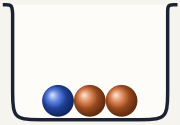
How long does it take to train an Elephant Random Walk?



Zheng Fang
Universität Zürich

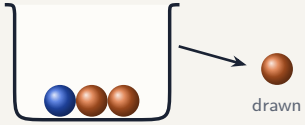
Al-Khwarizmi Webinar · 11th November 2026

Two-color Pólya urn



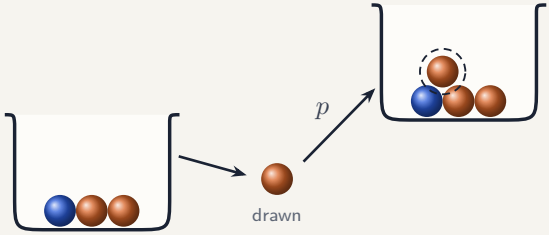
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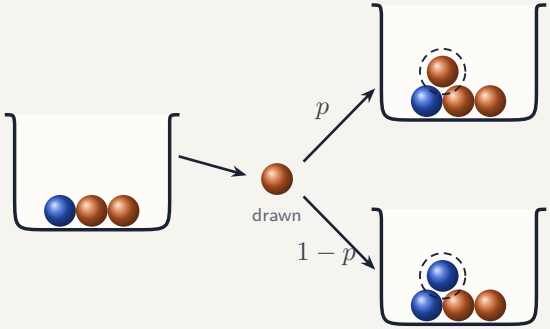
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2. With prob. p, add a ball of the **same** color.



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1. Draw a ball uniformly at random.
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3. With prob. $1-p$, add a ball of the **opposite** color.

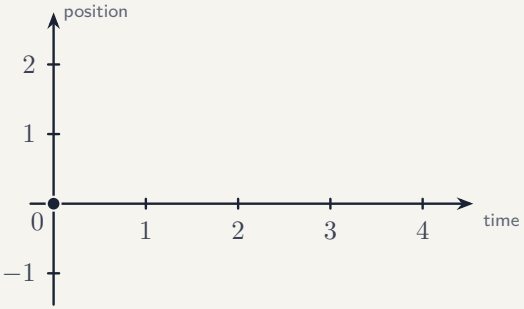


Step-reinforced random walk

At each step, the walk either

- chooses one previous step uniformly at random and repeats it, with prob. q,
- or draws an independent new step from a distribution, with prob. $1 - q$.

e.g. here we take Rademacher($\frac{1}{2}$): ± 1 with prob. $\frac{1}{2}$ each.

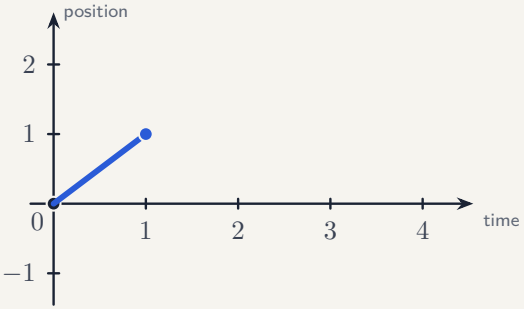


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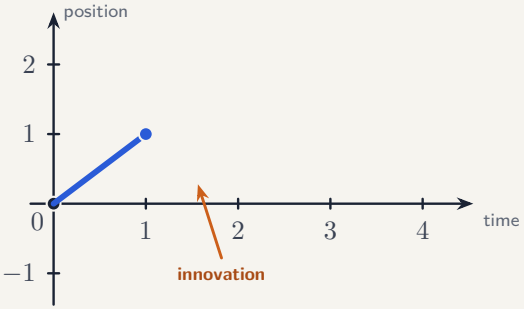


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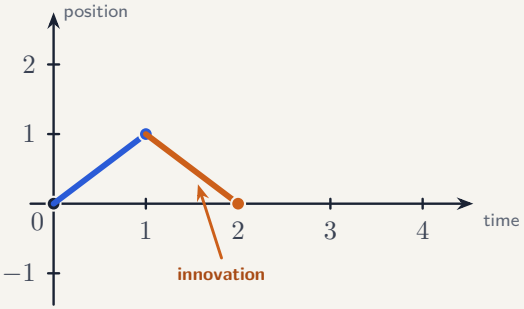


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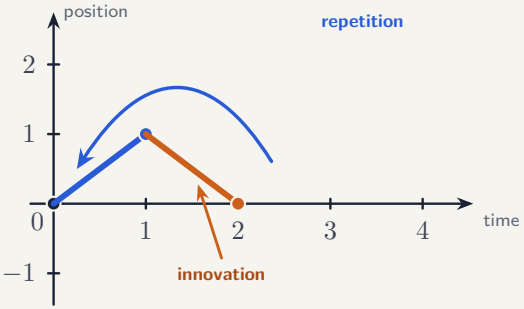


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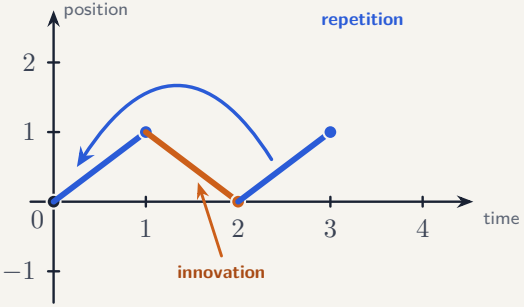


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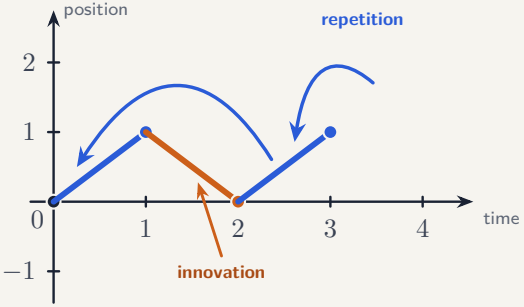


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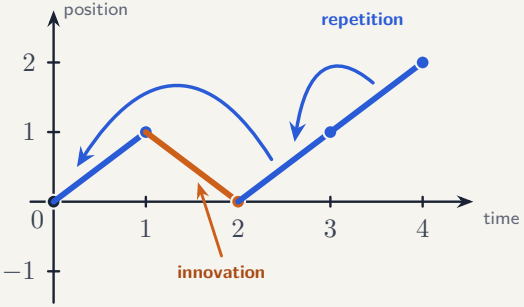


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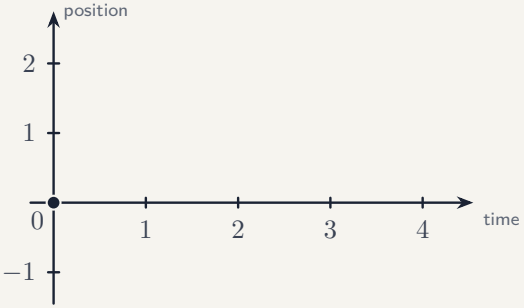
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Elephant random walk

Untrained ERW: $S(n) := \sum_{i=1}^{n} X_i$

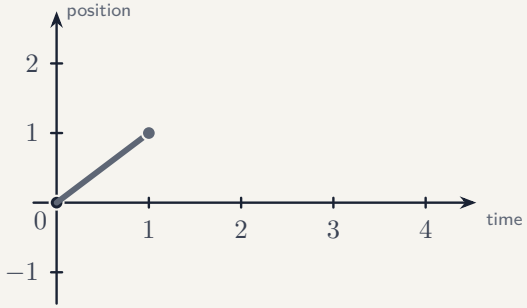
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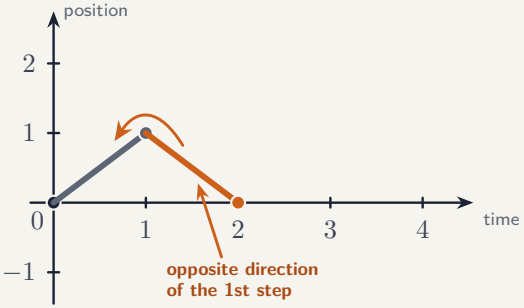
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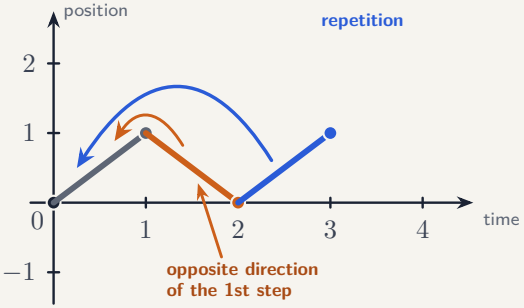
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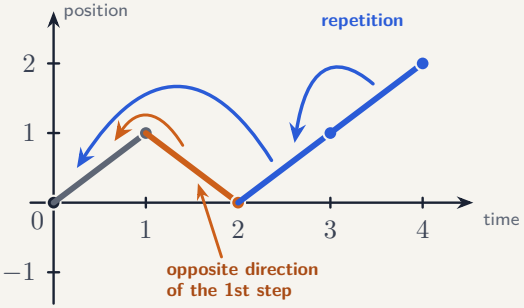
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Connection to the elephant random walk

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SRRW	Copy the selected past step	Fresh step from $\mathrm{Rad}(\frac{1}{2})$	$p = \frac{1+q}{2} \in \left(\frac{1}{2}, 1\right)$
Counterbalanced RW	Reverse the selected past step	Fresh step from $\mathrm{Rad}(\frac{1}{2})$	$p = \frac{1-q}{2} \in \left(0, \frac{1}{2}\right)$

Training the elephant

Set its first k steps to be all $+1$:

$$X_1 = X_2 = \cdots = X_k = +1,$$

and from time $k+1$ on, it evolves with the same rule as the usual ERW.

We call it the **trained ERW**, denoted

$$\big(S^{(k)}(n)\big)_{n\geq 0} := \Big(\sum_{i=1}^{n} X_i\Big)_{n\geq 0}.$$



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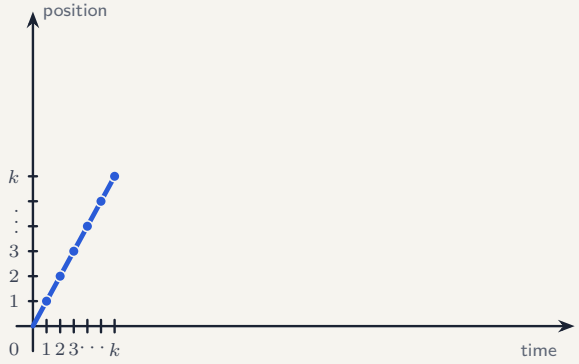
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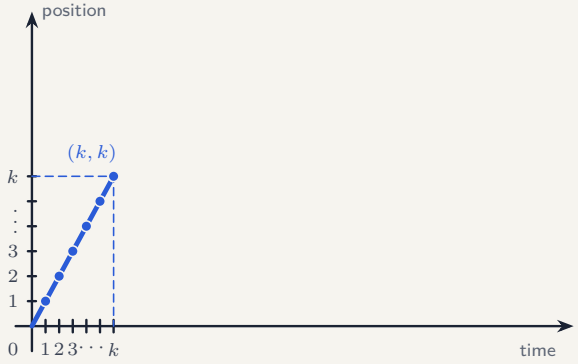
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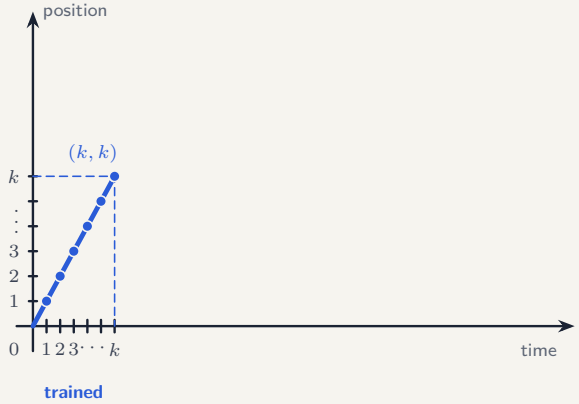
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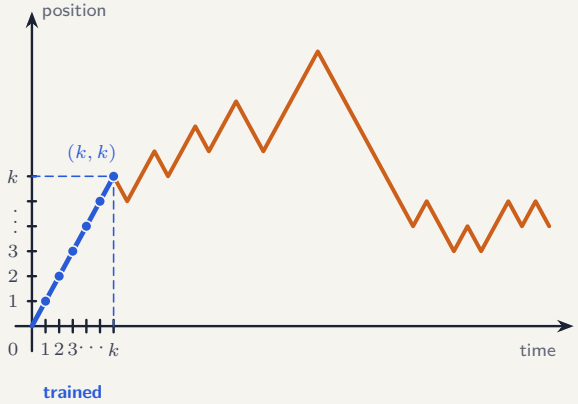
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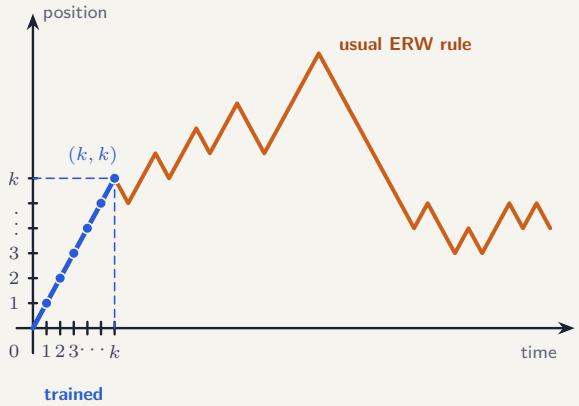
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GOAL

Study how training impacts the long-term behaviour of $S^{(k)}(n)$, when $k, n \to +\infty$ and $k \ll n$.

Especially: the first time $S^{(k)}(n)$ returns to 0 as $k \to \infty$.

Phase transition at $p = 3/4$

Both untrained & trained ERW exhibit a phase transition at $p = 3/4$, where $\mathbb{E}\big(S^2(n)\big)$ grows linearly for $p < 3/4$.

In the superdiffusive regime ($p > 3/4$):

$$\frac{S(n)}{n^{2p-1}} \xrightarrow{\text{a.s.}} L,$$

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where L is a non-degenerate random variable with mean 0.

If the elephant is trained to move to $+1$ in the 1st step, then its corresponding limit $L^{(1)}$ has positive mean. **Thus, the elephant is trained after the 1st step.**

Superdiffusive regime: why the mean is positive

Sketch

Since

$$\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) = \frac{2p-1}{n}\, S^{(1)}(n),$$

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Together with $S^{(1)}(1) = 1$,

$$\mathbb{E}\big(S^{(1)}(n)\big) = \prod_{j=1}^{n-1} \Big(1 + \frac{2p-1}{j}\Big) = \frac{\Gamma(n+2p-1)}{\Gamma(n)\,\Gamma(2p)} \sim \frac{n^{2p-1}}{\Gamma(2p)}.$$

Diffusive regime, $p < 3/4$

In the diffusive regime, $p < 3/4$, when the training is absent,

$$\Big(\frac{S(\lfloor nt \rfloor)}{\sqrt{n}}\Big)_{t\geq 0} \xrightarrow{(d)} \big(\hat{B}(t)\big)_{t\geq 0},$$

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We note $\hat{B}$ is recurrent at 0.

PLAN FOR THE REST OF THE TALK

We only deal with the diffusive regime and show that the elephant has to be trained roughly for

$$n^{\frac{3-4p}{4-4p}} \text{ steps}$$

to observe a noticeable difference from an untrained elephant at time n.

The critical training phase

Theorem 1

In the diffusive regime, $p < 3/4$, if

$$k(n) \sim (3-4p)^{-\frac{1}{4-4p}}\, n^{\frac{3-4p}{4-4p}} \qquad \text{as } n \to \infty,$$

then for all $0 < \varepsilon \leq \ell < \infty$,

$$\left(\frac{S^{(k(n))}(\lfloor nt \rfloor)}{\sqrt{n}}\right)_{t\in[\varepsilon,\ell]} \xrightarrow{(d)} \left(\hat{B}(t) + \frac{t^{2p-1}}{\sqrt{3-4p}}\right)_{t\in[\varepsilon,\ell]} \tag{1}$$

in the sense of the Skorokhod topology.

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The limit:

$$\hat{B}(t) + \frac{t^{2p-1}}{\sqrt{3-4p}} \overset{(d)}{=} \frac{t^{2p-1}}{\sqrt{3-4p}}\Big(B\big(t^{3-4p}\big) + 1\Big).$$

The limiting process near 0

Recall (Bertoin, 2021):

$$\limsup_{t\to 0^+} \frac{|\hat{B}(t)|}{\sqrt{2t\ln\ln(1/t)}} = \frac{1}{\sqrt{3-4p}} \qquad \text{a.s.}$$

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$p < 1/2$

The limiting process makes a quick return to 0 from ∞.

The question

How long does the elephant retain the effect of training before it enters the negative half-line?

We define the 1st return time to the origin by the trained ERW:

$$T^{(k)} := \inf\{\, n \geq k : S^{(k)}(n) = 0 \,\}. \tag{2}$$

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Remark. $T^{(k)}$ is the 1st time at which the elephant has the same prob. $\frac{1}{2}$ of making an upward or downward step. Nevertheless, the elephant does not lose all its memory at $T^{(k)}$, as the first step X_1 still gets recalled infinitely often.

The first return time

Theorem 2

In the diffusive regime, $p < 3/4$,

$$k^{-\frac{4-4p}{3-4p}}\, T^{(k)} \xrightarrow{(d)} (3-4p)^{\frac{1}{3-4p}}\, \tau^{\frac{1}{3-4p}}, \tag{3}$$

where τ is a Stable$(\frac{1}{2})$-distributed random variable with density

$$\frac{1}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2t}\right), \qquad t > 0.$$

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Remark. The quickest return to 0 for such trained ERW occurs at $p = 0$, which has return rate $k^{4/3}$, and not k as one might think.

Before proving, we recall Theorem 1

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Main steps of the proof

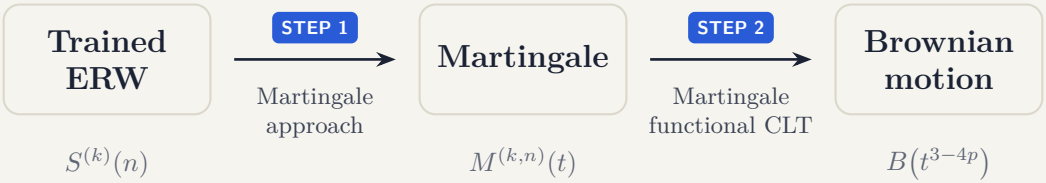
Trained ERW

$S^{(k)}(n)$

Main steps of the proof



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Step 1: a martingale

Show that

$$\left(M^{(k,n)}(t)\right)_{t\geq 0} := \left(a_{\lfloor k(n)\rfloor+\lfloor nt\rfloor}\, S^{(k(n))}\left(\lfloor k(n)\rfloor + \lfloor nt\rfloor\right) - a_{\lfloor k(n)\rfloor}\, S^{(k(n))}\left(\lfloor k(n)\rfloor\right)\right)_{t\geq 0}$$

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Sketch

Since

$$\mathbb{E}\big(S^{(k)}(n+1) \mid \mathcal{F}_n\big) = \Big(1 + \frac{2p-1}{n}\Big) S^{(k)}(n),$$

by setting $a_n := \dfrac{\Gamma(n)}{\Gamma(n+2p-1)}$, we note that for $n \geq k$, $\big(a_n S^{(k)}(n)\big)_{n\geq k}$ is a martingale.

Step 2: martingale functional CLT

We also identify the quadratic variation of $M^{(k,n)}$:

$$\left[M^{(k,n)}, M^{(k,n)}\right](t) = \sum_{s \le t} \left(\Delta M^{(k,n)}(s)\right)^2 \sim \frac{t^{3-4p}}{3-4p}\, n^{3-4p} \qquad \text{as } n \to \infty.$$

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Apply the martingale FCLT:

$$\left(\frac{M^{(k,n)}(t)}{\sqrt{\frac{n^{3-4p}}{3-4p}}} \right)_{t \ge 0} \xrightarrow{(d)} \left(B\left(t^{3-4p}\right) \right)_{t \ge 0} \tag{4}$$

in the sense of the Skorokhod topology.

The critical training phase

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Thus, we showed that for all $0 < \varepsilon \leq \ell < \infty$:

$$\Big(\frac{S^{(k(n))}(\lfloor nt \rfloor)}{\sqrt{n}}\Big)_{t \in [\varepsilon, \ell]} \xrightarrow{(d)} \Big(\hat{B}(t) + \frac{t^{2p-1}}{\sqrt{3-4p}}\Big)_{t \in [\varepsilon, \ell]},$$

which is (1).

Before proving, we recall Theorem 2

Theorem 2

In the diffusive regime, $p < 3/4$,

$$k^{-\frac{4-4p}{3-4p}}\, T^{(k)} \xrightarrow{(d)} (3-4p)^{\frac{1}{3-4p}}\, \tau^{\frac{1}{3-4p}}, \tag{3}$$

where τ is a Stable$(1/2)$-distributed random variable with density

$$\frac{1}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2t}\right), \qquad t > 0.$$

Main steps of the proof

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No early return

Exclude early returns on the $o(n)$-scale.

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Return times converge

The 1st return time of $S^{(k(n))}(\lfloor nt \rfloor)$ converges to that of the limiting process $\hat{B}(t) + \frac{t^{2p-1}}{\sqrt{3-4p}}$.

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Change of variable

Recover the 1st return time to 0 of $S^{(k)}(n)$ by a change of variable.

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Step 0

The convergence in Theorem 1 sees only the time scale of order n.
If $S^{(k(n))}\big(\lfloor k(n) \rfloor + o(n)\big) = 0$, after factoring n, we have $S^{(k(n))}\big(\lfloor k(n) \rfloor + n\, o(1)\big)$.
And $o(1)$ will fall out of $[\varepsilon, \ell]$ as $n \to \infty$.

Step 1: no quick return on the $o(n)$-scale

Show no quick return on the $o(n)$-scale, or equivalently

$$\lim_{\varepsilon \downarrow 0} \sup_{n \in \mathbb{N}} \mathbb{P}_{k(n)} \Big(\inf_{0 \le t \le \varepsilon} S^{(k(n))} \big(\lfloor k(n) \rfloor + \lfloor nt \rfloor \big) \le 0 \Big) = 0. \tag{5}$$

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Sketch

Rewrite the event with the martingale $M^{(k,n)}$:

$$\begin{aligned} &\mathbb{P}_{k(n)} \Big(\inf_{0 \le t \le \varepsilon} S^{(k(n))} \big(\lfloor k(n) \rfloor + \lfloor nt \rfloor \big) \le 0 \Big) \\ &\qquad = \mathbb{P} \Big(\sup_{0 \le t \le \varepsilon} \big(-M^{(k,n)}(t) \big) \ge a_{\lfloor k(n) \rfloor} \, S^{(k(n))} \big(\lfloor k(n) \rfloor \big) \Big). \end{aligned}$$

Step 1: the Azuma–Hoeffding bound

Sketch (continued)

And $\left|\Delta M^{(k,n)}(t)\right| \le 2\, a_{\lfloor k(n)\rfloor+\lfloor nt\rfloor+1}$, hence with bounded martingale differences we apply the Azuma–Hoeffding inequality:

$$\mathbb{P}_{k(n)}\Big(\inf_{0\le t\le\varepsilon} S^{(k(n))}\big(\lfloor k(n)\rfloor+\lfloor nt\rfloor\big)\le 0\Big)$$
$$\le \exp\Bigg(-\frac{1}{2}\,\underbrace{\frac{\big(a_{\lfloor k(n)\rfloor}\lfloor k(n)\rfloor\big)^2}{\sum_{j=1}^{\lfloor n\varepsilon\rfloor}\big(2a_{\lfloor k(n)\rfloor+j+1}\big)^2}}_{\asymp\,\varepsilon^{-(3-4p)}}\Bigg) \xrightarrow[\varepsilon\downarrow 0]{} 0,$$

which gives (5).

Step 2: convergence of the 1st return time

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$$k(n) \sim (3-4p)^{-\frac{1}{4-4p}}\, n^{\frac{3-4p}{4-4p}}, \qquad\qquad r(k) := (3-4p)^{\frac{1}{3-4p}}\, k^{\frac{4-4p}{3-4p}}.$$

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Denote

$$Z^{(k)}(t) := \frac{S^{(k)}\big(k + \lfloor r(k)t \rfloor\big)}{\sqrt{r(k)}},$$

$$Z(t) := \hat{B}(t) + \frac{t^{2p-1}}{\sqrt{3-4p}} \overset{(d)}{=} \frac{t^{2p-1}}{\sqrt{3-4p}}\Big(1 + B\big(t^{3-4p}\big)\Big).$$

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Theorem 1 states

$$\big(Z^{(k)}(t)\big)_{\varepsilon \le t \le \ell} \xrightarrow{(d)} \big(Z(t)\big)_{\varepsilon \le t \le \ell}.$$

Step 2: back to the original time scale

Since

$$\begin{aligned} T^{(k)} &= \inf\{n \geq k : S^{(k)}(n) \leq 0\} \\ &= k + r(k) \inf\{t \geq 0 : S^{(k)}(k + \lfloor r(k)t \rfloor) \leq 0\}, \end{aligned}$$

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we have

$$\begin{aligned} \frac{T^{(k)}}{r(k)} &= \frac{k}{r(k)} + \inf\{t \geq 0 : Z^{(k)}(t) \leq 0\} \\ &\xrightarrow{(d)} \inf\{t \geq 0 : Z(t) \leq 0\} = \tau^{\frac{1}{3-4p}}, \end{aligned} \tag{6}$$

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where $\tau := \inf\{u \geq 0 : B(u) + 1 = 0\}$. This is (3).

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Thank you for listening :)