$$E(X^n) = \langle \Phi_0, X^n \Phi_0 \rangle = \langle \Phi_0, (a^+ + a)^n \Phi_0 \rangle$$

$$= \sum_{(\varepsilon_1, \cdots, \varepsilon_n) \in \{+,-\}^n} \langle \Phi_0, a^{\varepsilon_1} \cdots a^{\varepsilon_n} \Phi_0 \rangle$$

Definition 4 The quantum moments of X are the following expectation values:

$$\langle \Phi_0, a^{\varepsilon n} \cdots a^{\varepsilon 1} \Phi_0 \rangle \tag{36}$$

Note that the expectation value (44) is zero if either $a^{\varepsilon 1} = a^-$ or $a^{\varepsilon n} = a^+$.
We want to look at the non–zero case, so $a^{\varepsilon n} = a^+$. Therefore

$$\langle \Phi_0, a^{\varepsilon 2p} \cdots a^{\varepsilon 2} a^{\varepsilon 1} \Phi_0 \rangle$$

Define

$$l_1 := \min\{j \in \{1, \ldots, 2p\} \setminus \{1, 2p\} \colon \varepsilon_{l_j} = -\}$$

one has

$$2 \leq l_1 \leq 2p - 1$$

Then

$$= \left\langle \Phi_0, a^{\varepsilon 2p} \cdots a^{\varepsilon_{l_p + 1}} \left((a)_{l_p} (a^+)_{l_p - 1} a^{+\,(l_p - 2)} \Phi_0 \right) \right\rangle$$

$$= \left\langle \Phi_0, a^{\varepsilon 2p} \cdots a^{\varepsilon_{l_p + 1}} \Omega_{\Lambda + 1} a^{+\,(l_p - 2)} \Phi_0 \right\rangle$$

$$= \omega_{l_1 - 1} \left\langle \Phi_0, a^{\varepsilon 2p} \cdots a^{\varepsilon_{l_p + 1}} a^{+\,(l_p - 2)} \Phi_0 \right\rangle$$

Define

$$r_1 := l_1 - 1 \ (\geq 1)$$

$$(l_1, r_1) := (l_1, l_p - 1) \quad ; \quad l_1 > r_1$$

$$l_2 := \min\{j \in \{1, \ldots, 2p\} \backslash \{\{1, 2p\} \cup \{l_1, r_1\}\} : \varepsilon_{l_j} = -\}$$

Then

$$\varepsilon_{l_2 - 1} = +$$

Define

$$r_2 := l_2 - 1 \ (< l_2)$$

Iterating

$$\langle \Phi_0, a^{\varepsilon_1} \cdots a^{\varepsilon_n} \Phi_0 \rangle = \prod_{j=1}^{p} \omega_{r_j^\varepsilon} \qquad (37)$$

$$(l_j^\varepsilon, r_j^\varepsilon)_{j=1}^p$$

non-crossing pair partition of $\{1, \ldots, 2p\}$.

The emergence of non-crossing pair partition in a Boson context is surprising!

The multi–dimensional case:
$\mathbb{R}^d$**–valued classical random variables**
d can be $= +\infty$.

Definition 5 Let:
– $(\Omega, \mathcal{F}, P)$ be a classical probability space;
– V be a **real** vector space.
A V**–valued classical random field** is a **(real) linear** map

$$X : v \in V \to X_v \tag{38}$$

$\in$ { real-valued random variables on $(\Omega, \mathcal{F}, P)$}

If $V \equiv \mathbb{R}$, one speaks of a random or stochastic **process**.

The **polynomial algebra of** X is

$$\mathcal{P}_V := \tag{39}$$

$$\{\mathbb{C}\text{–algebra generated by the } X_v \ , \ v \in V\}$$

is a commutative $*$**–algebra**, for the point–wise operations, with involution given by complex conjugation and with identity given by the constant function equal to 1.
Since X_v is real–valued, it is an **hermitian element** of $\mathcal{P}_V$:

$$X_v^* = X_v \qquad ; \qquad \forall v \in V$$

Theorem 5 For a semi–scalar product $\langle \cdot , \cdot \rangle$ on $\mathcal{P}$ the following statements are **equivalent**:
(i) There exists a state φ on $\mathcal{P}$ such that:

$$\varphi(f^*g) = \langle f, g\rangle \qquad ; \qquad f, g \in \mathcal{P} \tag{40}$$

(ii) The semi–scalar product $\langle \cdot , \cdot \rangle$ satisfies

$$\langle 1_{\mathcal{P}}, 1_{\mathcal{P}}\rangle_{\varphi} = 1 \tag{41}$$

and, for each $v \in V$, multiplication by the coordinate X_v is a symmetric linear operator on $\mathcal{P}$ with respect to $\langle \cdot , \cdot \rangle$, i.e.:

$$\langle X_v f, g\rangle = \langle f, X_v g\rangle \qquad ; \qquad \forall f, g \in \mathcal{P} \ , \ \forall v \in V \tag{42}$$

In this case the state φ is given by

$$\varphi(Q) := \langle 1_{\mathcal{P}}, Q1_{\mathcal{P}}\rangle \qquad ; \qquad Q \in \mathcal{P} \tag{43}$$

The canonical quantum moments of a classical real valued random variable

Let a^+, a, a^0, be the CAP operators of a real valued random variable X
and $\Phi_0 \in \mathcal{H}$ the associated vacuum vector. Then the quantum decomposition of X is

$$X = a^+ + a^0 + a$$

Therefore the **classical moments** of X are, for $n \in \mathbb{N}$

$$E(X^n) = \langle \Phi_0, X^n \Phi_0 \rangle = \langle \Phi_0, (a^+ + a^0 + a)^n \Phi_0 \rangle$$

$$= \sum_{(\varepsilon_1, \cdots, \varepsilon_n) \in \{+,0,-\}^n} \langle \Phi_0, a^{\varepsilon_1} \cdots a^{\varepsilon_n} \Phi_0 \rangle$$

Definition 6 The quantum moments of X are the following expectation values:

$$\langle \Phi_0, a^{\varepsilon_1} \cdots a^{\varepsilon_n} \Phi_0 \rangle \tag{44}$$

Remember that the quantum decomposition

$$X = a^{+} + a^{0} + a$$

is a shorthand notation for

$$X_v = a_v^{+} + a_v^{0} + a_v^{-} \quad , \quad v \in V \tag{45}$$

where

$$V \equiv \mathbb{R} e_1$$

and e_1 is a linear basis of V (i.e. $e_1 \neq 0$), so that

$$X_v = X_{v_1 e_1} \quad , \quad v_1 \in \mathbb{R}$$

Fact:
the above construction is functorial.
You can replace
$V = \mathbb{C} \cdot \Phi_0 \ \mapsto$ any real vector space V:
real valued random variable $\mapsto$ V–valued random field.

In physics this transition is called
second quantization.
In probability language, we can interpret the decomposition of the classical random field

$$X_v = a_v^+ + a_v^0 + a_v^- \quad , \quad v \in V \tag{46}$$

as an extension of the Ito decomposition of a stochastic process.
– a_v^+ , a_v^- is the diffusion part;
– a_v^0 is the jump part.
But, as already specified in the beginning of this talk, there is no time to discuss the multi–dimensional case.

The operator identity

$$X = a^+ + a^0 + a^- \tag{47}$$

is called: the

quantum decomposition of the classical random variable X and the operators a^+, a^-, a^0 are called the

CAP operators canonically associated to the classical random variable X

(**Creation, Annihilation, Preservation**).

The degree filtration on $\mathcal{P}_V$

The family of sub–spaces of $\mathcal{P}_V$: for $n \in \mathbb{N}$,

$$\mathcal{P}_{V,n]} := \mathcal{P}_{n]} := \text{ (V is fixed)} \tag{48}$$

$$\{\text{linear span of monomials of degree} \le n\}$$

is called **the degree filtration** in $\mathcal{P}_V$.
It is increasing in the sense that

$$\mathcal{P}_{V,n]} \subset \mathcal{P}_{V,n+1]} \subset \mathcal{P}_V \quad ; \quad \forall n \in \mathbb{N} \tag{49}$$

moreover

$$\bigcup_{n\in\mathbb{N}} \mathcal{P}_{V,n]} = \mathcal{P}_V \tag{50}$$

The Jacobi tri–diagonal relation and quantum decompositions

Given

– the Hilbert space

$$[\mathcal{P}] := \Gamma_X(X) \text{ (The Fock space of } X\text{)}$$

$$\begin{cases} \text{closure of } \mathcal{P} \text{ for the scalar product induced} \\ \text{by the distribution } P_X \text{ of } X \text{ (see eq. (1))} \end{cases}$$

– the **increasing filtration** of closed sub–spaces of $[\mathcal{P}]$, i.e. $\forall n \in \mathbb{N}$

$$[\mathcal{P}_{n]}] \text{ (closure of } \mathcal{P}_{n]}) \quad ; \quad [\mathcal{P}_{n]}] \subseteq [\mathcal{P}]_{n+1]}$$

with corresponding orthogonal projectors

$$P_{n]} : [\mathcal{P}] \to [\mathcal{P}_{n]}] \quad ; \quad \forall n \in \mathbb{N}$$

– for each $v \in V$, a linear hermitian operator

$$X_v = X_v^* : \mathcal{P} \subseteq_{subspace} [\mathcal{P}] \to [\mathcal{P}]$$

filtration increasing of degree $+1$

$$X_v(\mathcal{P}_{n]}) \subseteq [\mathcal{P}]_{n+1]} \tag{51}$$

Define the sequence (P_n) of orthogonal projectors:

$$P_n := P_{n]} - P_{n-1]} \quad ; \quad \forall n \in \mathbb{N} \tag{52}$$

with the notation

$$P_{-1} = P_{-1]} := 0 \tag{53}$$

Then the projectors in the sequence (P_n) are mutually orthogonal.

Theorem 6 The family $(X_v)_{v\in V}$ is a set) of linear operators on $[\mathcal{P}]$ with the following properties:

– Each X_v is **hermitian on** $\mathcal{P}$:

$$\langle \xi, X_v\eta\rangle = \langle X_v\xi, \eta\rangle \qquad ; \qquad \forall \xi, \eta \in \mathcal{P}$$

– For each $v \in V$ and $n \in \mathbb{N}$,

$$X_v P_{n]}\mathcal{P} \subseteq P_{n+1]}[\mathcal{P}] \tag{54}$$

Then:

(i) The coordinate independent 3–diagonal Jacobi identity

$$P_m X_v P_n = 0 \quad \text{if} \quad m \notin \{n-1, n, n+1\} \tag{55}$$

holds on $\mathcal{P}$ and is equivalent to

$$X_v = \sum_{n\in\mathbb{N}} P_{n+1} X_v P_n$$

$$+ \sum_{n\in\mathbb{N}} P_n X_v P_n + \sum_{n\in\mathbb{N}} P_{n-1} X_v P_n \tag{56}$$

This identity suggests the following definition.

Definition 7 The operators defined by

$$a_v^+ \quad \textbf{(creator with test function } v\textbf{)}$$

$$:= \sum_{n \in \mathbb{N}} P_{n+1} X_v P_n \tag{57}$$

$$a_v^- \quad \textbf{(annihilator with test function } v\textbf{)}$$

$$:= \sum_{n \in \mathbb{N}} P_{n-1} X_v P_n \tag{58}$$

$$a_v^0 \quad \textbf{(preservator with test function } v\textbf{)}$$

$$:= \sum_{n \in \mathbb{N}} P_n X_v P_n \tag{59}$$

are called the CAP operators associated to the classical random field X.

Corollary 1 The CAP operators associated to the classical random field X enjoy the following properties:

$$(a_v^0)^* = a_v^0 \quad ; \quad (a_v^+)^* := a_v^- \quad ; \quad \text{on} \quad \mathcal{P} \tag{60}$$

$$a_v^+(\mathcal{P}_n) \subseteq [\mathcal{P}]_{n+1} \tag{61}$$

$$a_v^0(\mathcal{P}_n) \subseteq [\mathcal{P}]_n \tag{62}$$

$$a_v^-(\mathcal{P}_n) \subseteq [\mathcal{P}]_{n-1} \tag{63}$$

The coordinate independent 3–diagonal Jacobi identity (55) is equivalent to the identity

$$X_v = a_v^+ + a_v^0 + a_v^- \tag{64}$$

Definition 8 The decomposition (64) is called the **canonical quantum decomposition** of X_v.

Type I commutation relations

Since the map $u \in V_{\mathbb{C}} \to a_u^+$ is **linear** in u. From

$$\partial\Omega(v,u) := [a_v^-, a_u^+] \tag{65}$$

it follows that, denoting $\mathcal{L}(\mathcal{D},\mathcal{H})$ the space of linear maps $\mathcal{D} \to \mathcal{H}$,

$$(v,u) \in V_{\mathbb{C}} \times V_{\mathbb{C}} \to \partial\Omega(v,u) \in \mathcal{L}(\mathcal{D},\mathcal{H})$$

is a **gradation preserving sesqui–linear** map from $V_{\mathbb{C}} \times V_{\mathbb{C}}$ to $\mathcal{L}(\mathcal{D},\mathcal{H})$.

Product states on $\mathcal{P}$

Here we illustrate,with a concrete example, why it is useful to consider usual quantum theory from a higher point of view

Fixing a basis $e \equiv (e_j)$, the Type I commutation relations become

$$V \ni v = \sum_{j \in D} v_j e_j \equiv (v_j)_{j \in D} \mathbb{R}^d$$

and one can consider the commutators

$$[a^-_{e_j}, a^+_{e_k}] \tag{66}$$

Usual boson quantum theory is formulated in such a way that one can always choose coordinates $e \equiv (e_j)$, in which CAP operators belonging to different degrees of freedom commute, namely:

$$j \neq k \ \Rightarrow [a_j, a^+_k] = 0 \tag{67}$$

Let us examine the implications of this conditions from the point of view of probabilistic quantization.

To this goal, we have to recall the notion of *product states on* $\mathcal{P}$.

Denote

$$D := \mathbb{N}^* \cup \{+\infty\} := (\mathbb{N} \setminus \{0\}) \cup \{+\infty\}$$

Fix a **linear basis** $e \equiv (e_j)_{j \in D}$ of V.
Then the X_{e_j} are classical (i.e. commutative) real valued random variables in the classical algebraic probability space $(\mathcal{P}, \mu)$ and they generate the whole algebra $\mathcal{P}$.

Definition 9 A state μ on $\mathcal{P}(\mathbb{R}^d)$ is said to be a **product state** if, there exists a linear basis $e = (e_j)$ of V such that, the real valued classical random variables

$$X_j := X_{e_j}$$

are μ–independent.
In this case we write

$$\mu =: \bigotimes_{j \in D} \mu_j$$

Recall that,

Definition 10 The real valued classical random variables

$$X_j := X_{e_j}$$

are called μ–independent if, with the notations,

$$\mathcal{P}(X_j) := \text{algebra generated by } X_j$$

$$\mu_j := \text{ the restriction of } \mu \text{ on } \mathcal{P}(X_j)$$

for any $n \in \mathbb{N}$, any $f_1, \ldots, f_d \in \mathcal{P}(\mathbb{R})$, one has:

$$\mu(f_1(X_1) \cdots f_d(X_d)) = \prod_{j \in D} \mu_j(f_j(X_j)) \quad (68)$$

expectation of a product
= product of the expectations

and, if $|D| = +\infty$, almost all the f_j ($d \in D$) are supposed to be identically equal to 1.

The following theorem is a quantum characterization of classical product states on $\mathcal{P}(\mathbb{R}^d)$.

Theorem 7 Let μ be a state on $\mathcal{P}(\mathbb{R}^d)$ and let

$$X_v = a_v^+ + a_v^0 + a_v^- \qquad ; \qquad v \in D \tag{69}$$

be the quantum decomposition of the classical $\mathbb{R}^d$–valued random field X with distribution μ. Then:

(i) μ is a product type state on $\mathcal{P}(\mathbb{R}^d)$ if and only if,

$$\forall j \neq k \in D \quad , \quad \forall \varepsilon, \eta \in \{+1, 0, -1\}$$

$$[a_j^\varepsilon, a_k^\eta] = 0 \tag{70}$$

(i.e., CAP operators corresponding to different degrees of freedom commute.)

In particular, for symmetric quantum fields

$$X_v = a_v^+ + a_v^- \qquad ; \qquad v \in D \qquad (71)$$

(the only ones considered up to now in quantum field theory)

the quantum characterization (70) of classical product states becomes

$$j \neq k \ \Rightarrow [a_j, a_k^+] = 0 \qquad (72)$$

Conclusion:

since, in contemporary quantum field theory, the implication (72) is always satisfied for some linear (in fact orthogonal) basis $e = (e_j)$ of V, it follows that:

contemporary quantum field theory cannot describe interacting (i.e. non–product) states on $\mathcal{P}(\mathbb{R}^d)$.

The above theorem naturally rises the following question.

If we want to describe truly interacting (i.e. non–product) states on $\mathcal{P}(\mathbb{R}^d)$ (which is the goal of quantum field theory), which kind of commutation relations shall we use?

The answer to this question is contained in the: Type II commutation relations.

Theorem 8 Let be given:
– a pre–Hilbert space H;
– an orthogonal gradation of H:

$$H = \bigoplus_{n \in \mathbb{N}} H_n;$$

– a family of operators
$a_j^{\pm} : H_n \to H_{n\pm 1}$, $a_j^0 : H_n \to H_n$, $(j \in \{1, \cdots, d\})$

$$a_j^0 = (a_j^0)^* \quad ; \quad a_j^- = (a_j^+)^* \quad ; \quad j \in \{1, \cdots, d\}$$

defined on a common dense domain $\mathcal{D}$ and such that the products of the form $a_j^{\varepsilon_j} a_k^{\varepsilon_k}$ are well defined on $\mathcal{D}$. Define the operators Y_j $(j \in \{1, \cdots, d\})$ on H by

$$Y_j := a_j^+ + a_j^0 + a_j^- , \qquad j \in \{1, \cdots, d\}. \tag{73}$$

Then the decomposition (73) is unique.

Moreover, the operators Y_j **commute on** $\mathcal{D}$ if and only if the operators a_j^+, a_j^0, a_j^- satisfy the following commutation relations on the same domain: for all $j, k \in \{1, \cdots, d\}$ such that $j < k$

$$[a_j^+, a_k^+] = 0 \tag{74}$$

$$[a_j^+, a_k^-] + [a_j^0, a_k^0] + [a_j^-, a_k^+] = 0 \tag{75}$$

$$[a_j^+, a_k^0] + [a_j^0, a_k^+] = 0 \tag{76}$$

The commutation relations (74), (75), (76), are called:

Type II commutation relations.
The reason why they have not yet been met in physics is explained by Theorem (7).
In fact in boson quantum theory up to now one always assumes that there are coordinates in which CAP operators belonging to different degrees of freedom commute.
But Theorem (7) tells us that this prejudice should be abandoned if we want to go beyond the family of product measures.

The following theorem extends to the multi–dimensional case, the quantum characterization of classical gaussian measures.

Theorem 9 Let V be a real vector space and let

$$(v, u) \in V_{\mathbb{C}} \times V_{\mathbb{C}} \mapsto \partial\omega(v, u) \in \mathbb{C}$$

be a non–trivial semi–scalar product on $V_{\mathbb{C}} \equiv V \dot{+} iV$ (the complexification of V) taking real values on $V \times V$.
For any classical random field X on V the following statements are equivalent.
(i) The CAP operators of X satisfy

$$a_v^0 = 0 \ , \ \forall v \in V \text{ (i.e. } X \text{ is symmetric)} \quad (77)$$

$$[a_v^-, a_u^+] = \partial\omega(v, u) \cdot 1 \text{ (Heisenberg CR)} \quad (78)$$

(ii) The quantum field canonically associated to X is the Fock Boson field over the Hilbert space $(V_{\mathbb{C},\partial\omega} \ , \ \langle \ \cdot \ , \ \cdot \ \rangle_{\partial\omega})$, obtained by completing $V_{\mathbb{C}}$ with the semi–scalar product $\partial\omega$.
(iii) X is the standard classical Gaussian field on the real Hilbert space $(V_{\partial\omega} \ , \ \langle \ \cdot \ , \ \cdot \ \rangle_{\partial\omega_V})$, obtained by completing V with the semi–scalar product $\partial\omega_V$ where $\partial\omega_V$ is the restriction of $\partial\omega$ on $V \times V$.

Conclusions

The purpose of this talk was to formulate the main theses advocated by the **probabilistic quantization program**, so to put everyone in the conditions to judge for themselves whether the more technical developments of this program really support these theses.

For 100 years the term quantization has been related to physics.

Now we understand that quantization is a universal phenomenon in classical probability.

Physics has shown that the quantum formalism is very powerful in dealing with physical phenomena.

Now we have to learn how to harness this power in favour of every field in which probability plays a role.

This practically includes every human

activity.

The term **quantum technology** has been understood up to now as: **applications of quantum theory to technology**.
But now we understand that, from the mathematical point of view, what in physics is called **quantization** should be called:
– **gaussian quantization** in the boson case;
– **Bernoulli quantization** in the fermion case;
These are **very special cases of probabilistic quantization**
and now we know that every measure with all moments canonically determines its own quantization rules.
So nowadays the term **quantum technology** should be understood as:
applications of probabilistic quantization to technology
(but see a caveat at the end of this introduction).

The following is one of the many non–trivial feedbacks for physics:
there are strong mathematical results proving that, in dimensions ≥ 2, in a truly interacting quantum theory **Heisenberg commutation relations must be replaced by the intrinsic (or canonical) probabilistic commutation relations associated with a given measure**
(see again the caveat at the end of this introduction).
The basic new idea is that
non–linearity is not only coded into the interaction potentials, but also into the commutation relations.

We have seen that:
any classical random variable with all moments (stochastic process, random field: so quantum field theory is also covered).
intrinsically generates **its own generalized**:
– Creation, Annihilation, Preservation (CAP) operators;
– quantum probability space;
– normal order
– commutation relations;
– Type II commutation relations (in dim ≥ 2);
– Fock space;
– Fock functor (second quantization);
– coherent vectors;
– momentum operator;
– free evolutions and equilibrium (or local equilibrium) states;
– generalized harmonic oscillators;
– ...
The term intrinsic here means that
nothing is put by hands, from the outside:
all mathematical structures are uniquely deduced from the definition of classical random variable with all moments.

In its more than 100 years of development, there have been several attempts:
(i) to generalize different aspects of quantum mechanics (QM): Birkhoff–von Neumann quantum logics, von Neumann projective geometries, Mackey's approach based on symmetries and group representations, Perelomov generalization of coherent states, ...
(ii) to reduce QM to some classical theory: various hydrodynamical models, Bohm-Vigier, Stochastic mechanics, correspondence between Poisson brackets and commutators, ...

But all these attempts concern very limited aspects of QT:
the probabilistic approach is the only one, to my knowledge, capable of deducing all the above mentioned characteristics of the usual quantum mechanical model in a much wider framework.

Implications for disciplines outside physics.

100 years of quantum Physics have shown that the quantum formalism is very powerful in dealing with physical phenomena.
Now we have to learn how to harness this power in favour of every field in which probability and quantitative data play a role:
biology, medicine, sociology, technology, ...
This practically includes every human activity.

Feedback for physics:
there are many of them, but they will not be discussed in my talk today.

A caveat to the above program comes from the fact that there are many examples, which show that not all commutation relations that encode nonlinear interactions arise from probabilistic quantization.

These example are not canonical, but they naturally arise from the stochastic limit of quantum theory.

The first example came from QED without dipole approximation, but after a few years it became clear that these new entangled (or module) commutation relations are a universal phenomenon associated to strongly non–linear interactions and are always accompanied by another universal phenomenon, namely the replacement of the usual boson diagrams by the non–crossing diagrams.

So, probabilistic quantization is not the end of the story (but who believes that, in science, there will ever be an *end of the story*?)
This is a delicate issue, but there will certainly not be time to discuss it in my talk.

For reasons of time I will discuss mainly the case of a single $\mathbb{R}$–valued random variable with all moments, but the theory works for every classical random field with all moments
(so quantum field theory is also covered).

If time allows, at the end of this talk, I will try to explain
– why there is a phase transition in difficulty from dimension 1 to dimensions ≥ 2;
– why this phase transition has not been noticed up to now in the literature,
– and which are the new qualitative features (with respect to standard quantization) that emerge in this transition.

Abstract

The emergence of Heisenberg commutation rule $[q, p] = i\hbar$ for position and momentum in boson quantum mechanics **(QM)**, and more generally of non–commutativity in QM, has been a mystery since the dawn of this theory and has remained so for almost 100 years.
The discovery of the **quantum decomposition of a classical random variable** gave rise to a line of research that, in little more that 25 years, has produced a natural solution to this problem, namely: *the whole quantum theory (**QT**) can be deduced from the combination of classical probability **(CP)** with the classical theory of orthogonal polynomials **(OP)***.
The importance of this new approach does not lie so much in the solution of the problem mentioned above, as in the fact that the QT used until now appears as a very special case of a much broader purely deductive theory, which

shows that *every classical random variable with all moments, is canonically associated to a new QT, which reduces to the standard boson one in the case of gaussian random variables and to the standard fermion one in the case of Bernoulli random variables*.

That's why we speak of **probabilistic quantization**.

In other words, for the first time in almost 100 years, the mathematical apparatus of QT appears in the perspective of a natural deduction and not as a strange, singular theory justified a posteriori by its enormous empirical success, but totally mysterious in its origins and meaning.

The fruitfulness of this new point of view is demonstrated by the fact that it has already led to the solution of several open problems in classical probability, in the classical theory of orthogonal polynomials and in physics itself, where it has provided powerful new tools for

the description of natural phenomena.
This poses an exciting challenge for everybody interested in classical probability and its applications. In fact, since we now know that **quantization is a universal classical probabilistic phenomenon** and that every classical random variable with all moments (or process or field, since the theory applies also to infinite dimensions) canonically produces a quantum theory, it follows that every field of science in which classical probability is involved (i.e. almost all), will have to learn how to exploit the benefits of this duality between classical and quantum description.
By exploiting the quantum mathematical formalism, the physicists have managed to produce fantastic results in the past 100 years: now this power is available to all those who use classical probability in any field of science and technology.
We need to learn to use these new mathematical tools.

Why mathematicians interested in applications should pay some attention to probabilistic quantization.

In the title of my talk, I speak of *mathematicians interested in applications* rather than *applied mathematicians* because in my opinion the, historically very recent, separation between so-called *pure* and so-called *applied* mathematicians is sociological rather than scientific and risks to kill the old tradition according to which a good mathematician has always been interested both in pure research and in problems arising outside mathematics. This duality was already present in the time when Aristotle criticised Plato for his overly purist attitude towards mathematics and, in various forms, it has survived to the present day.

History is full of examples of mathematical structures which were constructed without having in mind any concrete problem and which, sometimes centuries after, found important applications.

More frequently the development of mathematics chooses the opposite path, where some techniques or concepts developed to solve a specific problem prove, possibly with appropriate variations, applicable to large classes of problems, even very far from the one that originated them.

The techniques and concepts I will describe belong to the second category. In fact they were born from the solution of a very difficult and very specific problem of physics: **the stochastic limit of quantum electrodynamics without dipole approximation**.

I will not have time to describe what this problem consists of. The interested reader can find a description of it in **chapter 12** of the book

L. Accardi, Y.G. Lu, I. Volovich:
Quantum Theory and Its Stochastic Limit,
Springer (2002)

I only mention that the solution of this problem led LA and Yun Gang Lu to introduce the notions of IFS and IFM as an attempt to axiomatize the complex structures naturally arising from the stochastic limit of QED. In the subsequent years, these two notions developed in independent directions. In the following we only discuss the simple one: the notion of IFS.