Al-Khwarizmi Applied Mathematics Webinar

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Deformations through symmetry: Bridges to Life Sciences and Beyond

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Tunisian Academy of Sciences, Letters and Arts

Tuesday June 02, 2026

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TUNISIA

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Second area of research: Geometry
Emphasis on "Deformation theory of discontinuous groups". Deep connections with Topology, rich framework for rigidity and stability of geometric structures

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intersection of Analysis, Geometry, and Algebra,

and they continue to inspire a wide array of mathematical investigations.

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Plan of the Talk

1. What is Lie Theory?
2. Impact of Lie Theory on Life Sciences
3. Should we move from Groups to Homogeneous Spaces ?
4. Deformation Theory and outcomes
5. Representation Theory and Harmonic Analysis

What is Lie Theory

Lie and Galois

Wikipedia: Link

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He wanted to solve them with a single unified theory using

CONTINUOUS SYMMETRIES!!

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Lie wanted to replicate the success of Galois theory and made the analogy between differential and polynomial equations.

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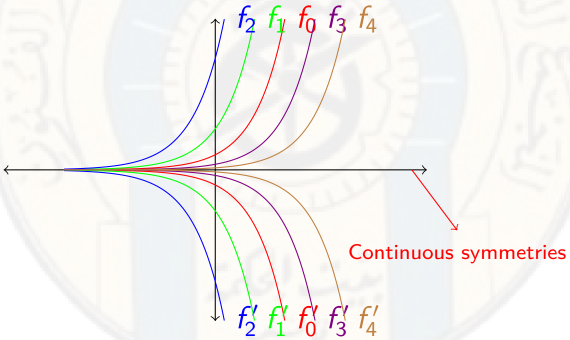

What is Lie Theory?

- As an easier concept, a Group G is a set where we can multiply. The product of two elements $x \cdot y$ is another element of G.
- This product is associative like for real numbers.
- It has an element called e like 1, $e \cdot x = x \cdot e$.
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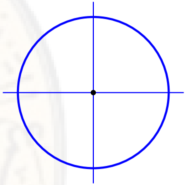

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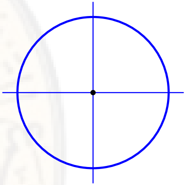

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Groups manifest themselves through symmetries! they most often appear as symmetries of objects, and multiplication is the composition of those symmetries.
A symmetry is any transformation which preserves an object

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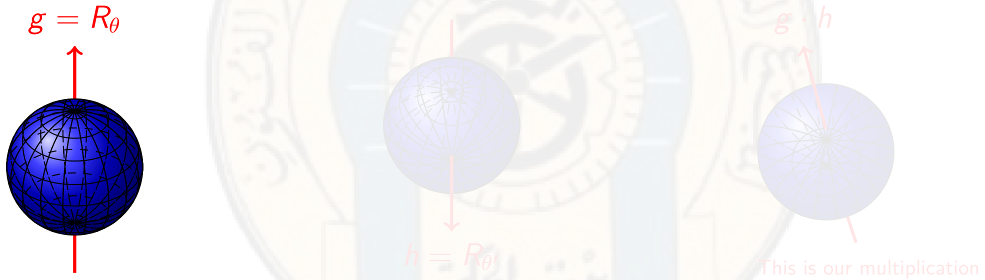

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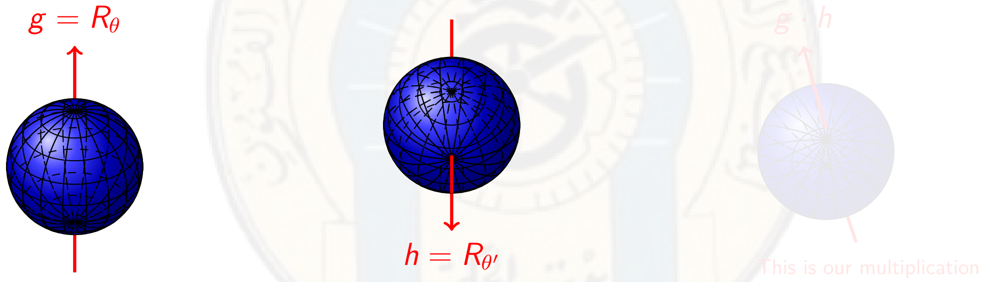

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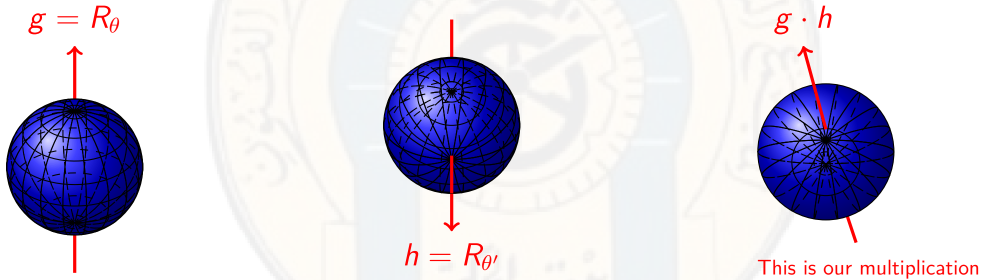

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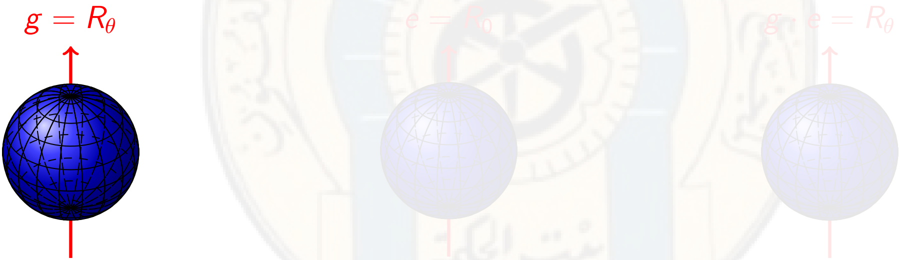

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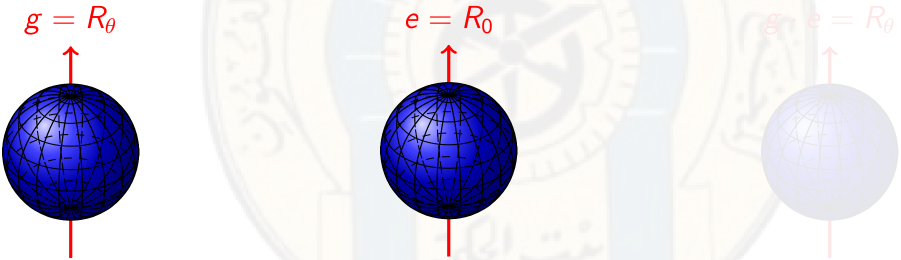

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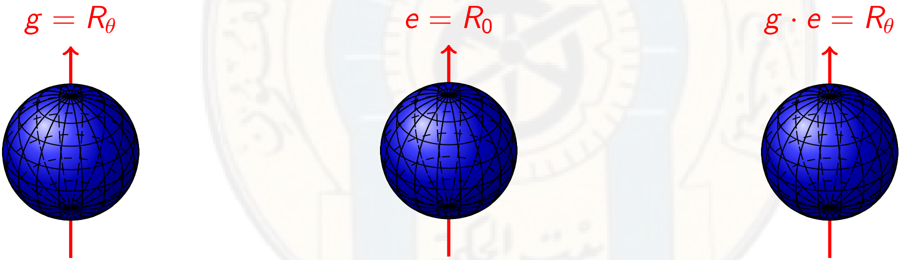
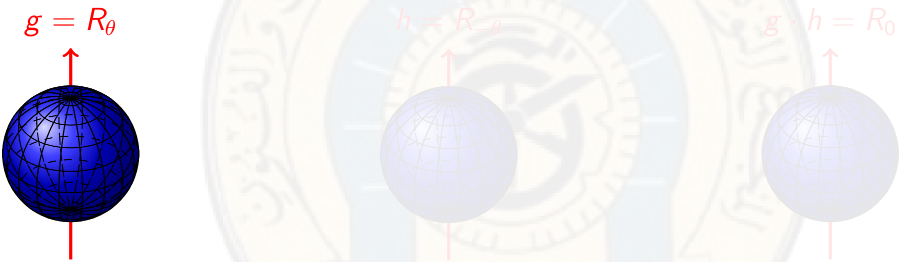


The inverse of a rotation R_θ is a rotation by the angle $-\theta$



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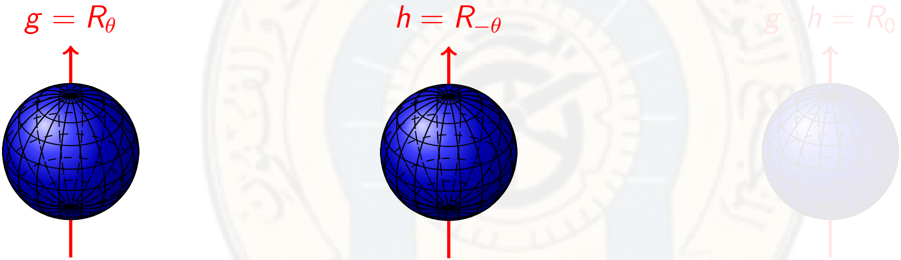

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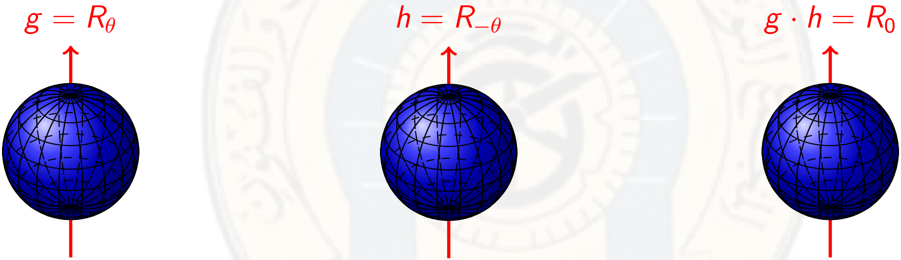

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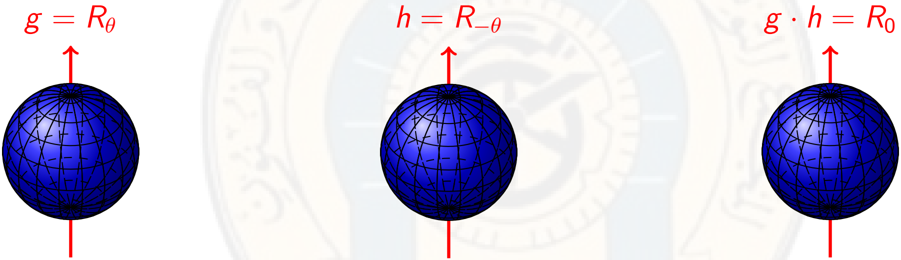

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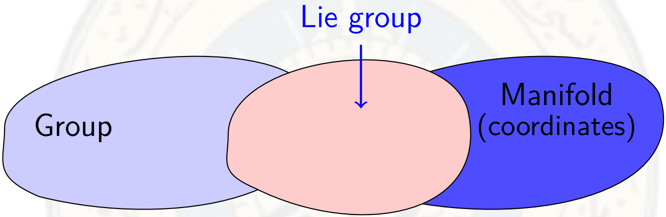

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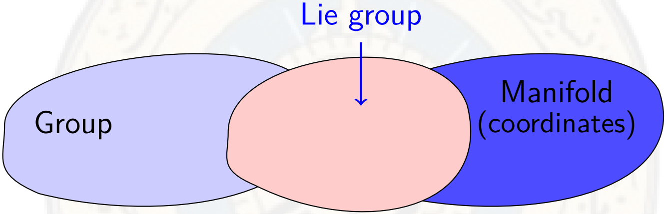

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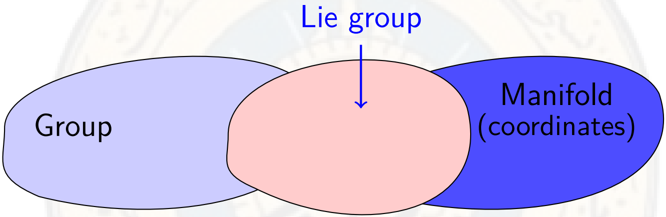

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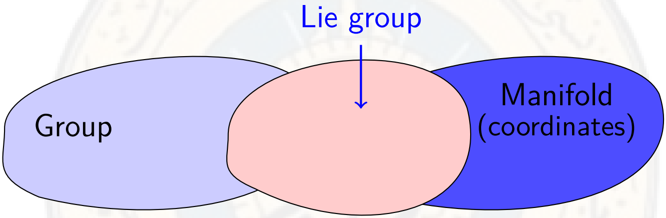

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Lie theory is the mathematical study of symmetry groups and their transformations. It appears wherever such symmetries are found in nature. It provides a powerful framework for analyzing the behavior of these symmetries.

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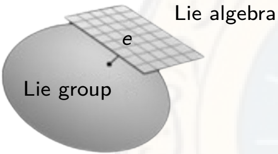

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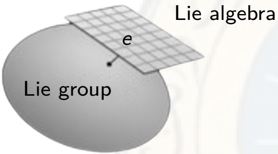

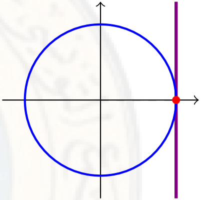

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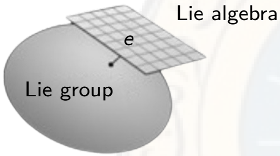

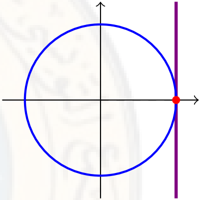

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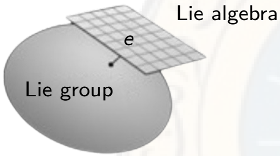

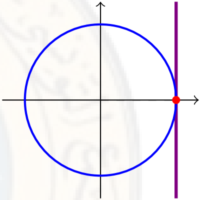

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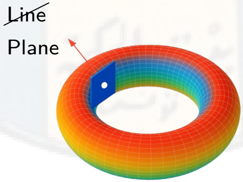

Example of The Surface of The Earth

Lie group Lie algebra

We badly need coordinates

❋ An illustrative example of a manifold is the surface of the Earth, and a natural thing to do is to impose a coordinate system on it. On Earth, these are the latitudes and longitudes, and the reason why this is useful is that the surface of the Earth is curved, but a coordinate system allows you to create a map, which is flat and much easier to manipulate than curved spaces. Lie's ideas were similar.

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Lie Groups in Frequent Uses

❋ The groups of rotations for higher dimensional spheres are the $SO_n(\mathbb{R})$, the rotations in n-dimensional space.

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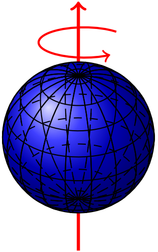


O_n

SO_n

U_n

SU_n

Lie groups

$$SO_n(\mathbb{R}) = \left\{R \in \mathcal{M}_n(\mathbb{R}) \mid R^T R = I,\ \det(R) = 1\right\}$$

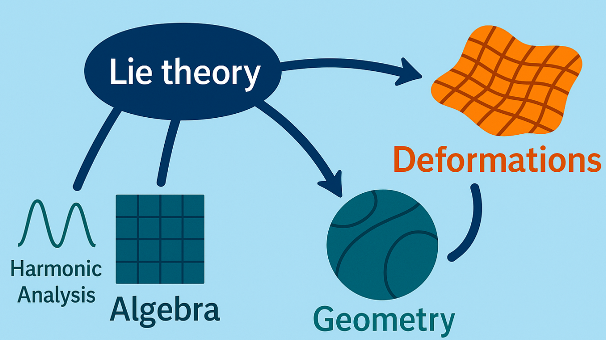
$$SU_n(\mathbb{C}) = \left\{U \in \mathcal{M}_n(\mathbb{C}) \mid \overline{U}^T U = I,\ \det(U) = 1\right\}$$

$$SL_n(\mathbb{C}|\mathbb{R}) = \{U \in \mathcal{M}_n(\mathbb{C}|\mathbb{R}) \mid \det(U) = 1\}$$

Impact of Lie Theory on Natural Sciences

It is Almost Perfectly Well Connected!

* Plenty of applications in more advanced areas in Mathematics.



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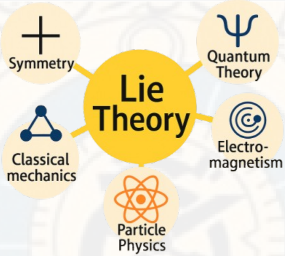
- Algebraic topology (e.g. principal bundles and characteristic classes),
- Algebraic geometry (e.g. algebraic groups and flag varieties),
- Combinatorics (e.g. root systems and Coxeter groups),
- Differential geometry (e.g. connections and Chern-Weil theory),
- Number theory (e.g. automorphic forms and Langlands program),
- Low-dimensional topology (e.g. quantum groups and Chern-Simons theory),
- Riemannian geometry (e.g. holonomy and symmetric spaces),
- Gauge theory.

Lie Theory is Ubiquitos in Physical Sciences



- ❋ U_1: Describes phase symmetry, crucial in electromagnetism.
- ❋ SU_2: Appears in weak interactions and in the theory of angular momentum.
- ❋ SU_3: Describes the strong force (Quantum Chromodynamics).
- ❋ SO_3: Governs rotational symmetries.
- ❋ $SL_2(\mathbb{C})$: Related to the Lorentz group and special relativity.

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M. Nakahara, Geometry, Topology and Physics, CRC Press, 2003.

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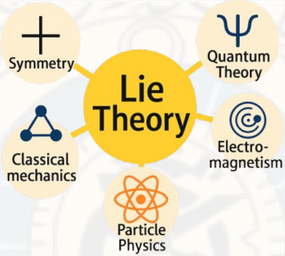

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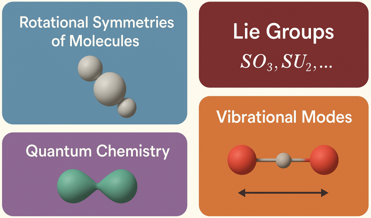
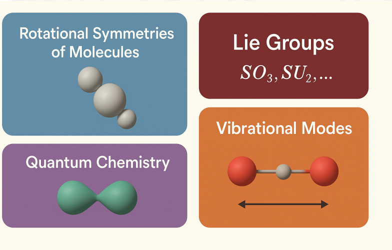
Lie Theory in Chemistry





❋ SO_3: Describes rotational symmetries of molecules, crucial for understanding rotational spectra.

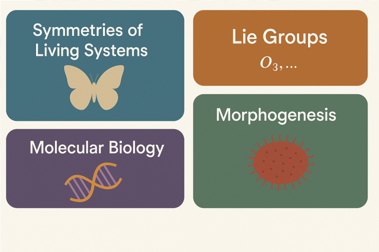
❋ SU_2: Used in describing the spin of electrons, especially in quantum chemistry.

❋ U_1: Phase symmetry group, useful in describing aspects of molecular orbitals.

❋ SO_2: Describes planar rotation symmetries, often applied in the study of planar molecules or vibrational modes.

F. A. Cotton, Chemical Applications of Group Theory, Wiley, 1990.
I. N. Levine, Quantum Chemistry, Pearson, 7th ed., 2013.

In Biology too?



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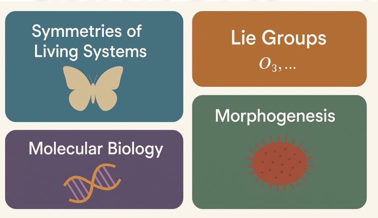


❋ SO_3: Describes rotational symmetries, especially in biomechanics and protein folding.

❋ SU_2: Appears in spin systems and sometimes in biological quantum systems (though this is more speculative).

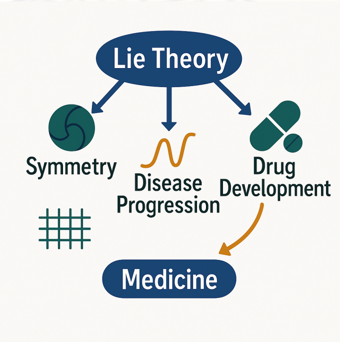
❋ SO_n: Can describe higher-dimensional symmetries, useful in understanding complex biological structures or dynamics in higher dimensions.

❋ Lie Algebras: These describe the local symmetries in biological systems, often applied to understand small deformations or changes in biological structures.

A . Goriely, The Mathematics and Mechanics of Biological Growth, Springer, 2017.

P. M. Wiggins, Introduction to Applied Biomechanics, Oxford University Press, 2009.

As for Medicine...



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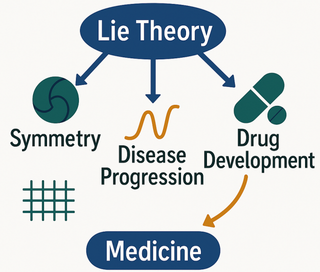
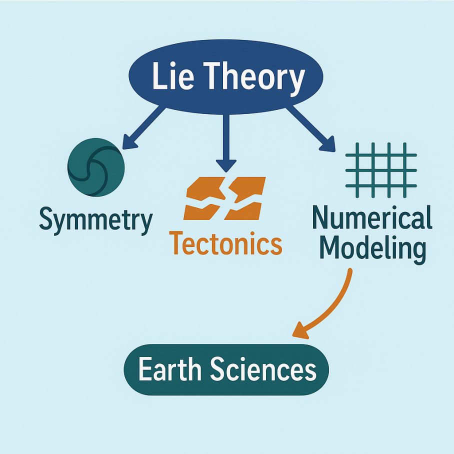


❋ SO_3: Used to describe rotational movements, essential in medical imaging, biomechanics, and surgical robotics.
❋ SE_3: Represents both rotations and translations, used in image registration, robotic surgery, and radiotherapy planning.
❋ Lie Algebras: Provide tools for understanding local symmetries in biological systems, especially useful in deformable image registration and neural modeling.

H. K. Ahuja, Mathematical Techniques in Medical Imaging, Springer, 2005.
A. Yezzi, Mathematical Methods in Medical Image Processing and Analysis, Cambridge University Press, 2013.

More Surprising in Earth Sciences?



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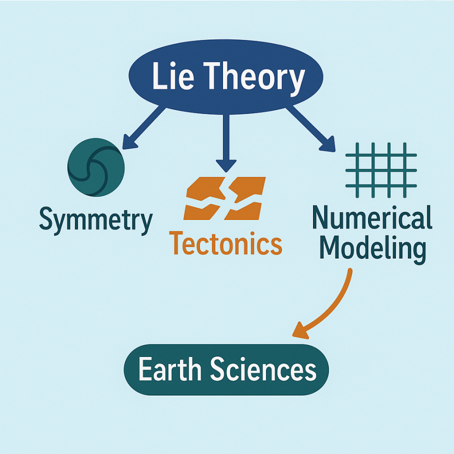

❋ Seismology and Wave Propagation.

❋ Dynamics of Geophysical Fluids.

❋ Crystallography and Mineralogy.

❋ Plate Tectonics and Geomechanics.

❋ Data Assimilation and Computational Geoscience.

❋ Geodesy and Gravimetry.

R. W. Clayton, Seismology: Theory and Practice, Springer, 2004.
G. R. Foulger, Plate Tectonics: An Insider's History of the Modern Theory of the Earth, Westview Press, 2003.

Should we move from Groups to Homogeneous Spaces ?

From Groups to Homogeneous Spaces

$$G \curvearrowright X$$

a group — a space

A group can be understood (and is often best understood) through the way it acts on or transforms other mathematical objects

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* A notable example is the permutation of polynomial's roots, which led Évariste Galois to the profound notion of Galois groups, capturing the symmetries underlying algebraic equations.

From Groups to Homogeneous Spaces

$$G \overset{\text{transitive}}{\curvearrowright} X$$

a group a space

A group can be understood (and is often best understood) through the way it acts on or transforms other mathematical objects

❋ A notable example is the permutation of polynomial's roots, which led Évariste Galois to the profound notion of Galois groups, capturing the symmetries underlying algebraic equations.

❋ If you think of G as a symmetry group, this means that any point of X can be reached from any other point by a symmetry.

❋ A transitive action of a group G defines the geometry of a space X. The space X that defines the geometry is called a homogeneous space.

$$X = G/H$$

where H is a closed subgroup of G defining the model geometry of X.

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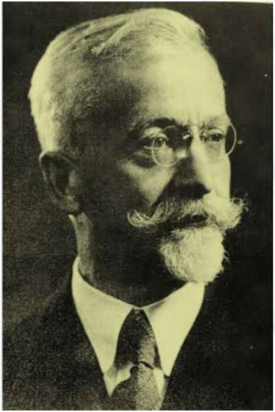
The Euclidean Motion groups

* Élie Cartan (1869-1951) was elected as a Foreign member of the Royal Society in 1947. He published between the two wars the following nice paragraph:

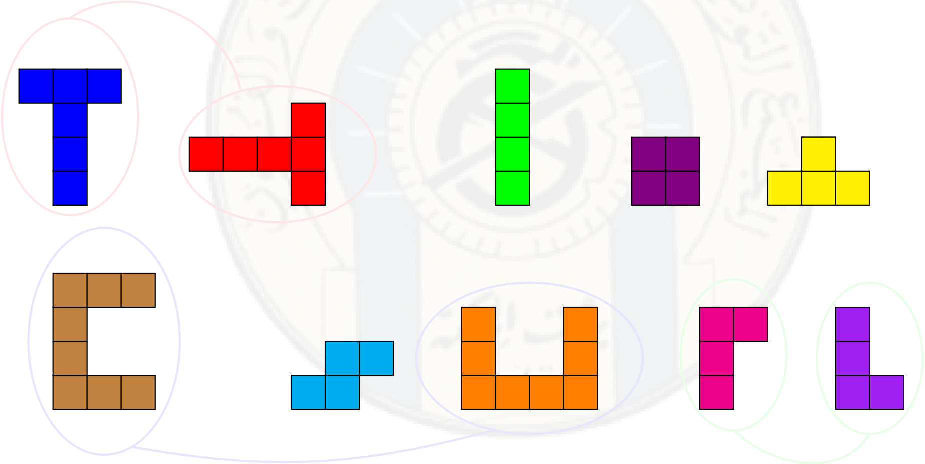


Bibm@th.net: Link

"Elementary geometry is essentially the theory of invariants of a certain group, the group of Euclidean motions: its purpose is indeed to study the properties of figures that are preserved by an arbitrary motion; stating that all motions form a group which expresses in precise language the axiom according to which two figures equal to a third are equal to each other".

Euclidean Motion groups

Two objects are said to be identical in form, if we can map each one onto the other by a motion:



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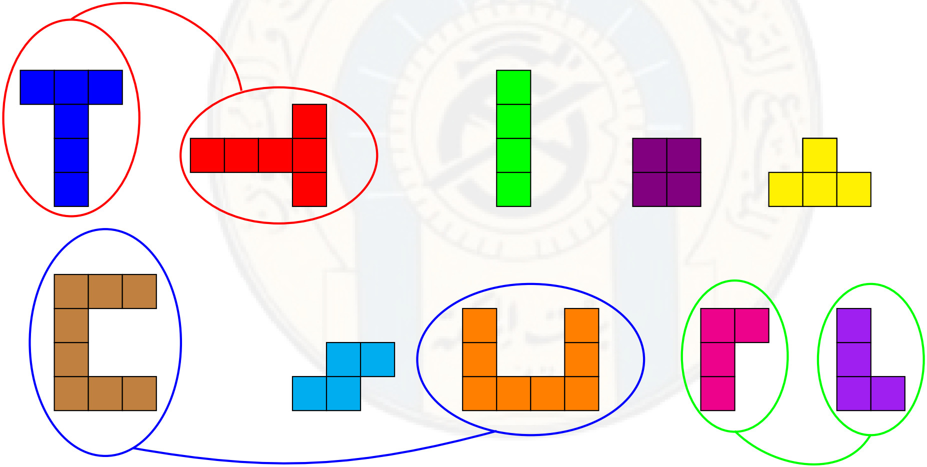
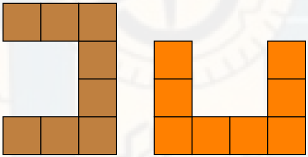


* Which two figures are identical in form?
  - parallel translations
  -
  -



Why Homogeneous Spaces? Groups are not Sufficient?

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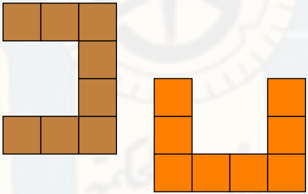

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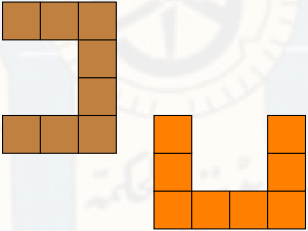

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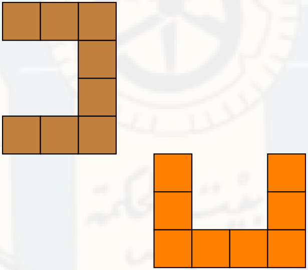

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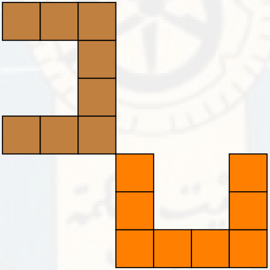

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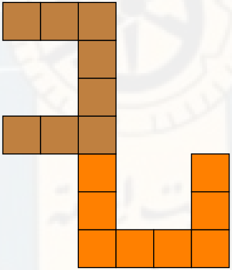

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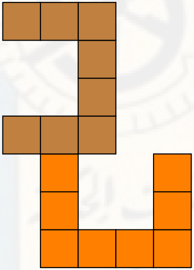

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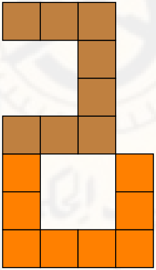

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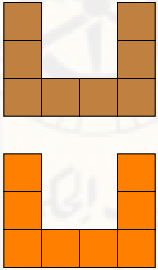

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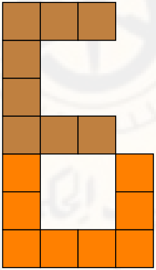

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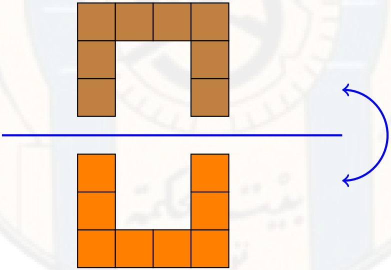

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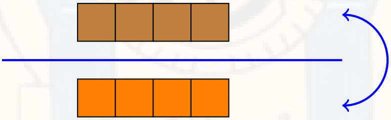

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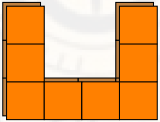
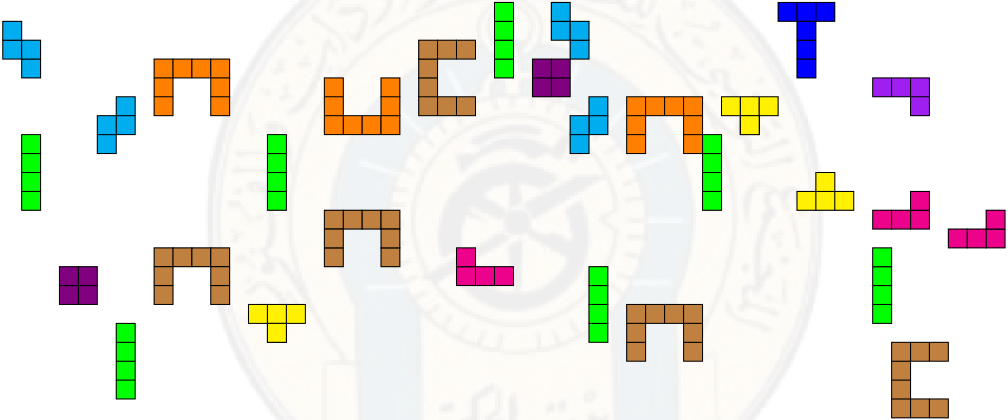
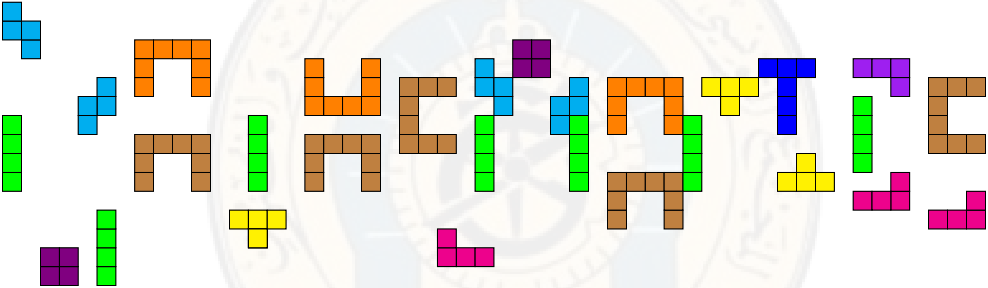
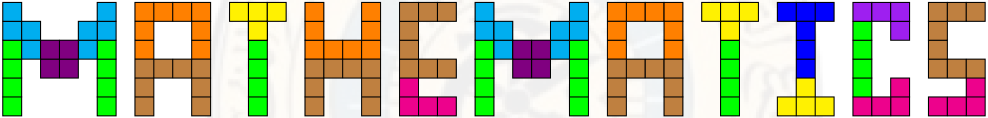







MATHEMATICS

$$E(2) = O(2) \ltimes \mathbb{R}^2$$

- The Euclidean motion group $E(2)$ acts transitively on the Euclidean space $\mathbb{E}^2$. It is the group of all transformations preserving its geometrical structure.

- So, $\mathbb{E}^2$ results a geometry! It is defined by the symmetry determined by the action of $E(2)$.

Homogeneous spaces lie at the heart of my research, just as they did nearly two centuries ago for Felix Klein, when he formulated his modern and unifying definition of geometry

The Role of Homogeneity: Felix Klein's Program

❋ The revolutionary Erlangen Program, proposed by (Felix Klein 1849 - 1925) in 1872, classifies geometries through the lens of transformation groups.

❋ A homogeneous space for a group is equivalent to the notion of geometry.

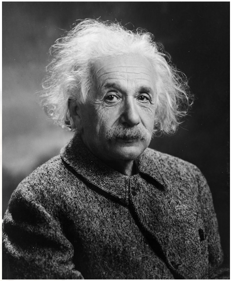
❋ It is a first time in history we have a very precise and deep understanding of what a geometry is.

Wikipedia Link

❋ This led to the classification of geometries by their symmetry groups:

- Euclidean geometry: Action of the Euclidean group
- Affine geometry: Action of the affine group (linear transformations + translations).
- Projective geometry: Action of the projective group (projective transformations).
- Riemannian geometry: Action of local isometries.

❋ The General Relativity program (1915) proposed by Albert Einstein (1879-1955) gave a physical realization to the spirit of the Erlangen Program by asserting that the laws of nature are invariant under general coordinate transformations. He modeled the spacetime as a pseudo-Riemannian manifold whose curvature reflects the distribution of mass and energy.



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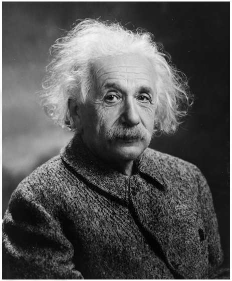

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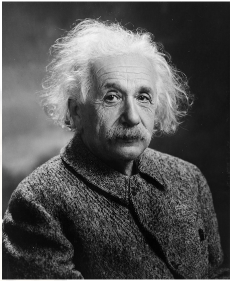


Wikipedia Link

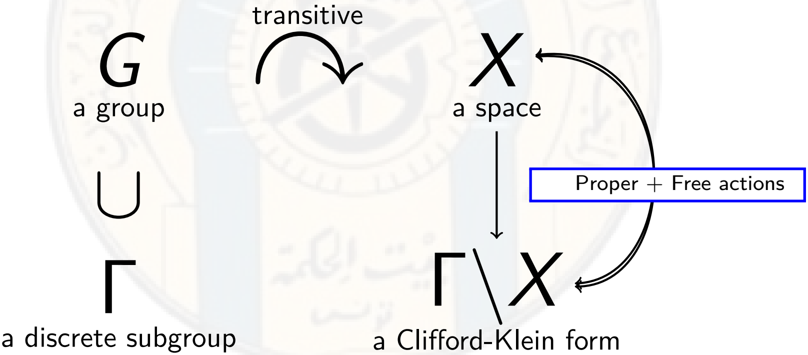
❋ Élie Cartan adds: Nature is not absolutely homogeneous !

❋ The insight of Charles Ehresmann (1905-1979) reflects the realistic complexity of geometric and physical systems: "No absolute homogeneity, but rather local homogeneity"

Wikipedia Link

Clifford-Klein forms, T. Kobayashi 1993

The local geometry of a space X is determined by a triplet: a Lie group G that acts locally, continuously, and transitively on X, a closed subgroup H of G defining the model geometry of X, and a discontinuous group Γ for the homogeneous space G/H that defines the Clifford-Klein form $\Gamma\backslash X = \Gamma\backslash G/H$.



$$E(n) \overset{\text{transitive}}{\curvearrowright} \mathbb{E}^n$$

the Euclidean group — the Euclidean space

The action of $E(n)$ defines the geometry of $\mathbb{E}^n$.

$E(n)$ the Euclidean group — transitive $\curvearrowright$ — $\mathbb{E}^n$ the Euclidean space

$\bigcup$ $\qquad\qquad\qquad\qquad$ $\downarrow$

$\mathbb{Z}^n$ a discrete subgroup $\qquad\qquad$ $\mathbb{T}^n$ a flat torus

$\mathbb{T}^n$ shares with $\mathbb{E}^n$ a local property "flatness".

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the same locally different globally

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Deformation Theory and Outcomes

Contributions...

Building on the foundational notions of groups, manifolds, geometry, and the profound interplay between them, my contributions focus on advancing the deformation theory of Clifford–Klein forms within the setting of solvable Lie groups and their compact extensions. This framework offers a concrete approach to tackling the local rigidity conjecture.

Erlangen's Program:

Geometry= Space + Group transitive action

Deformation= Geometry + Discontinuous Group Action

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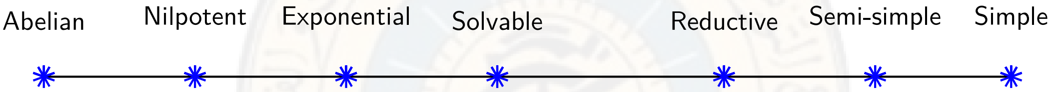
Our Framework

Lie groups



Our Framework

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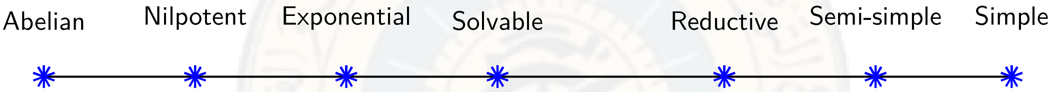

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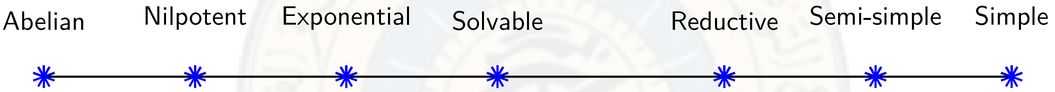

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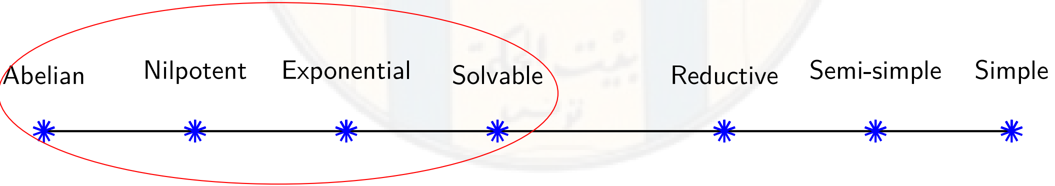

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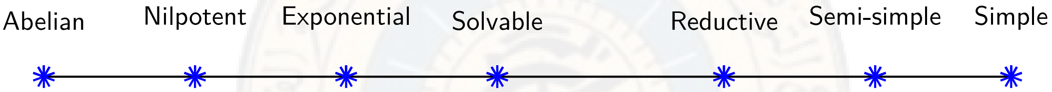

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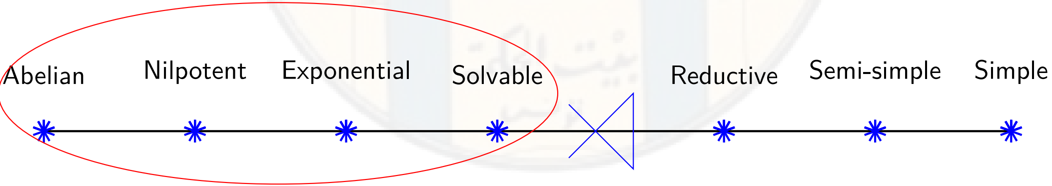

The Local Rigidity Conjecture

Local Rigidity Conjecture (A. Baklouti, Proc. Japanese Academy, 2011)

Let G be a 1-connected nilpotent Lie group and let Γ be a non-trivial discontinuous group of a homogeneous space X. Then any Clifford-Klein form $\Gamma\backslash X$ deforms continuously.

The Local Rigidity Conjecture

❋ In other words, there is an infinite continuous way to deform the discrete subgroup Γ, with the assumption that the geometric structures of the deformed Clifford-Klein forms $\Gamma\backslash X$ are preserved.

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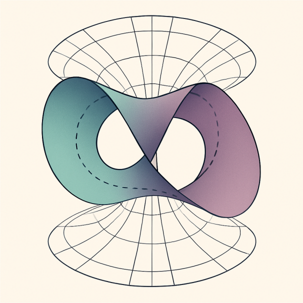

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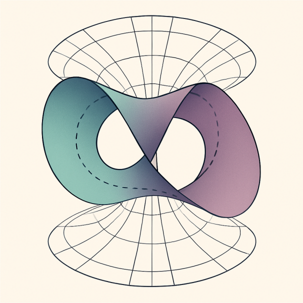

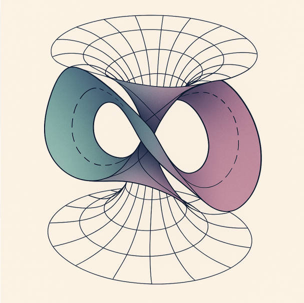

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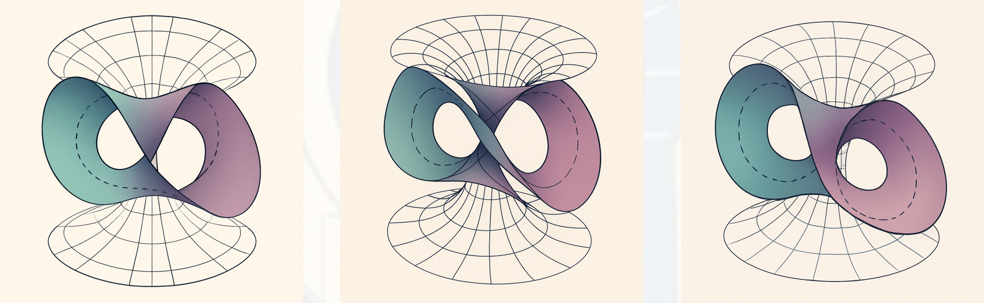

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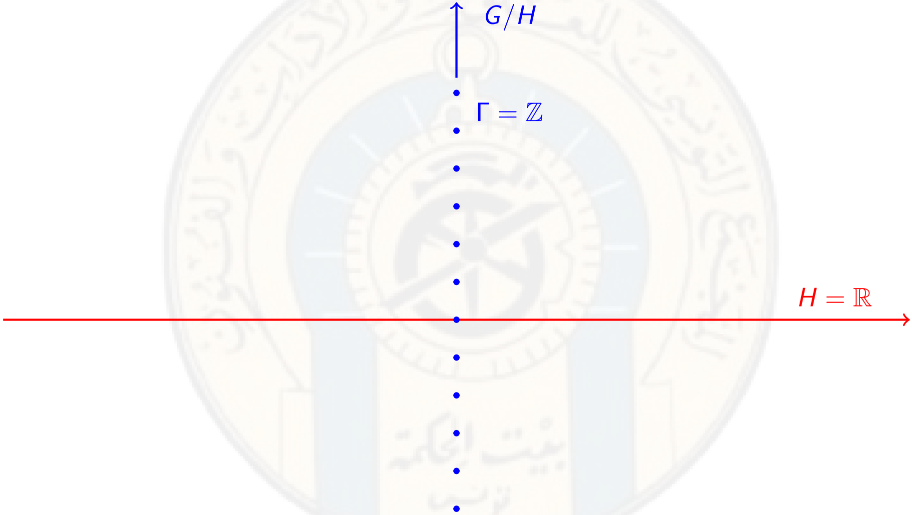
An example: $G \simeq \mathbb{R}^2$, the affine group of the real line



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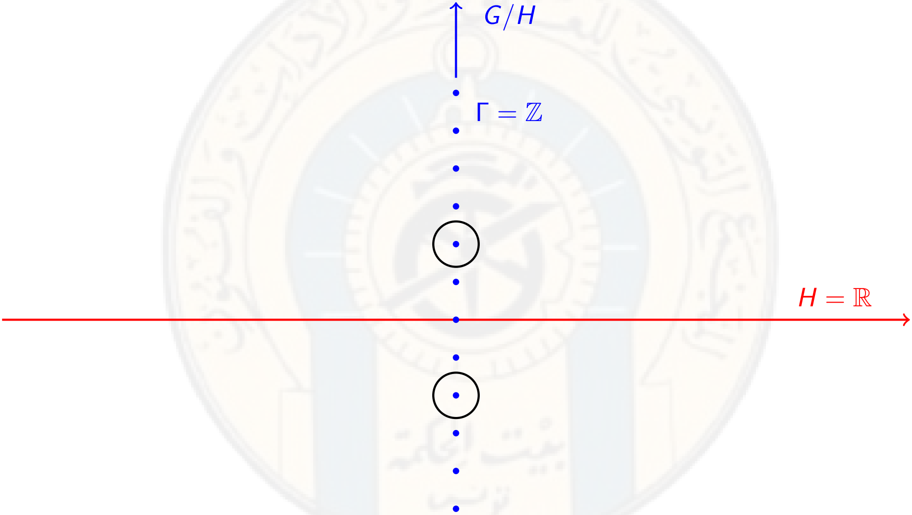
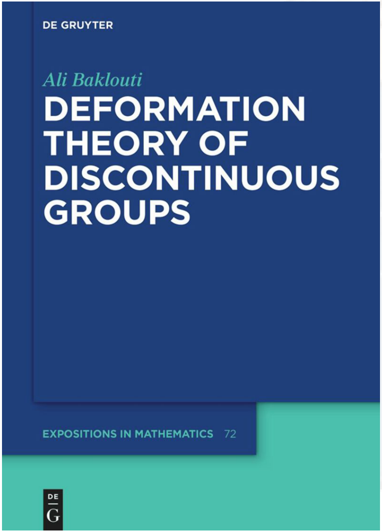


Only two deformations of the Torus

2022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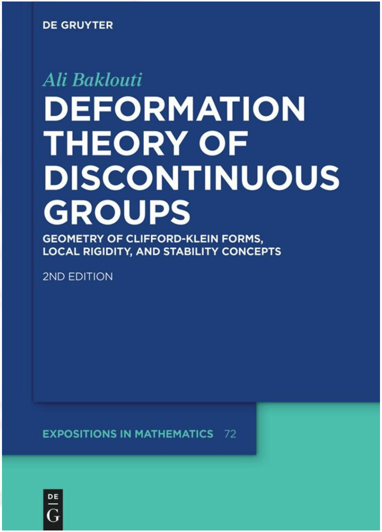
Ali Baklouti
DEFORMATION THEORY OF DISCONTINUOUS GROUPS
EXPOSITIONS IN MATHEMATICS 72

2025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GEOMETRY OF CLIFFORD-KLEIN FORMS, LOCAL RIGIDITY, AND STABILITY CONCEPTS
2ND EDITION
EXPOSITIONS IN MATHEMATICS 72

❋ Structure theory of solvable Lie groups, and their closed and discrete subgroups.

❋ Various geometric and topological concepts related to the deformation and moduli spaces of discontinuous group actions.

❋ The newest approaches and methods in deformation theory in the setting of compact extensions of solvable Lie groups.

❋ The local rigidity Conjecture in the setting of compact extensions of nilpotent Lie groups.

❋ Stability, Hausdorffness, and the Calabi-Markus phenomenon.

Some Applications of Deformation Theory in Life Sciences

❋ Physics (General Relativity & Liquid Crystals): Discrete Lie group deformations help model space-time singularities (e.g., black holes) and symmetry changes during phase transitions in liquid crystals. [Ref: Hawking & Ellis, 1973; de Gennes & Prost, 1993]

❋ Chemistry (Crystallography & Molecular Reactions): Symmetries of crystal lattices under stress (temperature, pressure) and molecular structure changes during chemical reactions are modeled with discrete groups. [Ref: Hahn, 2002 (International Tables for Crystallography)]

❋ Biology (Proteins & Biomolecular Structures): Folding, mutation, and molecular interactions are analyzed via symmetry deformations in 3D protein structures. [Ref: Goodsell & Olson, 2000]

❋ Medicine (Medical Imaging & Orthopedics): Deformation analysis supports anomaly detection in MRI/CT images and structural evaluation in musculoskeletal systems. [Ref: Maintz & Viergever, 1998; Viceconti et al., 2005]

Representation Theory and Harmonic Analysis

❋ The representation theory is a way to understand abstract mathematical objects by expressing them as transformations of vectors-basically using matrices, so that we can work with them more easily.

❋ More formally, a unitary representation of a group G is a pair $(\pi, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, and $\pi : G \to \mathcal{U}(\mathcal{H})$ is a group homomorphism into the group of unitary operators on $\mathcal{H}$.

For instance, the non-commutative analogue of the classical Fourier transform

$$f \in L^1(\mathbb{R}) : \ \widehat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-it\omega} dt; \ f \in L^1(G) \curvearrowright \widehat{f}(\pi) = \int_G f(g)\pi(g) dg.$$

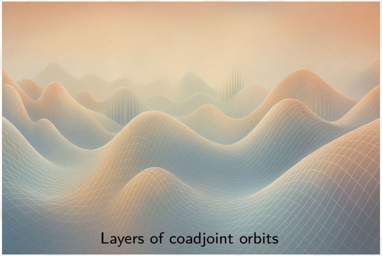
The non-commutative harmonic analysis includes the Plancherel formula for non-abelian groups, the non-commutative versions of uncertainty principles that govern the trade-off between localization in group and representation space, especially in the context of nilpotent or solvable Lie groups, where classical methods are insufficient.

Kirillov's Orbit Method for Exponential Lie Groups

* In Kirillov's orbit method, a unitary irreducible representation of an exponential solvable Lie group corresponds to a coadjoint orbit in the dual of the Lie algebra.

coadjoint representation

$$G \curvearrowright \mathfrak{g}^*$$

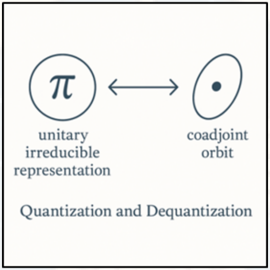


Layers of coadjoint orbits

* Coadjoint orbits (smooth manifolds of *odd* dimension $\Leftrightarrow$ symplectic forms).

Dequantization

⁎ The dequantization refers to the inverse process: recovering the orbit from its associated representation.



- Theory of Front Waves: (Roger Howe, late 1980's).
- Dequantization through primitive ideals: (Pedersen 1984, the setting of nilpotent Lie Groups).
- Theory of Moment Maps: (N. Wildberger, 1989).

The Zariski Closure Conjecture

❋ Dequantization by means of Poisson Characteristics Varieties (Bak-Dhieb-Manchon, 2005).

❋ For any unitary and irreducible representation π, we defined the Poisson characteristic variety $VA(\pi)$ as a variety of $\mathfrak{g}^*$, and we substantiated the following:

Conjecture (Zariski Closure Conjecture, 2005)

For an exponential solvable Lie group, $VA(\pi)$ coincides with the Zariski closure of the associated orbit of π in $\mathfrak{g}^*$, hence to the orbit itself in the setting where G is nilpotent.

* Two fundamental classes of representations:

- Induced representation $\tau = \mathrm{Ind}_H^G \chi$, χ is a character of $H \longleftrightarrow$ An algebra $D_\tau(G/H)$ of G-invariant differential operators on G/H
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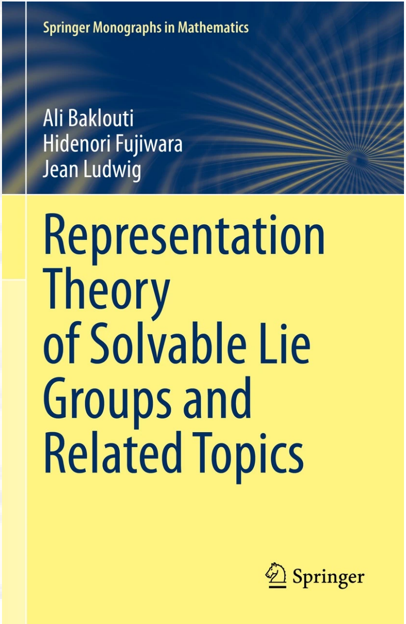
The representation theory of solvable Lie groups, with examples and solutions to long-standing problems related to τ and $\pi|_H$ and the related differential operators algebras, is subject of the following book (Springer-Verlag, 2022)



Beyond the theoretical framework, lie profound open problems whose influence and applications extend across numerous fields: mathematics, physics, and life sciences alike.

While a comprehensive treatment of these connections falls outside the scope of this talk, their relevance to certain theories, such as PDEs, will serve as an illustrative glimpse of their broader impact.

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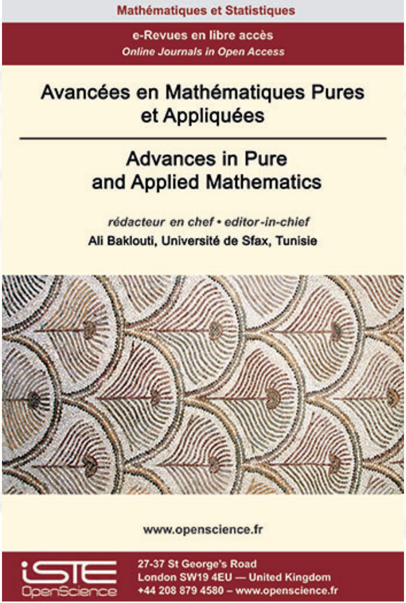
1. Benson-Ratcliff's Conjecture (1987): Counterexample (2007).
2. Corwin-Greenleaf's Conjecture (1992): Trans. Am. Math. Soc (2023).
3. Two parallel questions for restrictions (1994): J. Pure. App. Math (2004) and Rep. Theory: Am. Math. Soc (2022).
4. The Zariski-Closure Conjecture for Exponential Lie groups: A forthcoming paper (TMJ-2026), and a second in Progress.
5. The local rigidity Conjecture (2011). Still open in its generality.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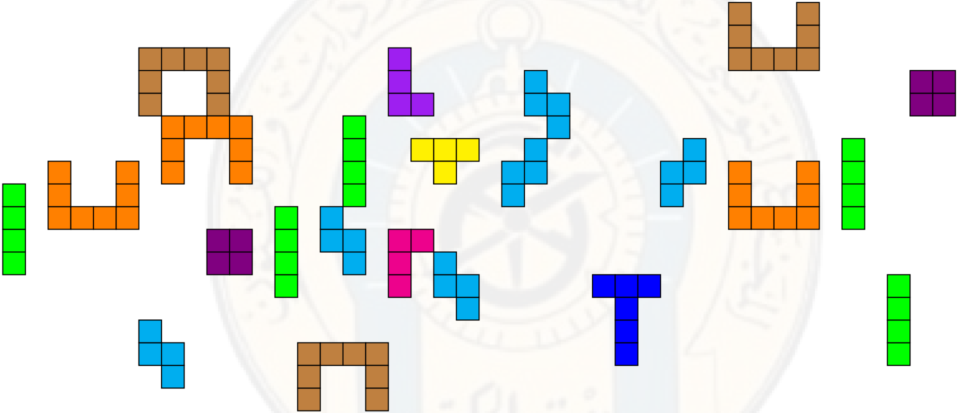

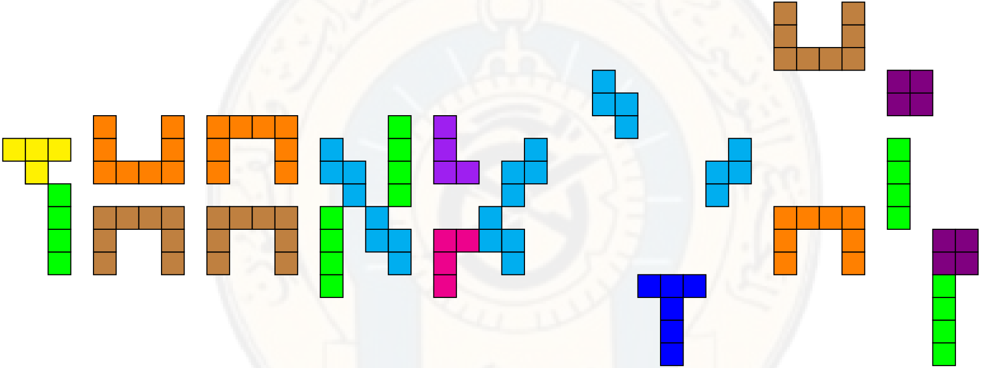

THANK YOU