

Ornstein-Uhlenbek operators induced by the Berezin-Weyl-Segal-Bargmann-Fock models

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Plan

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- 2 The Airault-Jendoubi-Ouerdiane approach
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- 5 Equivalent representations of the Heisenberg group on Twisted Fock spaces and perturbation of O-U operators
- 6 Ornstein-Uhlenbeck operator induced by the Engel geometric structure



Stochastic Models in Science

In 1827, the Botanist physicist R. Brown investigated the fertilization process in flowers and observed a rapid oscillatory motion of microscopic particles within pollen grains suspended in water.

1901, Louis Bachelier proposed the first application of Brownian motion in mathematical finance, modeling the evolution of prices in the financial market.

1905 Einstein interpreted quantitatively (physically) this phenomenon (BM) in terms of the heat equation.

1923, Wiener gave the first mathematical concept of Brownian motion (Wiener process)

1972, Black and Scholes used geometric Brownian motion as a functional of BM in concrete Stochastic Modelling: The financial Market.

Definition: The Brownian Motion (Wiener process)

A Brownian motion is a stochastic process $B = (B_t; t \geq 0)$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfies the following properties:

- ① For each $t > 0$ $B_t \sim \mathcal{N}(0, t)$ and $B_t \in \mathcal{F}_t$.
- ② Stationary increments: that is For any $t > s \geq 0$
 $B_t - B_s = B_{t-s} \sim \mathcal{N}(0, t - s)$.
- ③ Independent Increments: For all $n; t_1, t_2, \dots, t_n [0, +\infty)$
 with $(t_1 < t_2 < \dots < t_n)$, the random variables
 $(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})$ are independent.
- ④ Continuity: With probability 1, the function $t \mapsto B_t$ is
 continuous on $[0, +\infty)$.
 Nevertheless; with probability 1, the function $t \mapsto B_t$ is
 nowhere differentiable on $[0, +\infty)$.

The Heat semigroup

Let $(B_t)_{t \geq 0}$ be a Brownian motion on filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$.

The heat semi-group is the collection of operators $(H_t)_{t \geq 0}$ acting on $\mathcal{C}_0^2(\mathbb{R})$ by :

$$H_0 = Id, H_t(f)(x) = \mathbb{E}(f(x + B_t)) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(y) e^{-\frac{(y-x)^2}{2t}} dy; \quad t > 0 \quad (1)$$

The infinitesimal generator of the heat semi-group is given by

$$(L_0 f)(x) := \lim_{t \rightarrow 0} \frac{1}{t} (H_t(f) - f)(x) = \frac{1}{2} \frac{d^2}{dx^2} (f)(x) \quad (2)$$

for any $f \in \mathcal{C}_0^2(\mathbb{R})$,

The Euclidean Ornstein-Uhlenbeck phenomenon

De Grooth, B. G.,: A simple model for Brownian motion leading to the Langevin equation. Am. J. Phys (1999) describes the Euclidean Ornstein-Uhlenbeck phenomenon "The collisions between a Brownian particle and its surrounding molecules lead to the Langevin equation, the power spectrum of the stochastic force, and the equipartition of kinetic energy.

The Langevin equation

$$dX_t = -X_t dt + \sqrt{2} dB_t \quad (3)$$

where $(B_t)_t$ is the standard Brownian motion on $\mathbb{R}$.

Explicitly, we have

$$X_t = xe^{-t} + \sqrt{2} \int_0^t e^{s-t} dB_s \quad (4)$$

Definition

The Ornstein-Uhlenbeck semi-group is the collection of operators $(N_t)_{t \geq 0}$ acting on $\mathcal{C}_0^2(\mathbb{R})$ by :

$$N_t(f)(x) = \mathbb{E}_x(f(X_t)) = \int_{\mathbb{R}} f(e^{-t}x + \sqrt{1 - e^{-2t}}y) d\gamma(y), \quad (5)$$

where $d\gamma(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ is Gaussian probability measure on $\mathbb{R}$.

The family $(N_t)_{t \geq 0}$ satisfies the following properties:

- ① For any $f \in \mathcal{C}_0^2(\mathbb{R})$, $N_0 f = f$.
- ② For any $s, t \geq 0$ and $f \in \mathcal{C}_0^2(\mathbb{R})$, $N_s(N_t f) = N_{s+t} f$.
- ③ For any $f \in \mathcal{C}_0^2(\mathbb{R})$, the infinitesimal generator of the Ornstein-Uhlenbeck semi-group

$$(Lf)(x) := \lim_{t \rightarrow 0} \frac{1}{t} (N_t(f) - f)(x) = \left(\frac{d^2}{dx^2} - x \frac{d}{dx} \right) (f)(x).$$

Proposition

The Ornstein-Uhlenbeck semi-group and its generator L are self-adjoint in $L^2(\gamma)$.

In particular, they are *invariant w.r. to the Gaussian measure γ* ,

$$\int_{\mathbb{R}} (Lf)(x) d\gamma(x) = 0, \text{ for all } f \in \mathcal{C}_0^2(\mathbb{R}) \quad (6)$$

Complex O-U operators

The one-dimensional complex Ornstein-Uhlenbeck process $(Z_t)_{t \geq 0}$ was physically interpreted as the fluctuations of an electric charge in the presence of a magnetic field, it satisfied the following S. D. E (Cheng Yong, Liu Yong 2012)

$$dZ_t = -\sigma Z_t dt + \frac{1}{\sqrt{2}} dB_t, \quad (7)$$

where σ is a positive real number and $(B_t)_{t \geq 0}$ is an Euclidian Brownian motion. The infinitesimal generator of $(Z_t)_t$ is given by

$$\Delta^{OU, \sigma} = 2 \frac{\partial^2}{\partial z \partial \bar{z}} - \sigma \bar{z} \frac{\partial}{\partial \bar{z}} - \sigma z \frac{\partial}{\partial z}. \quad (8)$$

For $z = x + iy \in \mathbb{C}$, we recall the Cauchy-Riemann operators

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Unitary Berezin-Weyl Lie group representations on RKHS

Given a locally compact simply connected nilpotent Lie subgroup G of the complex linear group $GL_n(\mathbb{C})$ acting holomorphically on $\mathbb{C}$.

We consider Berezinian-Weyl (holomorphic multiplier) representations T_g satisfying the following conditions

1)

$$(T_g F)(z) = h_g(z) F(k_g(z)), \quad (9)$$

where $h_g : \mathbb{C} \rightarrow \mathbb{C}$ and $k_g : \mathbb{C} \rightarrow \mathbb{C}$ are holomorphic functions.

2) For any $g_1, g_2 \in G$ and $F \in \mathcal{H}(\mathbb{C})$, we have

$$(T_{g_1} \circ T_{g_2})F = C(g_1, g_2) T_{g_1 g_2} F$$

where the mapping $C : G \times G \rightarrow \mathbb{S}^1 = \{z \in \mathbb{C}, |z| = 1\}$ is called the Weyl factor.

Conditions 1.) and 2.) together imply the 2-cocycle relations

$$k_{g_2}(k_{g_1}(z)) = k_{g_1 g_2}(z), \quad (10)$$

$$h_{g_1}(z) h_{g_2}(k_{g_1}(z)) = C(g_1, g_2) h_{g_1 g_2}(z). \quad (11)$$

3) The representations T_g is unitary into a reproducing kernel Hilbert space (in abbreviation RKHS) of holomorphic functions $\mathcal{HL}^2(\mathbb{C}, \mu)$, where μ is a probability measure on $\mathbb{C}$ absolutely continuous with respect to the Lebesgue measure $d\lambda$, that is $d\mu(z) = \theta(z)\lambda(z)$, where θ is a continuous non negative function on $\mathbb{C}$. Throughout this development, the space $\mathcal{HL}^2(\mathbb{C}, \mu) = \mathcal{H}(\mathbb{C}) \cap L^2(\mu)$ endowed with the inner product

$$\langle F_1, F_2 \rangle = \int_{\mathbb{C}} F_1(z) \overline{F_2(z)} d\mu(z)$$

for $F_1, F_2 \in \mathcal{HL}^2(\mathbb{C}, \mu)$, is assumed to be a RKHS. We shall say that μ is *unitarising* measure of T_g .

Since the Lebesgue measure is rotation invariant on $\mathbb{C}$, it is convenient to investigate automorphisms of $\mathbb{C}$ having the form $k_g(z) = z + s(g)$, where $s : G \rightarrow (\mathbb{C}, +)$ is a group homomorphism. We consider T_g of the form

$$(T_g F)(z) = e^{\alpha(g) + \beta(g) \cdot z} F(z + s(g)), \quad (12)$$

where $\beta, s : G \rightarrow (\mathbb{C}, +)$ are group homomorphisms and a mapping $\alpha : G \rightarrow \mathbb{C}$.

Proposition

The representation T_g in (12) is a Weyl representation if and only if

$$C(g_1, g_2) = e^{\alpha(g_1) + \alpha(g_2) - \alpha(g_1 g_2) + \beta(g_2) s(g_1)}, \quad g_1, g_2 \in G. \quad (13)$$

Furthermore, if there exists a positive continuous map $\Theta : \mathbb{C} \rightarrow \mathbb{R}^+$ satisfies

$$\Theta(z - s(g)) = e^{-2 \operatorname{Re}\{\alpha(g) + \beta(g) \cdot (z - s(g))\}} \Theta(z), \quad \forall g \in G, \quad \forall z \in \mathbb{C}. \quad (14)$$

Then, the representation (12) is unitary into $\mathcal{H}L^2(\mathbb{C}, \Theta(z) d\lambda(z))$.

The problem of unitarising-invariant measure

Our central goal is to prove that

A **unitarising measures** μ for a multiplier representation of a Lie group is also **invariant for a second order differential operators** on D of Ornstein-Uhlenbeck type, denoted Δ^{OU} , invariant with respect to μ , i.e

$$\int_D \Delta^{OU} \Phi(z, \bar{z}) \, d\mu(z) = 0 \quad (15)$$

for all Φ in a suitable class of test functions.

Infinitesimal Representations

Definition

Let G be a Lie group and $\mathcal{G}$ its Lie algebra. For $X \in \mathcal{G}$ and $F \in \mathcal{H}(\mathbb{C})$, the infinitesimal representation $\rho(X)$ of T_g is defined by

$$(\rho(X)F)(z) = \frac{d}{dt}\bigg|_{t=0} T_{g_t} F(z). \quad (16)$$

where $t \mapsto g_t := e^{tX}$ be a differentiable one-parameter subgroup on G such that $g_0 = I_n$.

infinitesimal representation of

$$(T_g F)(z) = e^{\alpha(g) + \beta(g) \cdot z} F(z + s(g))$$

Let $u : GL_n(\mathbb{C}) \rightarrow \mathbb{C}$ be a smooth map, we denote by $du_{I_n}(X) := Du(I_n)(X)$, its differential at the identity matrix I_n . Let $z \in \mathbb{C}$ and $X \in \mathcal{G}$; we put

$$b(X)(z) = \frac{d}{dt}\bigg|_{t=0} h_{g_t}(z),$$

then

infinitesimal representation of

$$(T_g F)(z) = h_g(z) F(k_g(z))$$

Proposition

$$\rho(X) = H(X) + b(X)I, \quad (17)$$

where $H(X)$ is the holomorphic vector field,

$$H(X) = ds_{I_n}(X) \frac{\partial}{\partial z} \quad (18)$$

and $b(X)$ is the scalar entire function on $\mathbb{C}$

$$b(X)(z) = d\alpha_{I_n}(X) + d\beta_{I_n}(X) z. \quad (19)$$

We introduce a class of second-order differential operators, called of Ornstein-Uhlenbeck type, invariant with respect to the unitarising measure of an holomorphic representation T_g of a Lie group G with Lie algebra $\mathcal{G}$.

Definition

Let T_g be a unitary representation of a Lie group into $\mathcal{H}L^2(\mathbb{C}, \mu)$. We say that a second-order differential operator Δ^{OU} is of Ornstein-Uhlenbeck type if it satisfies the following conditions:

- 1 Δ^{OU} has the form

$$\Delta^{OU} = A(z, \bar{z}) \frac{\partial^2}{\partial z \partial \bar{z}} + B(z, \bar{z}) \frac{\partial}{\partial z} + C(z, \bar{z}) \frac{\partial}{\partial \bar{z}}, \quad (20)$$

where $A(z, \bar{z})$, $B(z, \bar{z})$ and $C(z, \bar{z})$ are continuous functions on $\mathbb{C}$.

- 2 Δ^{OU} is invariant w.r.t the unitarising measure μ of the representation T_g .

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- 2 Δ^{OU} is invariant w.r.t the unitarising measure μ of the representation T_g .

Canonical Induction

Definition

Let T_g be a unitary representation of a Lie group G into the Hilbert space $\mathcal{H}L^2(\mathbb{C}, \mu)$ and ρ its infinitesimal representation. We say that the geometric structure $(G, \mathbb{C}, T_g, d\mu, \rho)$ induced an Ornstein-Uhlenbeck type operators Δ^{OU} (phenomenon) if there exists an Ornstein-Uhlenbeck type operator Δ^{OU} generated by $\rho(X)$, $H(X)$ or/and their complex conjugate, for some $X \in \mathcal{G}$.

Canonical Induction

In the sense that if $(v_1, \dots, v_N)$ is a basis of the Lie algebra $\mathcal{G}$, there exist constants $\alpha_{j,k}, \beta_{j,k}, \gamma_{j,k}, \delta_{j,k}, \sigma_{j,k}, \theta_{j,k}$ such that the Ornstein-Uhlenbeck type operator Δ^{OU} can be expressed as

$$\begin{aligned} \Delta^{OU} = & \sum_{j,k=1}^N \{ \alpha_{j,k} \rho(v_j) \overline{H(v_k)} + \beta_{j,k} \overline{\rho(v_j)} \overline{H(v_k)} \\ & + \gamma_{j,k} \rho(v_j) H(v_k) + \delta_{j,k} \overline{\rho(v_j)} H(v_k) \\ & + \sigma_{j,k} \rho(v_j) \overline{\rho(v_k)} + \theta_{j,k} \overline{\rho(v_j)} \rho(v_k) \}. \end{aligned} \tag{21}$$

Canonical Induction

Let G be a Lie group with Lie algebra $\mathfrak{g}$ of dimension $n \geq 2$, endowed with a basis $(X_j)_{j=1,\dots,n}$, and let T_g (as in (12)) be a unitary Berezin-Weyl representation into the reproducing kernel Hilbert space

$$\mathcal{H}L^2(\mathbb{C}, d\mu(z) := e^{V(z,\bar{z})} d\lambda(z)),$$

where $V(z, \bar{z}) = -az\bar{z} - b(z^2 + \bar{z}^2)$, $a > 0$, $b \in \mathbb{R}$ with an infinitesimal representation

$$\rho(X_j) = s_j \frac{\partial}{\partial z} + b_j(z)I, \quad (22)$$

where

$$s_j = ds_{l_n}(X_j); \quad b_j(z) = b(X_j)(z) = \alpha_j + \beta_j \cdot z, \quad \forall j \in \{1, \dots, n\}, z \in \mathbb{C}. \quad (23)$$

Canonical Induction: $\dim(\text{Im}(\rho)) \geq 2$

Assume that $d_\rho := \dim(\text{Im}(\rho)) \geq 2$ and there exist real constants $(C_{j,k})_{j,k=1,\dots,n}$ satisfying the following two conditions

1

$$\sum_{j,k=1}^n C_{j,k} s_j s_k = 0. \quad (24)$$

2

$$2 \sum_{j,k=1}^n C_{j,k} \overline{s_k} \text{Re}(b_j(z)) = \left(\sum_{j,k=1}^n C_{j,k} s_j \overline{s_k} \right) \frac{\partial V(z, \overline{z})}{\partial z}. \quad (25)$$

Then,

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Then,

Canonical Induction: $\dim(\text{Im}(\rho)) \geq 2$

Theorem: S. J 26

The complex second-order differential operator

$$\Delta^{ou} = \frac{\partial^2}{\partial z \partial \bar{z}} + \frac{\partial V(z, \bar{z})}{\partial z} \frac{\partial}{\partial \bar{z}} \quad (26)$$

$$= \sum_{j,k=1}^n C_{j,k} (\rho(X_j) + \overline{\rho(X_j)}) \circ \overline{H(X_k)} \quad (27)$$

is an Ornstein-Uhlenbeck operator induced by the quintuple $(G, \mathbb{C}, T_g, \mu, \rho)$.

Canonical Induction: $\dim(\text{Im}(\rho)) = 1$

We prove the induction phenomenon when the infinitesimal representation of the Lie algebra has only one generator

$$\rho(X_{j_0}) = s_{j_0} \frac{\partial}{\partial z} + b_{j_0}(z)I, \quad s_{j_0} \neq 0. \quad (28)$$

where $b_{j_0}(z) = \alpha_{j_0} + \beta_{j_0} z$ and such that $\beta_{j_0} \neq 0$.

We introduce the following differential operators

$$H_{j_0} = s_{j_0} \frac{\partial}{\partial z}, \quad M_{j_0} := \rho(X_{j_0}) - H_{j_0} = b_{j_0}(z)I, \quad (29)$$

$$P_{j_0} := M_{j_0} \circ H_{j_0} = b_{j_0}(z)s_{j_0} \frac{\partial}{\partial z}, \quad (30)$$

$$Q_{j_0} := \overline{M_{j_0}} \circ H_{j_0} = (s_{j_0} \overline{\alpha_{j_0}} + s_{j_0} \overline{\beta_{j_0}} \overline{z}) \frac{\partial}{\partial \overline{z}}. \quad (31)$$

Canonical Induction

Theorem: S. J 26

The real second-order differential operator

$$\Delta^{OU} = 2 \frac{\partial^2}{\partial z \partial \bar{z}} + \frac{\partial V(z, \bar{z})}{\partial z} \frac{\partial}{\partial \bar{z}} + \frac{\partial V(z, \bar{z})}{\partial \bar{z}} \frac{\partial}{\partial z} \quad (32)$$

is an Ornstein-Uhlenbeck type operator induced by the quintuple $(G, \mathbb{C}, T_g, \mu, \rho)$ if and only if

$$\frac{\alpha_{j_0}}{\beta_{j_0}} \Delta_{\mathbb{C}} V(z, \bar{z}) + \frac{\overline{\alpha_{j_0}}}{\beta_{j_0}} \frac{\partial^2}{\partial z^2} V(z, \bar{z}) = 0. \quad (33)$$

Canonical Induction

In this case, Δ^{OU} has the decomposition

$$\Delta^{OU} = \frac{2}{|s_{j_0}|^2} H_{j_0} \overline{H_{j_0}} + 2\operatorname{Re}(c_{j_0}^{(1)} P_{j_0} + c_{j_0}^{(2)} Q_{j_0}), \quad (34)$$

where, $c_{j_0}^{(1)} = \frac{\Delta_C V(z, \bar{z})}{\beta_{j_0} s_{j_0}}$, $c_{j_0}^{(2)} = \frac{\frac{\partial^2 V(z, \bar{z})}{\partial z^2}}{s_{j_0} \beta_{j_0}}$ and $\operatorname{Re}(P) = \frac{1}{2}(P + \bar{P})$.

The Heisenberg group $H_3 = \mathbb{C} \times \mathbb{R}$

The Heisenberg group H_3 is the set $\mathbb{C} \times \mathbb{R}$ equipped with the multiplication

$$\{a_1, b_1\} \star \{a_2, b_2\} = \{a_1 + a_2, b_1 + b_2 + \operatorname{Im}(\overline{a_1} a_2)\},$$

where $a_1, a_2 \in \mathbb{C}$; $b_1, b_2 \in \mathbb{R}$ and $\operatorname{Im}(z)$ denotes the imaginary part of the complex number z . The group H_3 can be realized as a matrix group

$$H_3 \simeq \left\{ \begin{pmatrix} 1 + \frac{1}{2}|a|^2 + ib & \bar{a} & -\frac{1}{2}|a|^2 - ib \\ a & 1 & -a \\ \frac{1}{2}|a|^2 + ib & \bar{a} & 1 - \frac{1}{2}|a|^2 - ib \end{pmatrix} ; a \in \mathbb{C}, b \in \mathbb{R} \right\},$$

A basis of the Lie algebra $\mathfrak{h}_3$ of H_3 is given by

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & -i & 0 \end{pmatrix}, X_3 = \begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{pmatrix}, \quad (35)$$

with the Lie bracket $[X_1, X_2] = X_1X_2 - X_2X_1 = 2X_3$.

In addition, the following matrices

$$g_t^{(1)} = e^{tX_1} = \begin{pmatrix} 1 + \frac{t^2}{2} & t & -\frac{t^2}{2} \\ t & 1 & -t \\ \frac{t^2}{2} & t & 1 - \frac{t^2}{2} \end{pmatrix},$$

$$g_t^{(2)} = e^{tX_2} = \begin{pmatrix} 1 + \frac{t^2}{2} & -it & -\frac{t^2}{2} \\ it & 1 & -it \\ \frac{t^2}{2} & -it & 1 - \frac{t^2}{2} \end{pmatrix},$$

$$g_t^{(3)} = e^{tX_3} = \begin{pmatrix} 1 + it & 0 & -it \\ 0 & 1 & 0 \\ it & 0 & -it \end{pmatrix}, \quad t \geq 0, \quad (36)$$

constitute smooth curves (one parameter subgroups) on H_3 .

Bargmann Representations of the Heisenberg group

The following result can be found in the Monograph of J. Faraut and K. Harzallah "Deux cours d'analyse harmonique Ecole d'Ete Tunis 1984."

Bargmann Representations of the Heisenberg group

Proposition

Let σ be a non zero real number. The Heisenberg group H_3 admits two inequivalent classes of holomorphic unitary representations on the Segal-Bargmann space

$\mathcal{HL}^2(\mathbb{C}, dm_\sigma(z) := \frac{|\sigma|}{\pi} e^{-|\sigma|z\bar{z}} d\lambda(z))$ given by

- 1 For $\sigma > 0$, $g = (a, b) \in H_3$

$$(T_g^\sigma F)(z) = e^{\sigma(ib - \frac{1}{2}|a|^2 - \bar{a}z)} F(z + a), \quad (37)$$

- 2 For $\sigma < 0$, $g = (a, b) \in H_3$

$$(N_g^\sigma F)(z) := (T_{(\bar{a}, -b)}^{-\sigma} F)(z) = e^{\sigma(ib + \frac{1}{2}|a|^2 + az)} F(z + \bar{a}). \quad (38)$$

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1 For $\sigma > 0$, $g = (a, b) \in H_3$

$$(T_g^{\sigma+} F)(z) = e^{\sigma(ib - \frac{1}{2}|a|^2 - \bar{a}z)} F(z + a), \quad (37)$$

2 For $\sigma < 0$, $g = (a, b) \in H_3$

$$(N_g^{\sigma-} F)(z) := (T_{(\bar{a}, -b)}^{-\sigma} F)(z) = e^{\sigma(ib + \frac{1}{2}|a|^2 + az)} F(z + \bar{a}). \quad (38)$$

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Infinitesimal representation on the Lie algebra $\mathfrak{h}_3$

We denote by $\rho^{\sigma+}$ and $\rho^{\sigma-}$ respectively the infinitesimal representations associated to the holomorphic representations $T_g^{\sigma+}$ in (37) and $N_g^{\sigma-}$ in (38). On the basis (X_1, X_2, X_3) , we have For $\sigma > 0$,

$$\rho^{\sigma+}(X_1) = \frac{\partial}{\partial z} - \sigma z l, \quad (39)$$

$$\rho^{\sigma+}(X_2) = i \frac{\partial}{\partial z} + i \sigma z l, \quad (40)$$

$$\rho^{\sigma+}(X_3) = i \sigma l. \quad (41)$$

and for $\sigma < 0$,

$$\rho^{\sigma-}(X_1) = \frac{\partial}{\partial Z} + \sigma Z l, \quad (42)$$

$$\rho^{\sigma-}(X_2) = -i \frac{\partial}{\partial Z} + i \sigma Z l, \quad (43)$$

$$\rho^{\sigma-}(X_3) = i \sigma l. \quad (44)$$

Induction of Ornstein-Uhlenbeck operators

Denote by

$$\Delta_{j,k} = (\rho(X_j) + \overline{\rho(X_j)}) \circ \overline{H(X_k)}. \quad (45)$$

Theorem

Let σ be a non zero real number. The geometric structures $(H_3, \mathbb{C}, T_g^\sigma, \rho^\sigma, dm_\sigma)$ and $(H_3, \mathbb{C}, N_g^\sigma, \rho^\sigma, dm_\sigma)$ induce the complex Ornstein-Uhlenbeck operator

$$\Delta^{OU,\sigma} = \frac{1}{2}\Delta_{1,1} + \frac{1}{2}\Delta_{2,2}, \quad (46)$$

$$= \frac{\partial^2}{\partial z \partial \bar{z}} - |\sigma|^2 \bar{z} \frac{\partial}{\partial \bar{z}}. \quad (47)$$

The Twisted Fock spaces

Let $\sigma \in \mathbb{R} \setminus \{0\}$, T_g^σ be the representation of H_3 on the Segal-Bargmann space $\mathcal{HL}^2(\mathbb{C}, \frac{|\sigma|}{\pi} e^{-|\sigma|z\bar{z}} d\lambda(z))$. For $\sigma_1 > 0, \sigma_2 > 0$, we consider the probability measure

$$d\mu_{\sigma_1, \sigma_2}(z) = \frac{\sqrt{\sigma_1 \sigma_2}}{\pi} e^{-(\sigma_1 x^2 + \sigma_2 y^2)} d\lambda(z), \quad z = x + iy.$$

We set $|\sigma| = \frac{\sigma_1 + \sigma_2}{2}$, $\sigma_1 > 0$, $\sigma_2 > 0$. The multiplication operator

$$M_{\sigma_1, \sigma_2} : \mathcal{HL}^2(\mathbb{C}, \frac{|\sigma|}{\pi} e^{-|\sigma|z\bar{z}} d\lambda(z)) \longrightarrow \mathcal{HL}^2(\mathbb{C}, d\mu_{\sigma_1, \sigma_2}(z))$$

defined by

$$M_{\sigma_1, \sigma_2} F := \frac{\sqrt{\sigma_1 + \sigma_2}}{\sqrt{2}(\sigma_1 \sigma_2)^{\frac{1}{4}}} e^{\frac{\sigma_1 - \sigma_2}{4} z^2} F$$

yields a unitary isomorphism. Furthermore, M_{σ_1, σ_2} intertwines the representations $T_g^{\sigma+}$ in (37) and $N_g^{\sigma-}$ in (38) respectively to unitary multiplier representations $E_g^{\sigma_1, \sigma_2}$ and $\Gamma_g^{\sigma_1, \sigma_2}$ on $\mathcal{HL}^2(\mathbb{C}, d\mu_{\sigma_1, \sigma_2}(z))$ by the relations

$$E_g^{\sigma_1, \sigma_2} = M_{\sigma_1, \sigma_2} T_g^{\sigma+} M_{\sigma_1, \sigma_2}^{-1} \text{ and } \Gamma_g^{\sigma_1, \sigma_2} = M_{\sigma_1, \sigma_2} N_g^{\sigma-} M_{\sigma_1, \sigma_2}^{-1}. \quad (48)$$

Explicitly, we get

representations of the Heisenberg group on Twisted Fock spaces

Proposition

Let σ be a non zero real number such that $|\sigma| = \frac{\sigma_1 + \sigma_2}{2}$, $\sigma_1 > 0, \sigma_2 > 0$. We put $\bar{\sigma} = \frac{\sigma_1 - \sigma_2}{2}$. Unitary representations of H_3 on the Twisted Fock space $\mathcal{HL}^2(\mathbb{C}, d\mu_{\sigma_1, \sigma_2}(z))$ are given by

① For $\sigma > 0$, $g = (a, b) \in H_3$, $F \in \mathcal{HL}^2(\mathbb{C}, d\mu_{\sigma_1, \sigma_2}(z))$

$$(E_g^{\sigma_1, \sigma_2} F)(z) = e^{\sigma(ib - \frac{1}{2}|a|^2) - \frac{\bar{\sigma}}{2}a^2 - (\sigma\bar{a} + \bar{\sigma}a)z} F(z + a). \quad (49)$$

② For $\sigma < 0$, $g = (a, b) \in H_3$ and $F \in \mathcal{HL}^2(\mathbb{C}, d\mu_{\sigma_1, \sigma_2}(z))$

$$(\Gamma_g^{\sigma_1, \sigma_2} F)(z) = e^{\sigma(ib + \frac{1}{2}|a|^2) - \frac{1}{2}\bar{\sigma}a^2 + (a\sigma - \bar{\sigma}\bar{a})z} F(z + \bar{a}). \quad (50)$$

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② For $\sigma < 0$, $g = (a, b) \in H_3$ and $F \in \mathcal{HL}^2(\mathbb{C}, d\mu_{\sigma_1, \sigma_2}(z))$

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Proposition

The infinitesimal representations associated to $E_g^{\sigma_1, \sigma_2}$ and $\Gamma_g^{\sigma_1, \sigma_2}$

① For $\sigma > 0$,

$$\rho^\sigma(X_1) = \frac{\partial}{\partial z} - \sigma_1 z l, \quad (51)$$

$$\rho^\sigma(X_2) = i \frac{\partial}{\partial z} + i \sigma_2 z l. \quad (52)$$

$$\rho^\sigma(X_3) = i \sigma l. \quad (53)$$

② For $\sigma < 0$,

$$\rho^\sigma(X_1) = \frac{\partial}{\partial z} - \sigma_1 z l, \quad (54)$$

$$\rho^\sigma(X_2) = -i \frac{\partial}{\partial z} - i \sigma_2 z l. \quad (55)$$

$$\rho^\sigma(X_3) = i \sigma l. \quad (56)$$

Perturbation of the complex Ornstein-Uhlenbeck operators

Theorem

Let σ be a non zero real number such that $|\sigma| = \frac{\sigma_1 + \sigma_2}{2}$, $\sigma_1 > 0, \sigma_2 > 0$. We put $\bar{\sigma} = \frac{\sigma_1 - \sigma_2}{2}$. The complex Ornstein-Uhlenbeck operators

$$\Delta^{OU, \sigma} = \frac{1}{2} \Delta_{1,1} + \frac{1}{2} \Delta_{2,2}, \quad (57)$$

$$= \frac{\partial^2}{\partial z \partial \bar{z}} - (\bar{\sigma} z + |\sigma| \bar{z}) \frac{\partial}{\partial \bar{z}} \quad (58)$$

is induced simultaneously by the geometric structures $(H_3, \mathbb{C}, E_g^{\sigma_1, \sigma_2}, \rho, d\mu_{\sigma_1, \sigma_2}(z))$ and $(H_3, \mathbb{C}, \Gamma_g^{\sigma_1, \sigma_2}, \rho, d\mu_{\sigma_1, \sigma_2}(z))$.

From L^2 -representations to unitary Holo representations of the Engel group G_4

Let G_4 denotes $\mathbb{R}^4$ with the multiplication law

$$\{a_1, b_1, c_1, d_1\} \star \{a_2, b_2, c_2, d_2\}$$

$$= \{a_1 + a_2, b_1 + b_2, c_1 + c_2 + a_1 b_2, d_1 + d_2 + a_1(c_2 + \frac{a_1 b_2}{2})\}$$

The group G_4 is identified to a closed subgroup of $GL_4(\mathbb{R})$.

$$G_4 \simeq \left\{ \begin{pmatrix} 1 & a & \frac{a^2}{2} & d \\ 0 & 1 & a & c \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix} ; a, b, c, d \in \mathbb{R} \right\}.$$

A basis of the Lie algebra $\mathfrak{G}_4$ of G_4 is given by

$$X_1 = E_{1,2} + E_{2,3}, X_2 = E_{3,4}, X_3 = E_{2,4}, X_4 = E_{1,4} \quad (59)$$

Consider the unitary irreducible representation of G_4 on $L^2(\mathbb{R}, du)$ given by

$$(W_g^\tau f)(u) = e^{2i\pi\tau(c + \frac{b}{2}u)} f(u + a), \quad (60)$$

for some $\tau \in \mathbb{R}$.

Let $j \in \{1; 2\}$ and $g_j = (a_j, b_j, c_j, d_j) \in G_4$. The representation $W_g^{(\tau)}$ satisfies the relation

$$W_{g_1}^{(\tau)} \circ W_{g_2}^{(\tau)} = e^{-i\pi\tau a_1 b_2} W_{g_1 \star g_2}^{(\tau)}. \quad (61)$$

M. Gordina and J. Haga: Lévy processes on a 3 step nilpotent Lie group; Potential Analysis, 2012.

The Segal-Bargmann transform

Proposition

For any $f \in L^2(\mathbb{R}, du)$, let $B_1 f$ be the function on $\mathbb{C}$ given by

$$B_1(f)(z) = \pi^{-\frac{1}{4}} e^{-\frac{z^2}{2}} \int_{\mathbb{R}} f(u) e^{\sqrt{2}z \cdot u - \frac{u^2}{2}} du. \quad (62)$$

The operator $f \mapsto B_1(f)$ is a unitary from $L^2(\mathbb{R}, du)$ onto $\mathcal{HL}^2(\mathbb{C}, d\gamma)$. For any $F \in \mathcal{HL}^2(\mathbb{C}^d, d\gamma(z))$, the inverse of the Segal-Bargmann transform is given by

$$(B_1^{-1}F)(u) = \pi^{-\frac{1}{4}} e^{-\frac{u^2}{2}} \int_{\mathbb{C}} F(z) e^{\sqrt{2}u \bar{z} - \frac{z^2}{2}} d\gamma(z). \quad (63)$$

V. Bargmann 1961.

Weyl representations of the Engel group G_4

Let $g = (a, b, c, d) \in G_4$, $\tau \in \mathbb{R}$. We consider the representation W_g^τ in (60) and we put

$$T_g^\tau = B_1 W_g^\tau B_1^{-1}.$$

Theorem

T_g^τ is a unitary representation of G_4 on $\mathcal{F}_1^2(\mathbb{C})$. Explicitly, we have

$$(T_g^\tau F)(z) = e^{2i\pi\tau c - \frac{a^2}{2} + \frac{1}{4}(i\pi\tau b - a)^2 + \frac{1}{\sqrt{2}}(i\pi\tau b - a)z} F\left(z + \frac{1}{\sqrt{2}}(i\pi\tau b + a)\right). \quad (64)$$

Proposition

On the basis (X_1, X_2, X_3, X_4) , we have

$$(\rho(X_1)F)(z) = -\frac{1}{\sqrt{2}}zF(z) + \frac{1}{\sqrt{2}}F'(z), \quad (65)$$

$$(\rho(X_2)F)(z) = \frac{i\pi\tau}{\sqrt{2}}zF(z) + \frac{i\pi\tau}{\sqrt{2}}F'(z), \quad (66)$$

$$(\rho(X_3)F)(z) = 2i\pi\tau F(z), \quad (67)$$

$$(\rho(X_4)F)(z) = 0. \quad (68)$$

Theorem

If the real number $\tau \neq 0$, the second order differential operator

$$\Delta^{OU} = \pi^2 \tau^2 \Delta_{1,1} + \Delta_{2,2}, \quad (69)$$

$$= \frac{\partial^2}{\partial z \partial \bar{z}} - \bar{z} \frac{\partial}{\partial \bar{z}} \quad (70)$$

is a complex Ornstein-Uhlenbeck type operator induced by the geometric structure $(G_4, \mathbb{C}, T_g^\tau, \rho^\tau, d\gamma(z))$.

If $\tau = 0$; the geometric structure $(G_4, \mathbb{C}, T_g^{(0)}, \rho^{(0)}, d\gamma(z))$ induces S. O. D. O of O-U type.

$$\Delta_R^{OU} = 4H(X_1)\overline{H(X_1)} + 2(\overline{\rho(X_1)} - \overline{H(X_1)})\overline{H(X_1)} \quad (71)$$

$$+ 2(\rho(X_1) - H(X_1))H(X_1) \quad (72)$$

$$= 2 \frac{\partial^2}{\partial z \partial \bar{z}} - \bar{z} \frac{\partial}{\partial \bar{z}} - z \frac{\partial}{\partial z} \quad (73)$$

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In Progress

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