How long does it take to train an Elephant Random Walk ?

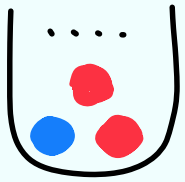
@ Al-Khwarizmi Webinar

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- Models with reinforcement (memory)

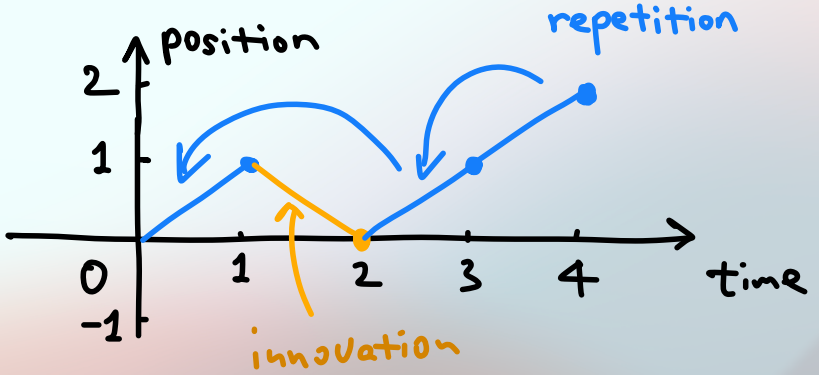
  - Two-color Pólya Urn model :



With prob. p add ball of same color or add an opposite color ball with prob. 1-p.

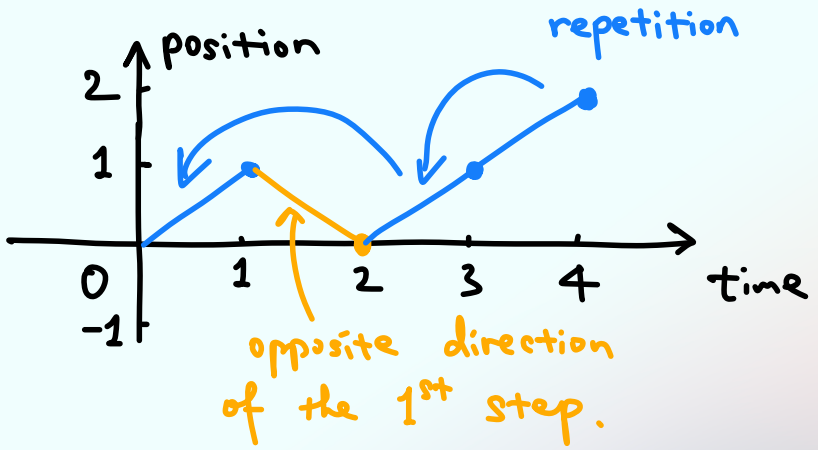
  - Step-reinforced Random Walk :
    - either choose one uniform previous step and repeat it with prob. q
    - or draw an independent new step from a distribution with prob 1-q.

    e.g. here we take Radmacher($\frac{1}{2}$)



2

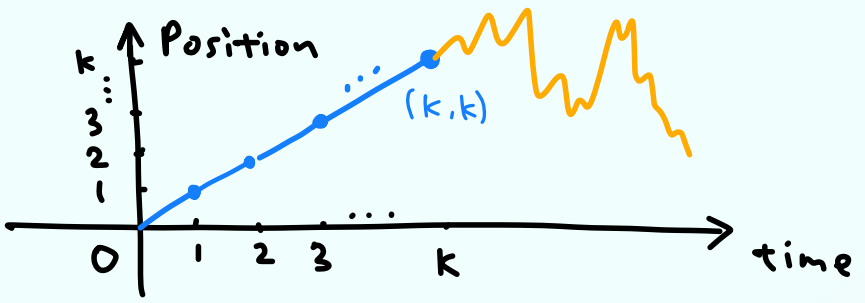
- Elephant Random Walk (untrained ERW: $S(n) := \sum_{i=1}^{n} X_i$)
  - Starts from 0 & with unit increment
  - Chooses one previous step uniformly
  - either repeat with prob. $p \in [0,1]$
  - or repeat the opposite direction of the chosen step with prob. $1-p$



Remark: [Bertoin 2024 JEMS]

	w/ prob. q	w/ prob. $1-q$	ERW parameter
SRRW	Copy the selected past step	Fresh step from $\mathrm{Rad}(\frac{1}{2})$	$p = \frac{1+q}{2} \in (\frac{1}{2}, 1)$
Counterbalanced RW	Reverse the selected past step	Fresh step from $\mathrm{Rad}(\frac{1}{2})$	$p = \frac{1-q}{2} \in (0, \frac{1}{2})$

- Training the elephant



Set its first k steps to be all $+1$. i.e $X_1 = X_2 = \cdots = X_k = +1$, and from time $k+1$ on, it evolves with the same rule as the usual ERW. And we call it the trained ERW, which we denote later as $\left(S^{(k)}(n)\right)_{n\geq 0} := \left(\sum_{i=1}^{n} X_i\right)_{n\geq 0}$

Goal: Study how training impacts the long term behaviour of $S^{(k)}(n)$, when $k, n \to +\infty$ & $k \ll n$. Especially the first time $S^{(k)}(n)$ returns 0 as $k \to \infty$.

4

Both untrained & trained ERW exhibit a phase transition at $p = \frac{3}{4}$, where $\mathbb{E}(S^2(n))$ grows linearly.

In the **superdiffusive regime** $(p > \frac{3}{4})$

$$\frac{S(n)}{n^{2p-1}} \xrightarrow{a.s} L$$

where L is a non-degenerate random variable with mean 0.

If the elephant is trained to move into +1 in the 1st step, then its corresponding limit $L^{(1)}$ have positive mean. Thus, the elephant is trained after the 1st step.

Since $\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) = \frac{2p-1}{n} S^{(1)}(n)$ then $\mathbb{E}(S^{(1)}(n+1)) = \left(1 + \frac{2p-1}{n}\right)\mathbb{E}(S^{(1)}(n))$

Together with $S^{(1)}(1) = 1$, $\mathbb{E}(S^{(1)}(n)) = \prod_{j=1}^{n-1}\left(1 + \frac{2p-1}{j}\right) = \frac{\Gamma(n+2p-1)}{\Gamma(n)\,\Gamma(2p)} \sim \frac{n^{2p-1}}{\Gamma(2p)}$

5

In the diffusive regime, $p < \frac{3}{4}$, when the training is absent

$$\left(\frac{S(\lfloor nt \rfloor)}{\sqrt{n}} \right)_{t \geq 0} \xrightarrow{(d)} \left(\hat{B}(t) \right)_{t \geq 0}$$

where $\hat{B}$ is a centered Gaussian process and

$$\hat{B}(t) \overset{(d)}{=} \frac{t^{2p-1}}{\sqrt{3-4p}} B\left(t^{3-4p}\right)$$

we note $\hat{B}$ is recurrent at 0.

Plan for the rest of the presentation:

we only deal with the diffusive regime and show that the elephant has to be trained roughly for $n^{\frac{3-4p}{4-4p}}$ steps to observe noticeable difference from an untrained elephant at time n.

6

Theorem 1 In the diffusive regime, $p < \frac{3}{4}$, if

$$k(n) \sim (3-4p)^{-\frac{1}{4-4p}} n^{\frac{3-4p}{4-4p}} \quad \text{as} \quad n \to \infty$$

then $\forall\, 0 < \varepsilon \le l < \infty$

$$\left(\frac{S^{(k(n))}(\lfloor nt \rfloor)}{\sqrt{n}} \right)_{t \in [\varepsilon, l]} \xrightarrow{(d)} \left(\hat{B}(t) + \frac{t^{2p-1}}{\sqrt{3-4p}} \right)_{t \in [\varepsilon, l]}$$

in the sense of Skorokhod topology.

$$\overset{(d)}{=} \frac{t^{2p-1}}{\sqrt{3-4p}} \left(\hat{B}(t^{3-4p}) + 1 \right)$$

Recall (Bertoin, 2021)

$$\limsup_{t \to 0^+} \frac{|\hat{B}(t)|}{\sqrt{2t \ln\ln \frac{1}{t}}} = \frac{1}{\sqrt{3-4p}} \quad \text{a.s.}$$

$\Rightarrow \hat{B}$ is dominated by the correction term near 0.

- $p > \frac{1}{2}$, the limiting process makes a 1st excursion with > 0 length
- $p < \frac{1}{2}$, the limiting process makes a faster return to 0 from ∞

7

Question: How long does the elephant retain the effect of training before it enters the negative half line?

We define the 1st return time to the origin by trained ERW

$$T^{(k)} := \inf\{n \geq k : S^{(k)}(n) = 0\}$$

Remark: $T^{(k)}$ is the 1st time at which the elephant has the same prob. $1/2$ of making an upward on downward step. Nevertheless, the elephant does not lose all its memory at $T^{(k)}$, as the first step X_1 still gets recalled infinitely often.

8

Theorem 2 In the diffusive regime, $p < \frac{3}{4}$

$$k^{-\frac{4-4p}{3-4p}} T^{(k)} \xrightarrow{(d)} (3-4p)^{\frac{1}{3-4p}} \tau^{\frac{1}{3-4p}}$$

where τ is a Stable $(\frac{1}{2})$ - distributed random variable with density

$$\frac{1}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2t}\right), \quad t > 0$$

Remark: the quickest return to 0 for such trained ERW occurs at $p = 0$, which has return rate $k^{\frac{4}{3}}$ and not k as one might think

9

Before proving, we recall Theorem 1

In the diffusive regime, $p < \frac{3}{4}$, if

$$k(n) \sim (3-4p)^{-\frac{1}{4-4p}} n^{\frac{3-4p}{4-4p}} \quad \text{as} \quad n \to \infty$$

then $\forall\, 0 < \varepsilon \leq l < \infty$

$$\left(\frac{S^{(k(n))}(\lfloor nt \rfloor)}{\sqrt{n}} \right)_{t \in [\varepsilon, l]} \xrightarrow{(d)} \left(\hat{B}(t) + \frac{t^{2p-1}}{\sqrt{3-4p}} \right)_{t \in [\varepsilon, l]}$$

in the sense of Skorokhod topology.

10

The main steps of the proof for Theorem 1 include :

- Martingale approach (trained ERW $\rightsquigarrow$ martingale)
- Martingale Functional CLT (martingale $\rightsquigarrow$ Brownian Motion)

Step 1 : Show $\left(M^{(k,n)}(t)\right)_{t\geq 0} := \left(a_{\lfloor k(n)\rfloor+\lfloor nt\rfloor} S^{(k(n))}(\lfloor k(n)\rfloor+\lfloor nt\rfloor) - a_{\lfloor k(n)\rfloor} S^{(k(n))}(\lfloor k(n)\rfloor)\right)_{t\geq 0}$

is a martingale starting from 0 .

Sketch : Since $\mathbb{E}\left(S^{(k)}(n+1) \mid \mathcal{F}_n\right) = \left(1+\frac{2p-1}{n}\right) S^{(k)}(n)$,

By setting $a_n := \frac{\Gamma(n)}{\Gamma(n+2p-1)}$, we note for $n\geq k$, $\left(a_n S^{(k)}(n)\right)_{n\geq k}$ is a martingale .

11

We also identify the quadratic variation of $M^{(k,n)}$:

$$[M^{(k,n)}, M^{(k,n)}](t) = \sum_{s \le t} \left(\Delta M^{(k,n)}_{\lfloor ns \rfloor}\right)^2 \sim \frac{t^{3-4p}}{3-4p} n^{3-4p} \quad \text{as} \quad n \to \infty$$

- Step 2 : Apply martingale FCLT

$$\left(\frac{M^{(k,n)}(t)}{\sqrt{\frac{n^{3-4p}}{3-4p}}} \right)_{t \ge 0} \xrightarrow{(d)} \left(B(t^{3-4p}) \right)_{t \ge 0}$$

in the sense of Skorokhod topology.

We require $a_{\lfloor k(n) \rfloor} S^{(k(n))}(\lfloor k(n) \rfloor) \sim \sqrt{\frac{n^{3-4p}}{3-4p}}$ to obtain the critical traing phase $k(n) \sim (3-4p)^{-\frac{1}{4-4p}} n^{\frac{3-4p}{4-4p}}$.

Thus, we showed $\forall 0 < \varepsilon \le l < \infty$: $\left(\frac{S^{(k(n))}(\lfloor nt \rfloor)}{\sqrt{n}} \right)_{t \in [\varepsilon, l]} \xrightarrow{(d)} \left(\hat{B}(t) + \frac{t^{2p-1}}{\sqrt{3-4p}} \right)_{t \in [\varepsilon, l]}$

12

Before proving, we recall Theorem 2

In the diffusive regime, $p < \frac{3}{4}$

$$k^{-\frac{4-4p}{3-4p}} T^{(k)} \xrightarrow{(d)} (3-4p)^{\frac{1}{3-4p}} \tau^{\frac{1}{3-4p}}$$

where τ is a Stable $(\frac{1}{2})$ - distributed random variable with density

$$\frac{1}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2t}\right), \quad t > 0$$

13

Next, we show the main steps behind the proof of Theorem 2.

- Excludes early returns on $o(n)$ - scale.
- The 1st return time of $S^{(k(n))}(\lfloor nt \rfloor)$ converges to that of the limiting process $\hat{B}(t) + \frac{t^{2p-1}}{\sqrt{3-4p}}$.
- Recover the 1st return time to 0 of $S^{(k)}(n)$ by a change of variable.

Step 0: The convergence in Theorem 1 sees only time scale of order n.

If $S^{(k(n))}(\lfloor k(n) \rfloor + o(n)) = 0$, after factoring n, we have $S^{(k(n))}(\lfloor k(n) \rfloor + n(o(1)))$

And $o(1)$ will fall out of $[\varepsilon, 1]$ as $n \to \infty$

14

Step 1 : Show NO quick return on $o(n)$ - scale, or equivalently

$$\limsup_{\varepsilon \downarrow 0} \sup_{n \in \mathbb{N}} \mathbb{P}_{k(n)} \left(\inf_{0 \le t \le \varepsilon} S^{(k(n))}(\lfloor k(n) \rfloor + \lfloor nt \rfloor) \le 0 \right) = 0$$

$$\mathbb{P}_{k(n)} \left(\inf_{0 \le t \le \varepsilon} S^{(k(n))}(\lfloor k(n) \rfloor + \lfloor nt \rfloor) \le 0 \right) = \mathbb{P} \left(\sup_{0 \le t \le \varepsilon} \left(-M^{(k,n)}(t) \right) \ge -a_{\lfloor k(n) \rfloor} S^{(k(n))}(\lfloor k(n) \rfloor) \right)$$

And $|\Delta M^{(k,n)}(t)| \le 2 a_{\lfloor k(n) \rfloor + \lfloor nt \rfloor + 1}$, hence with bound martingale differences

we apply Azuma - Hoeffding inequality,

$$\mathbb{P}_{k(n)} \left(\inf_{0 \le t \le \varepsilon} S^{(k(n))}(\lfloor k(n) \rfloor + \lfloor nt \rfloor) \le 0 \right) \le \exp\left(-\frac{1}{2} \underbrace{\frac{\left(a_{\lfloor k(n) \rfloor} \lfloor k(n) \rfloor \right)^2}{\sum_{j=1}^{\lfloor n\varepsilon \rfloor} \left(2 a_{\lfloor k(n) \rfloor + j + 1} \right)^2}}_{\asymp \varepsilon^{-(3-4p)}} \right) \xrightarrow{\varepsilon \downarrow 0} 0$$

15

Step 2 : Show the convergence of the 1st return time.

Recall the critical training phase

$$k(n) \sim (3-4p)^{-\frac{1}{4-4p}} n^{\frac{3-4p}{4-4p}}$$

inverting it gives $r(k) := (3-4p)^{\frac{1}{3-4p}} k^{\frac{4-4p}{3-4p}}$

Denote $Z^{(k)}(t) := \frac{S^{(k)}(k+\lfloor r(k)t \rfloor)}{\sqrt{r(k)}}$ & $Z(t) := \hat{B}(t) + \frac{t^{2p-1}}{\sqrt{3-4p}} \overset{(d)}{=} \frac{t^{2p-1}}{\sqrt{3-4p}}(1+B(t^{3-4p}))$

Theorem 1 states $(Z^{(k)}(t))_{\varepsilon \le t \le l} \xrightarrow{(d)} (Z(t))_{\varepsilon \le t \le l}$

Since $T^{(k)} = \inf\{n \ge k : S^{(k)}(n) \le 0\} = k + r(k) \inf\{t \ge 0 : S^{(k)}(k+\lfloor r(k)t \rfloor) \le 0\}$

We have $\frac{T^{(k)}}{r(k)} = \frac{k}{r(k)} + \inf\{t \ge 0 : Z^{(k)}(t) \le 0\} \xrightarrow{(d)} \inf\{t \ge 0 : Z(t) \le 0\} = \tau^{\frac{1}{3-4p}}$

where $\tau := \inf\{u \ge 0 : B(u) + 1 = 0\}$

Thank You for listening :)

16