Maximal height of non-intersecting Brownian motions

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- P. J. Forrester (Dept. of Math., Melbourne)
- S. N. Majumdar (LPTMS, Orsay)
- J. Rambeau (Inst. Theor. Phys., Cologne)
- J. Randon-Furling (Univ. Paris 1, Paris)

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References:

- G. S., S. N. Majumdar, A. Comtet, J. Randon-Furling, Phys. Rev. Lett. **101**, 150601 (2008)
- J. Rambeau, G. S., Europhys. Lett. **91**, 60006 (2010); Phys. Rev. E **83**, 061146 (2011)
- P. J. Forrester, S. N. Majumdar, G. S., Nucl. Phys. B **844**, 500 (2011)

Height of rooted planar tree

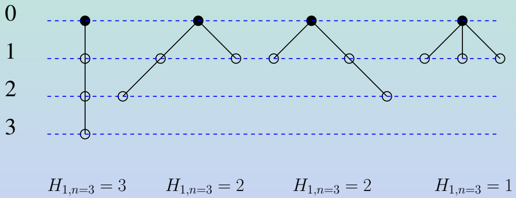
THE AVERAGE HEIGHT OF PLANTED PLANE TREES

N. G. de Bruijn
Technological University
Eindhoven, The Netherlands

D. E. Knuth†
Stanford University
Stanford, California

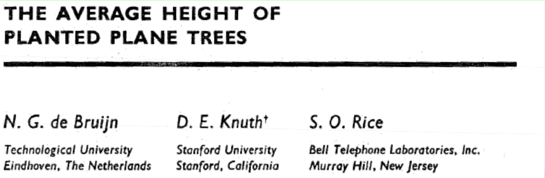
S. O. Rice
Bell Telephone Laboratories, Inc.
Murray Hill, New Jersey

- Height of rooted plane trees $H_{1,n}$ with $n+1$ nodes



Q : $\langle H_{1,n} \rangle$ for large n ?

Height of rooted planar tree



THE AVERAGE HEIGHT OF PLANTED PLANE TREES

N. G. de Bruijn — Technological University, Eindhoven, The Netherlands

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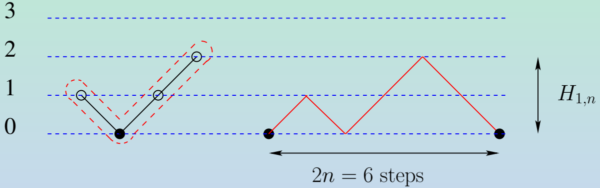
S. O. Rice — Bell Telephone Laboratories, Inc., Murray Hill, New Jersey

- Mapping between rooted plane trees and Dyck paths



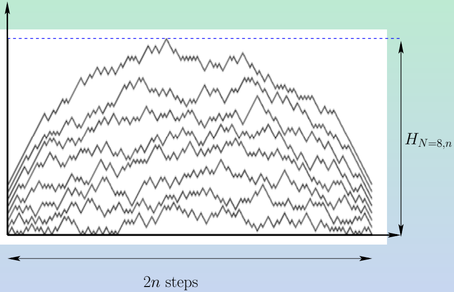
$$\lim_{n\to\infty} \frac{\langle H_{1,n} \rangle}{\sqrt{2n}} = \langle H_1 \rangle = \sqrt{\frac{\pi}{2}}$$

Generalization to N non-intersecting Dyck paths

Watermelon uniform random generation with applications

Nicolas Bonichon*, Mohamed Mosbah
LaBRI-Université Bordeaux 1, 351 Cours de la Libération, 33405 Talence, France

Theoretical Computer Science 307 (2003) 241 – 256



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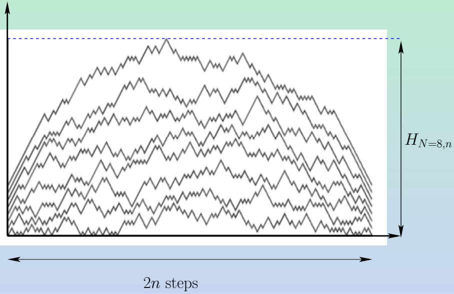

$\Longrightarrow$ numerical estimate for $\langle H_{N=8,n} \rangle$

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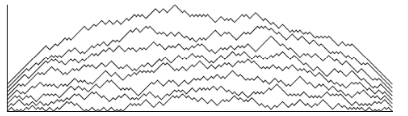

$$\frac{\langle H_{N,n}\rangle_{\mathrm{num}}}{\sqrt{2n}} = \langle H_N\rangle_{\mathrm{num}} \sim \sqrt{1.67\,N}$$

Generalization to N non-intersecting Dyck paths

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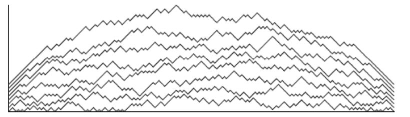

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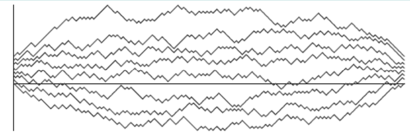

$$\frac{\langle H_{N,n}\rangle_{\rm num}}{\sqrt{2n}} = \langle H_N\rangle_{\rm num} \sim \sqrt{0.82\,N}$$

Generalization to N non-intersecting Dyck paths

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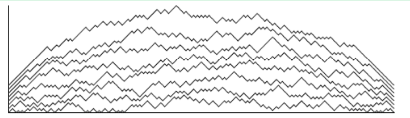

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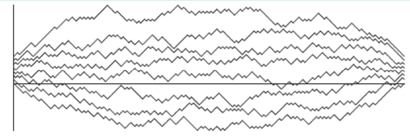


$$\frac{\langle H_{N,n}\rangle_{\rm num}}{\sqrt{2n}} = \langle H_N\rangle_{\rm num} \sim \sqrt{0.82\, N}$$

Q : can one compute $\langle H_N \rangle$?

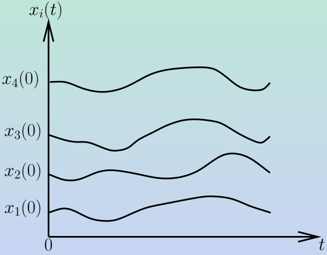
Non-intersecting Brownian motions in 1d

- N Brownian motions in one-dimension

$$\dot{x}_i(t) = \zeta_i(t)\ ,\ \langle \zeta_i(t)\zeta_j(t') \rangle = \delta_{i,j}\delta(t-t')$$
$$x_1(0) < x_2(0) < ... < x_N(0)$$

- Non-intersecting condition

$$x_1(t) < x_2(t) < ... < x_N(t)\ ,\quad \forall t \geq 0$$



Non-intersecting Brownian motions in 1d

- N Brownian motions in one-dimension

$$\dot{x}_i(t) = \zeta_i(t) \; , \; \langle \zeta_i(t) \zeta_j(t') \rangle = \delta_{i,j} \delta(t - t')$$
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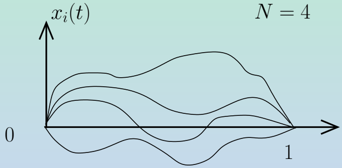

watermelons

Non-intersecting Brownian motions in 1d

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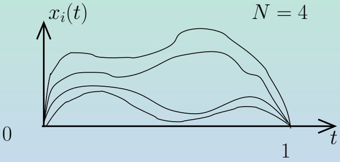


watermelons "with a wall"

Vicious walkers in physics

THE JOURNAL OF CHEMICAL PHYSICS VOLUME 48, NUMBER 5 1 MARCH 1968

Soluble Model for Fibrous Structures with Steric Constraints

P.-G. DE GENNES

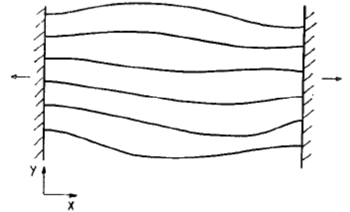
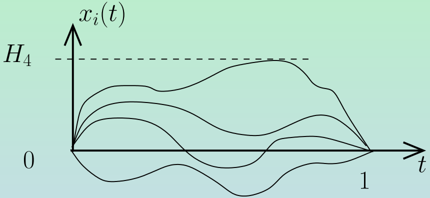


FIG. 1. Model for a two-dimensional fiber structure. The component chains are assumed to be attached to two plates I and F and placed under tension. The chains are bent by thermal fluctuations. Different chains cannot intersect each other.

Vicious walkers in physics and maths

- P. G. de Gennes, *Soluble Models for fibrous structures with steric constraints* (1968)
- M. E. Fisher, *Walks, Walls, Wetting and Melting* (1984)
- D. J. Grabiner, *Brownian motion in a Weyl chamber, non-colliding particles, and random matrices* (1999)
- C. Krattenthaler, A. J. Guttmann, X. G. Viennot, *Vicious walkers, friendly walkers and Young tableaux* (2000)
- P. L. Ferrari, K. Johansson, N. O'Connell, M. Praehofer, H. Spohn, C. Tracy, H. Widom... *Stochastic growth models, directed polymers* (from 2000)
- ...

Extreme statistics of vicious walkers

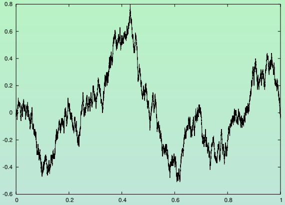


Maximal height of watermelons

$$x_1(t) < x_2(t) < ... < x_N(t)$$
$$H_N = \max_{\tau}[x_N(\tau), 0 \leq \tau \leq 1]$$
$$\langle H_N \rangle = ?$$

Extreme statistics of Brownian motion

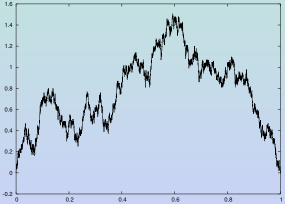
- Brownian bridge



$$H_1 = \max_{\tau}[x(\tau), 0 \leq \tau \leq 1]$$

$$\langle H_1 \rangle = \sqrt{\frac{\pi}{8}}$$

- Brownian excursion



$$H_1 = \max_{\tau}[x(\tau), 0 \leq \tau \leq 1]$$

$$\langle H_1 \rangle = \sqrt{\frac{\pi}{2}}$$

de Bruijn, Knuth, Rice '72

Numerical computation of $\langle H_N \rangle$

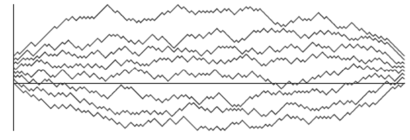
N. Bonichon, M. Mosbah (2003)

Watermelon uniform random generation with applications

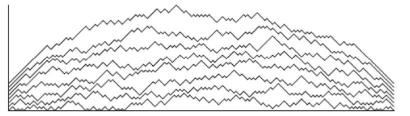
Nicolas Bonichon*, Mohamed Mosbah
LaBRI-Université Bordeaux 1, 351 Cours de la Libération, 33405 Talence, France

Theoretical Computer Science 307 (2003) 241 – 256

$$H_N = \max_\tau [x_N(\tau), 0 \leq \tau \leq 1]$$



$$\langle H_N \rangle_{\text{num}} \sim \sqrt{0.82\, N}$$

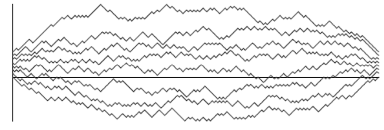


$$\langle H_N \rangle_{\text{num}} \sim \sqrt{1.67\, N}$$

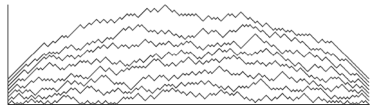
Main results

1 Connection between watermelons and random matrices

$\Longrightarrow$ exact asymptotic results for $\langle H_N \rangle$ for $N \gg 1$



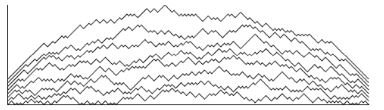
$$\langle H_N \rangle \sim \sqrt{N}$$



$$\langle H_N \rangle \sim \sqrt{2N}$$

Main results

2 Exact results for the full distribution of H_N



Cumulative distribution

$$F_N(M) = \mathrm{Proba}[H_N \leq M]$$

$$F_N(M) \to \mathcal{F}_1\left(2^{11/6} N^{1/6} \left| M - \sqrt{2N} \right|\right) \ , \ N \to \infty$$

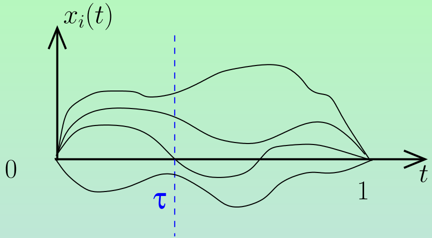
$\mathcal{F}_1$ is the Tracy-Widom distribution for GOE random matrices

Outline

1. Vicious walkers and random matrices
2. Connection with stochastic growth models
3. Exact computation using Feynman-Kac formula
4. Large N limit using discrete orthogonal polynomials
5. Conclusion

Outline

Non intersecting Brownian motions and RMT



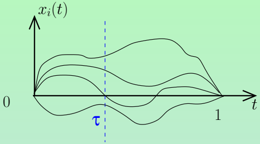
- Joint probability of $x_1(\tau), x_2(\tau), \cdots, x_N(\tau)$ at fixed time τ

$$P_{\rm joint}(x_1, x_2, \cdots, x_N, \tau) \propto \prod_{i<j=1}^{N} (x_i - x_j)^2 e^{-\frac{1}{\sigma^2(\tau)} \sum_{i=1}^N x_i^2}$$

$$\sigma(\tau) = \sqrt{2\tau(1-\tau)}$$

- The rescaled positions $\frac{x_i}{\sigma(\tau)}$ are distributed like the eigenvalues of random matrices of the Gaussian Unitary Ensemble (GUE, $\beta = 2$)

Non intersecting Brownian motions and RMT



- $H \equiv H(t)$, $N \times N$ Hermitian matrices from GUE:

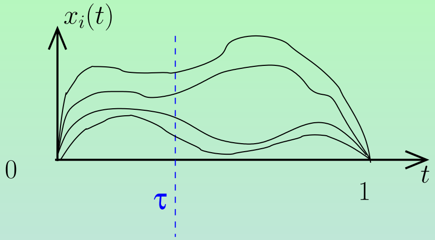
$$H_{mn}(t) = \begin{cases} \frac{1}{\sqrt{2}} \left(B_{mn}(t) + i\,\tilde{B}_{mn}(t) \right) \ , & m < n \ , \\ B_{mm}(t) \ , & m = n \\ \frac{1}{\sqrt{2}} \left(B_{nm}(t) - i\,\tilde{B}_{nm}(t) \right) \ , & m > n \ , \end{cases}$$

where $B_{m,n}, \tilde{B}_{m,n}$ independent Brownian bridges

- Eigenvalues of $H(t)$

$\{\lambda_1(t) < \lambda_2(t) < \cdots < \lambda_N(t)\} \stackrel{d}{=} \{x_1(t) < x_2(t) < \cdots < x_N(t)\}$

Non intersecting Brownian motions and RMT



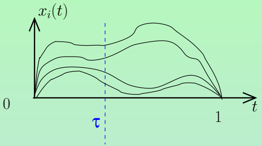
- Joint probability of $x_1(\tau), x_2(\tau), \cdots, x_N(\tau)$ at fixed time τ

$$P_{\rm joint}(\mathbf{x}, \tau) \propto \prod_{i=1}^{N} x_i^2 \prod_{1 \leq i < j \leq N} (x_i^2 - x_j^2)^2 e^{-\frac{\mathbf{x}^2}{\sigma^2(\tau)}}$$

- The rescaled positions $\frac{x_i}{\sigma(\tau)}$ are distributed like the eigenvalues of random matrices of the Bogoliubov-de Gennes type (class C)

A. Atland, M. R. Zirnbauer '96

Non intersecting Brownian motions and RMT



- $C(t)$, $2N \times 2N$ Herm. rand. mat. of Bologoliubov-de-Gennes type

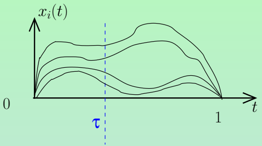
$$C(t) = \begin{pmatrix} H(t) & S(t) \\ S^\dagger(t) & -{}^tH(t) \end{pmatrix}$$ A. Atland, M. R. Zirnbauer '96

  $H(t)$ is $N \times N$ Hermitian, $S(t)$ is $N \times N$ complex symmetric, where the entries are Brownian bridges

- $C(t)$ is symplectic, $C(t) \in \mathrm{sp}(2N)$, *i.e.*:

$${}^tC(t)\,J + J\,C(t) = 0 \;,\; J = \begin{pmatrix} \mathbb{O}_p & \mathbb{I}_p \\ -\mathbb{I}_p & \mathbb{O}_p \end{pmatrix}$$

Non intersecting Brownian motions and RMT



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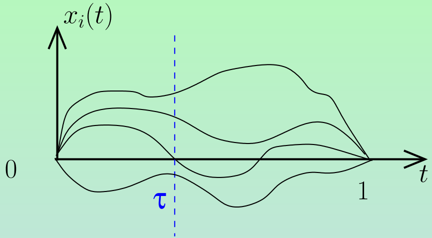
$H(t)$ is $N \times N$ Hermitian, $S(t)$ is $N \times N$ complex symmetric, where the entries are Brownian bridges

- Positive eigenvalues of $C(t) \stackrel{d}{=}$ positions of walkers M. Katori et al. '03

$$\lambda_N > \cdots > \lambda_2 > \lambda_1 > -\lambda_1 > -\lambda_2 \cdots > -\lambda_N \quad , \quad \lambda_i > 0$$

Non intersecting Brownian motions and RMT



- Joint probability of $x_1(\tau), x_2(\tau), \cdots, x_N(\tau)$ at fixed time τ

$$P_{\rm joint}(x_1, x_2, \cdots, x_N, \tau) \propto \prod_{i<j=1}^{N} (x_i - x_j)^2 e^{-\frac{1}{\sigma^2(\tau)} \sum_{i=1}^N x_i^2}$$

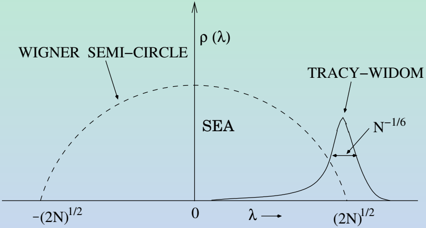
$$\sigma(\tau) = \sqrt{2\tau(1-\tau)}$$

- The rescaled positions $\frac{x_i}{\sigma(\tau)}$ are distributed like the eigenvalues of random matrices of the Gaussian Unitary Ensemble (GUE, $\beta = 2$)

Non intersecting Brownian motions and RMT

- The rescaled positions $\frac{x_i}{\sigma(\tau)}$ are distributed like the eigenvalues of random matrices of Gaussian Unitary Ensemble (GUE, $\beta = 2$)
- Mean density $\rho(\lambda)$ of eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_N$ for GUE

$$\rho(\lambda) = \frac{1}{N}\sum_{\alpha=1}^{N}\langle\delta(\lambda - \lambda_\alpha)\rangle$$

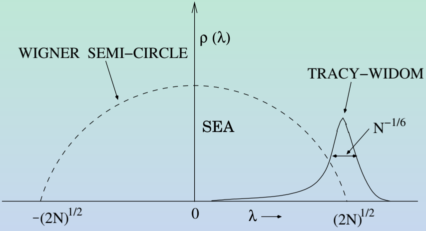


Non intersecting Brownian motions and RMT

- The rescaled positions $\frac{x_i}{\sigma(\tau)}$ are distributed like the eigenvalues of random matrices of Gaussian Unitary Ensemble (GUE, $\beta = 2$)
- Largest eigenvalue of random matrices from GUE

$$\begin{aligned} \lambda_{\max} &= \max_{1 \leq i \leq N} \lambda_i \\ &= \sqrt{2N} + \frac{N^{-\frac{1}{6}}}{\sqrt{2}} \chi_2 \\ \Pr[\chi_2 \leq \xi] &= \mathcal{F}_2(\xi) \end{aligned}$$

Tracy – Widom distribution



Non intersecting Brownian motions and RMT

- The rescaled positions $\frac{x_i}{\sigma(\tau)}$ are distributed like the eigenvalues of random matrices of Gaussian Unitary Ensemble (GUE, $\beta = 2$)
- Largest eigenvalue of random matrices from GUE

$$\lambda_{\max} = \max_{1 \leq i \leq N} \lambda_i$$

$$= \sqrt{2N} + \frac{N^{-\frac{1}{6}}}{\sqrt{2}} \chi_2$$

$$\Pr[\chi_2 \leq \xi] = \mathcal{F}_2(\xi)$$

Tracy – Widom distribution

$$\mathcal{F}_2(\xi) = \exp\left[-\int_{\xi}^{\infty} (s-\xi) q^2(s)\, ds\right]$$

where $q(s)$ satisfies Painlevé II

$$q''(s) = s\,q(s) + 2q^3(s)$$

$$q(s) \sim \mathrm{Ai}(s)\,,\; s \to \infty$$

C. Tracy, H. Widom '94

Watermelons in the limit of large N

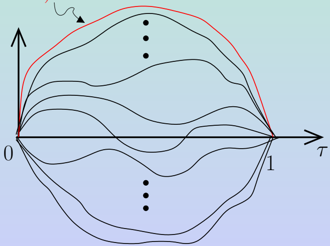
- Consequences for watermelons without wall for large N

$$\frac{x_N(\tau)}{\sqrt{2\tau(1-\tau)}} \sim \sqrt{2N} + \frac{N^{-1/6}}{\sqrt{2}}\chi_2$$

$$\text{Proba}[\chi_2 \leq \xi] = \mathcal{F}_2(\xi),$$ Tracy-Widom distribution for $\beta = 2$

- When $N \to \infty$, $x_N(\tau)$ reaches a deterministic elliptic shape

$x_N(\tau) \sim 2\sqrt{N}\sqrt{\tau(1-\tau)}$



Asymptotic behavior of $\langle H_N \rangle$: without wall

- Consequences for watermelons without wall for large N

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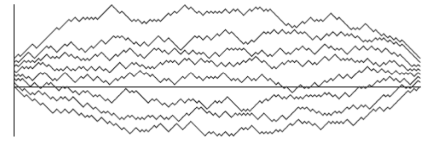
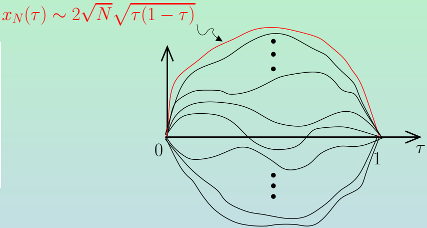




- The maximal height is reached for $\tau = 1/2$

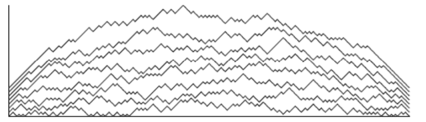
$$H_N = \max_{\tau}[x_N(\tau), 0 \leq \tau \leq 1]$$

$$\langle H_N \rangle = \langle x_N(\tau = \frac{1}{2}) \rangle \sim \sqrt{N} \quad \textit{vs.} \quad \langle H_N \rangle_{\rm num} \sim \sqrt{0.82\, N}$$

Asymptotic behavior of $\langle H_N \rangle$: with wall

- Consequences for watermelons without wall for large N

$$\frac{x_N(\tau)}{\sqrt{2\tau(1-\tau)}} \sim 2\sqrt{N} + \frac{N^{-1/6}}{2^{2/3}}\chi_2$$



$\mathrm{Proba}[\chi_2 \leq \xi] = \mathcal{F}_2(\xi)$,
Tracy-Widom distribution for $\beta = 2$

- The maximal height is reached for $\tau = 1/2$

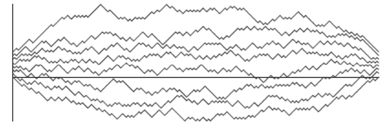
$$H_N = \max_{\tau}[x_N(\tau), 0 \leq \tau \leq 1]$$

$$\langle H_N \rangle = \langle x_N(\tau = \frac{1}{2}) \rangle \sim \sqrt{2N} \;\; \textit{vs.} \;\; \langle H_N \rangle_{\mathrm{num}} \sim \sqrt{1.67\, N}$$

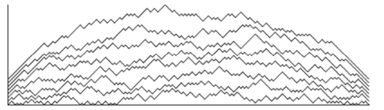
Summary I

- Connection between watermelons and random matrices

$\Longrightarrow$ exact asymptotic results for $\langle H_N \rangle$ for $N \gg 1$



$$\langle H_N \rangle \sim \sqrt{N}$$

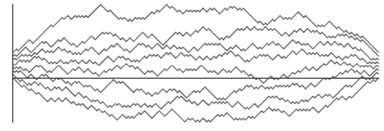


$$\langle H_N \rangle \sim \sqrt{2\,N}$$

Summary I

- Connection between watermelons and random matrices

$\Longrightarrow$ exact asymptotic results for $\langle H_N \rangle$ for $N \gg 1$



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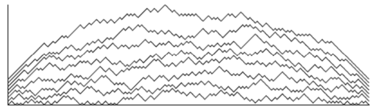


$$\langle H_N \rangle \sim \sqrt{2N}$$

What about the fluctuations of H_N ?

Fluctuations of H_N in the limit of large N

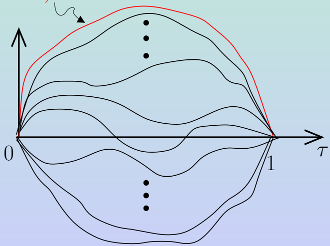
- Consequences for watermelons without wall for large N

$$\frac{x_N(\tau)}{\sqrt{2\tau(1-\tau)}} \sim \sqrt{2N} + \frac{N^{-1/6}}{\sqrt{2}}\chi_2$$

$$\text{Proba}[\chi_2 \leq \xi] = \mathcal{F}_2(\xi), \text{ Tracy-Widom distribution for } \beta = 2$$

- When $N \to \infty$, $x_N(\tau)$ reaches a deterministic elliptic shape

$x_N(\tau) \sim 2\sqrt{N}\sqrt{\tau(1-\tau)}$



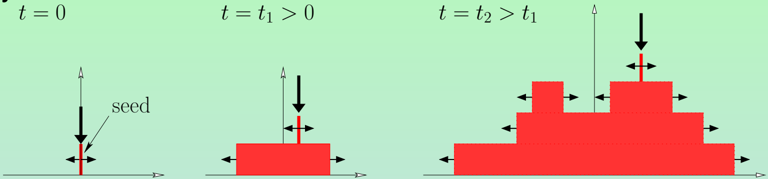
Fluctuations

$$x_N(\tau = 1/2) - \sqrt{N} \sim N^{-\frac{1}{6}}$$

Outline

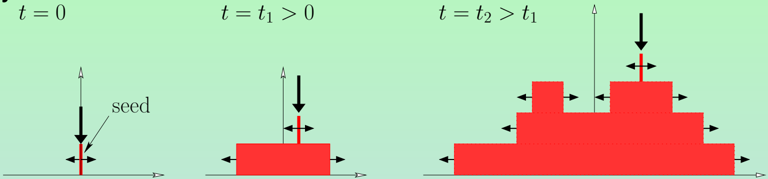
Curved growing interface : the PNG droplet

- Polynuclear Growth Model

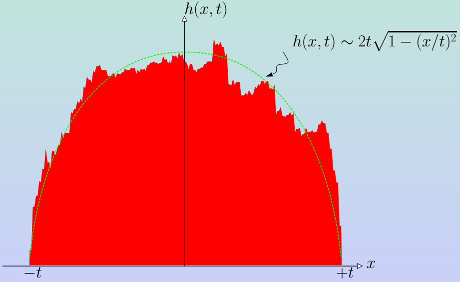


Curved growing interface : the PNG droplet

- Polynuclear Growth Model



- At large time t the profile becomes droplet-like



Curved growing interface : the PNG droplet

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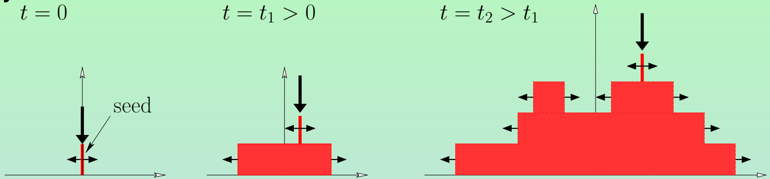

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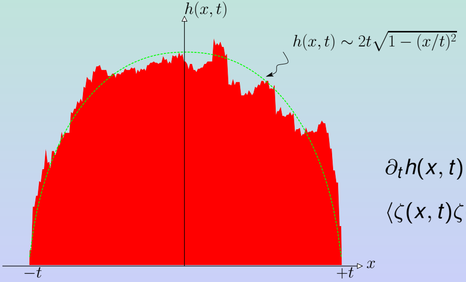


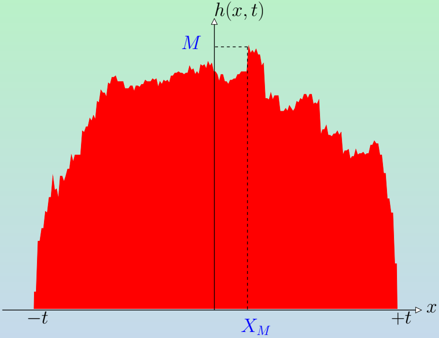
Fluctuations : KPZ equation

$$\partial_t h(x,t) = \nu \nabla^2 h(x,t) + \frac{\lambda}{2}(\nabla h(x,t))^2 + \zeta(x,t)$$

$$\langle \zeta(x,t)\zeta(x',t')\rangle = D\delta(x-x')\delta(t-t')$$

Curved growing interface : the PNG droplet

- Fluctuations : focus on extreme statistics



Curved growing interface : the PNG droplet

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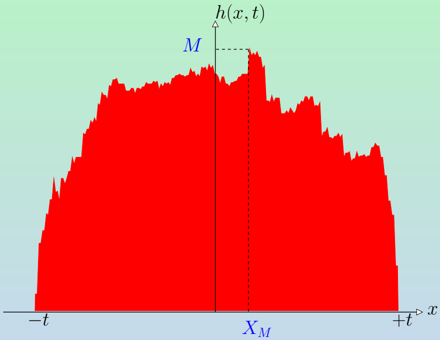
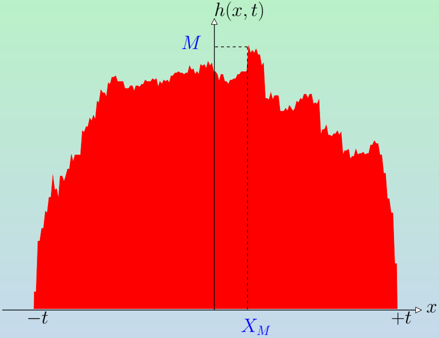


KPZ scaling

$$M - 2t \sim t^{1/3}$$
$$X_M \sim t^{2/3}$$

Curved growing interface : the PNG droplet

- Fluctuations : focus on extreme statistics



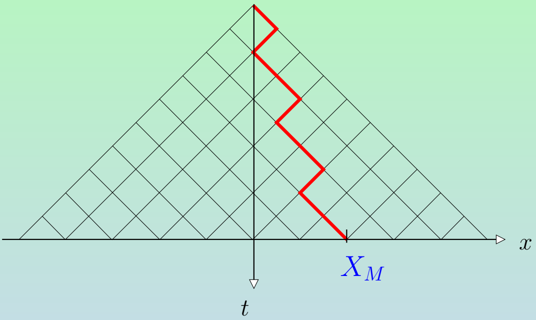
KPZ scaling

$$M - 2t \sim t^{1/3}$$
$$X_M \sim t^{2/3}$$

What is the joint distribution of M, X_M ?

Connection with the Directed Polymer (DPRM)

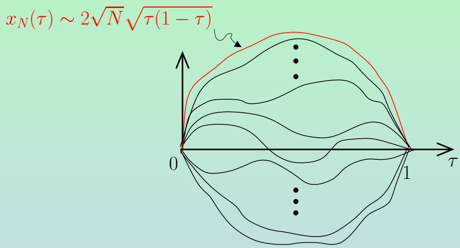
- DP in random media with one free end ("point to line")

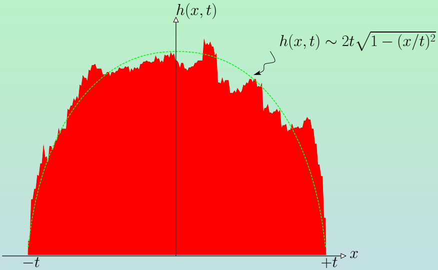


$$E(\mathcal{C}) = \sum_{\langle i,j \rangle \in \mathcal{C}} \epsilon_{ij}$$

- $M \equiv -$Energy of the optimal polymer
- $X_M \equiv$ Transverse coordinate of the optimal polymer

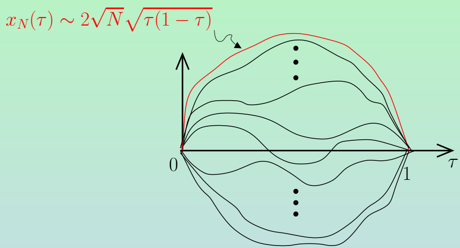
Vicious walkers and PNG droplet





$$x_N \iff h$$
$$\tau \iff x$$
$$N \iff t$$

watermelons



PNG droplet

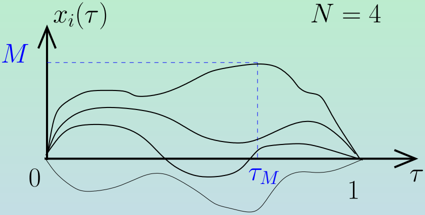
$$\frac{h(u t^{\frac{2}{3}}, t) - 2t}{t^{\frac{1}{3}}} \overset{d}{=} \frac{2\left[x_N(\frac{1}{2} + \frac{u}{2}N^{-\frac{1}{3}}) - \sqrt{N}\right]}{N^{-\frac{1}{6}}} \overset{d}{=} \mathcal{A}_2(u) - u^2$$

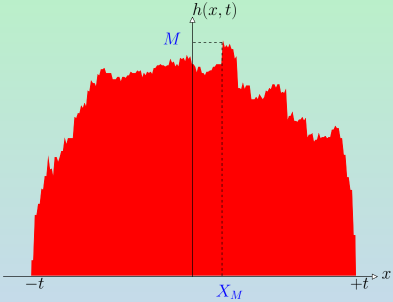
Prähofer & Spohn '00

$\mathcal{A}_2(u) \equiv$ Airy$_2$ process

Vicious walkers and PNG droplet

- Use this correspondence to study extreme statistics of PNG



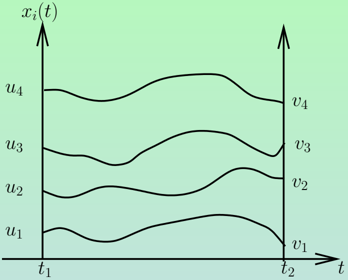


HERE: • exact computation of the distribution $P_N(M)$

• $P_N(M)$ in the $N \to \infty$ limit

Outline

Transition probability



$$\mathcal{P}_N(v_1, v_2, \cdots, v_N, t_2 | u_1, u_2, \cdots, u_N, t_1) = ?$$

Karlin-Mc Gregor (1959), Lindström (1973), Gessel-Viennot (1985)

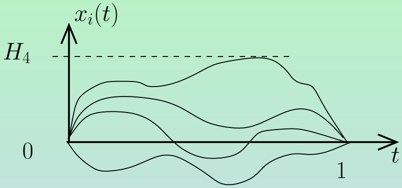
- The case $N = 2$: a reflection principle

$$\begin{aligned} \mathcal{P}_2(v_1, v_2, t_2 | u_1, u_2, t_1) &= \mathbf{P}_1(v_1, t_2 | u_1, t_1) \mathbf{P}_1(v_2, t_2 | u_2, t_1) \\ &\quad - \mathbf{P}_1(v_2, t_2 | u_1, t_1) \mathbf{P}_1(v_1, t_2 | u_2, t_1) \\ &= \mathbf{P}_2(v_1, v_2, t_2 | u_1, u_2, t_1) - \mathbf{P}_2(v_2, v_1, t_2 | u_1, u_2, t_1) \end{aligned}$$

where $\mathbf{P}_2(.|.) \equiv$ transition probability for two free particles (allowed to cross each other)

Watermelons configurations : regularization procedure

- Brownian motion has an infinite density of zero-crossings



Such configurations are ill-defined for Brownian motion :

$$x_i(0) = x_{i+1}(0)$$
$$\text{AND } x_i(t = 0^+) < x_{i+1}(t = 0^+)$$

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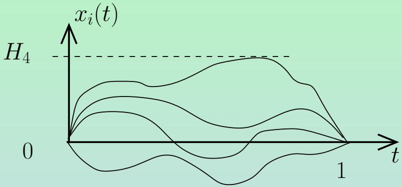
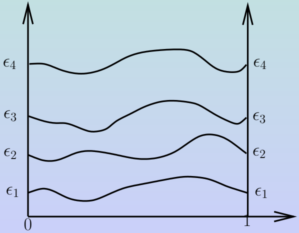


Such configurations are ill-defined for Brownian motion :

$$x_i(0) = x_{i+1}(0)$$
$$\text{AND } x_i(t = 0^+) < x_{i+1}(t = 0^+)$$

- A need for regularization : introduce cut-offs ϵ_i's

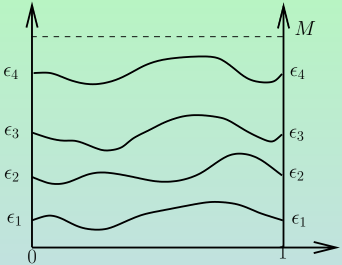


Only at the end take the limit
$$\epsilon_i \to 0$$

Distribution of the maximal height : without wall

- Cumulative distribution of the maximal height

$$F_N(M) = \Pr\left[x_N(\tau) \leq M\ ,\ \forall 0 \leq \tau \leq 1\right]$$



- Path integral (Feynman-Kac formula) for free fermions

$$F_N(M) = \frac{2^{-\binom{N}{2}}}{\prod_{j=0}^{N-1} j!} \det_{1 \leq i,j \leq N} \left[(-1)^{i-1} H_{i+j-2}(0) - H_{i+j-2}(\sqrt{2}M) e^{-2M^2}\right]$$

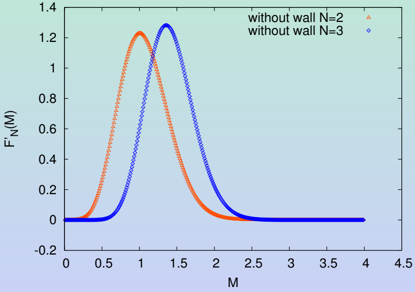
where $H_n(M) \equiv$ Hermite Polynomials   see also T. Feierl '08

Distribution of the maximal height : without wall

$$F_N(M) = \frac{2^{-\binom{N}{2}}}{\prod_{j=0}^{N-1} j!} \det_{1\leq i,j\leq N} \left[(-1)^{i-1} H_{i+j-2}(0) - H_{i+j-2}(\sqrt{2}M) e^{-2M^2}\right]$$

- Shape and asymptotic behavior

$$F_N(M) \propto M^{N^2+N} \; , \; M \to 0$$
$$1 - F_N(M) \sim e^{-2M^2} \; , \; M \to \infty$$



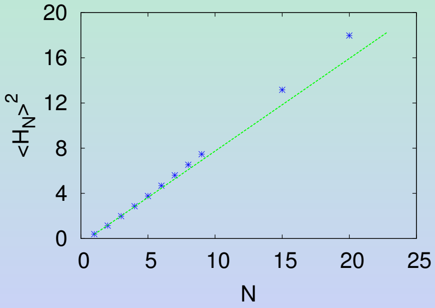
Exact value of the $\langle H_N \rangle$

$$
\begin{aligned}
\langle H_1 \rangle &= \frac{\sqrt{\pi}}{4}\sqrt{2} \\
\langle H_2 \rangle &= \frac{\sqrt{\pi}}{4}\left(1+\sqrt{2}\right) \\
\langle H_3 \rangle &= \frac{\sqrt{\pi}}{96}\left(45+36\sqrt{2}-8\sqrt{6}\right) \\
\langle H_4 \rangle &= \frac{\sqrt{\pi}}{20736}\left(17091+5184\sqrt{2}-1888\sqrt{6}\right) \\
&\ldots
\end{aligned}
$$

Comparison with numerics by Bonichon & Mosbah

Numerical estimate by Bonichon & Mosbah

$$\langle H_N \rangle^2_{\text{num}} \sim 0.82N$$



Exact behavior at large N

$$\langle H_N \rangle^2 \sim N$$

Distribution of the maximal height : with a wall

- Cumulative distribution of the maximal height

G. S., S. N. Majumdar, A. Comtet, J. Randon-Furling '08

$$F_N(M) = \frac{A_N}{M^{2N^2+N}} \sum_{n_1,\cdots,n_N=0}^{+\infty} \prod_{i=1}^{N} n_i^2 \prod_{1\leq j<k\leq N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2M^2}\sum_{i=1}^N n_i^2}$$

$$A_N = \frac{\pi^{2N^2+N}}{2^{N^2-N/2}\prod_{j=0}^{N-1}\Gamma(2+j)\Gamma(\frac{3}{2}+j)} \text{ cf Selberg integral}$$

see also T. Feierl '08 and '12, N. Kobayashi *et al.* '08

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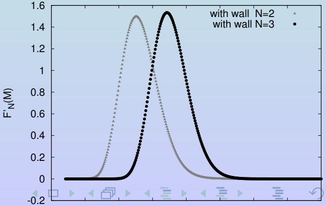
see also T. Feierl '08 and '12, N. Kobayashi *et al.* '08

- Shape and asymptotic behavior

$$F_N(M) \sim \frac{\alpha_N}{M^{2N^2+N}} e^{-\frac{\pi^2}{12M^2}N(N+1)(2N+1)} \;, \; M \to 0$$

$$1 - F_N(M) \sim e^{-2M^2} \;, \; M \to \infty$$



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see also T. Feierl '08 and '12, N. Kobayashi *et al.* '08

What about the asymptotic behavior of $F_N(M)$ for $N \to \infty$?

Outline

Discrete orthogonal polynomials

- Large N analysis of

$$F_N(M) = \frac{A_N}{M^{2N^2+N}} \sum_{n_1,\cdots,n_N=0}^{+\infty} \prod_{i=1}^{N} n_i^2 \prod_{1\le j<k\le N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2M^2}\sum_{i=1}^N n_i^2}$$

- Discrete orthogonal polynomials

$$\sum_{n=-\infty}^{\infty} p_k(n) p_{k'}(n) e^{-\frac{\pi^2}{2M^2}n^2} = \delta_{k,k'} h_k \ ,$$

$$p_k(n) = n^k + \cdots$$

- Useful expression for asymptotic anaysis

$$F_N(M) = \frac{B_N}{M^{2N^2+N}} \prod_{k=1}^{N} h_{2k-1}$$

Large N limit for $F_N(M)$

- For large N, in the "double-scaling limit" P. J. Forrester, S. N. Majumdar, G. S. '11

$$\frac{d^2}{dt^2} \log F_N\left(\sqrt{2N}(1 + t/(2^{7/3}N^{2/3}))\right) = -\frac{1}{2}\left(q^2(t) - q'(t)\right)$$
$$q''(t) = 2q^3(t) + tq(t)\ ,\ q(t) \sim \mathrm{Ai}(t)\ ,\ t \to \infty$$

i.e.

$$F_N(M) \to \mathcal{F}_1\left(2^{11/6}p^{1/6}\left|M - \sqrt{2N}\right|\right)$$
$$\mathcal{F}_1(t) = \exp\left(-\frac{1}{2}\int_t^\infty \left((s-t)\,q^2(s) - q(s)\right)\,ds\right)$$
$$\equiv \text{Tracy-Widom distribution for GOE}$$

Large N limit for $F_N(M)$

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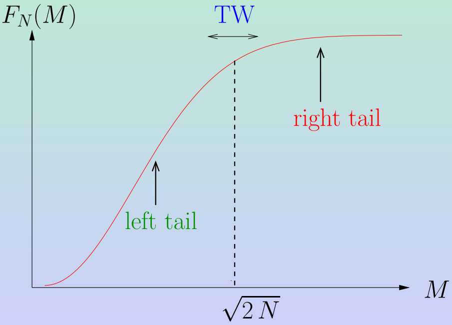

Outline

Conclusion

- Exact results for $\langle H_N \rangle$ for large N
- Exact result for distribution of the maximal height H_N using path integral techniques
- Connections with stochastic growth models for large N
- Large N limit

$$F_N(M) \to \mathcal{F}_1\left(2^{11/6} N^{1/6} \left| M - \sqrt{2N} \right|\right) \ , \ N \to \infty$$

see also Liechty'11 for a recent rigorous proof