Partitions of direct products of complete graphs into independent dominating sets

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Domination in graphs

- Let $G = (V, E)$ be a finite undirected graph without loops. A set $S \subseteq V$ is called a *dominating set* of G if for every vertex $v \in V \setminus S$ there exists a vertex $u \in S$ such that u is adjacent to v.
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- The minimum cardinality of a dominating set in a graph G is called the *domination number* of G, and is denoted $\gamma(G)$.
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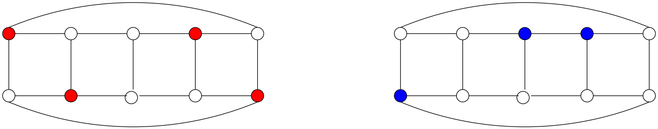

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Mathematical History of Domination in Graphs

- ☒ In 1862 C. F. De Jaenisch studied the problem of determining the minimum number of queens which are necessary to cover (or dominate) an $n \times n$ chessboard.
- ☒ In 1892 W. W. Rouse Ball reported that chess enthusiast in the late 1800s studied, among others, the following problems:
  - ⋆ Covering: what is the minimum number of chess pieces of a given type which are necessary to cover / attack / dominate every square of an $n \times n$ board ? (Ex. of min. dominating set).
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- In 1958 C. Berge defined for the first time the concept of the domination number of a graph (see also O. Ore 1962).
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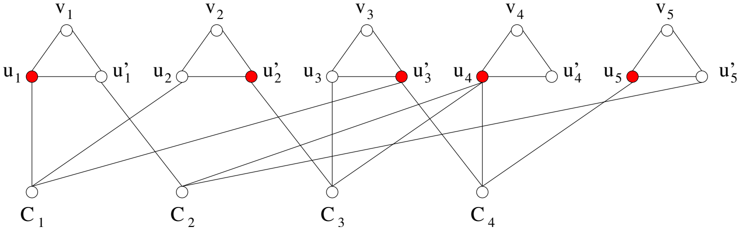

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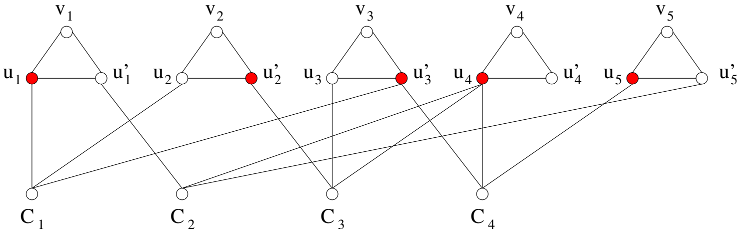

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$\star$ Trivially, $id(G) \leq \delta(G) + 1$, where $\delta(G)$ denote the minimum degree of any vertex in G.

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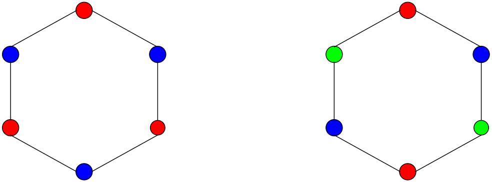

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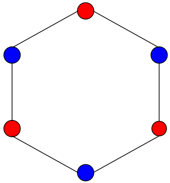

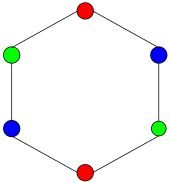

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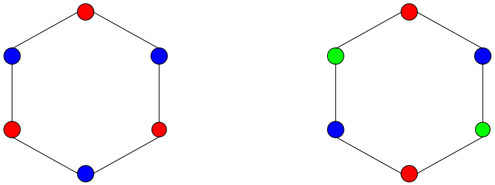

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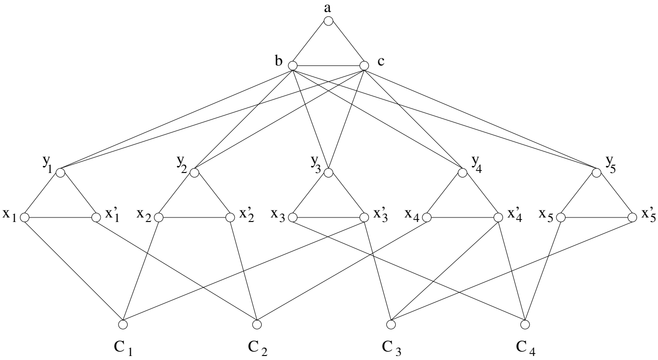

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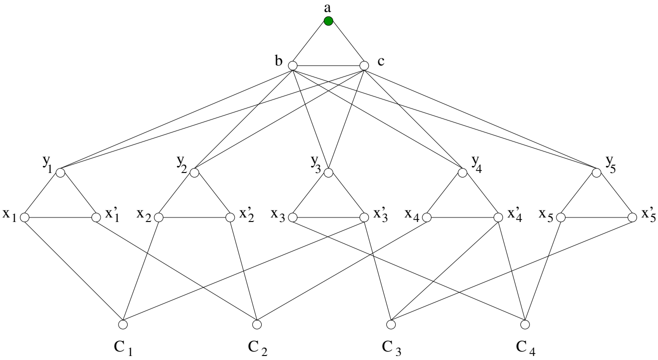

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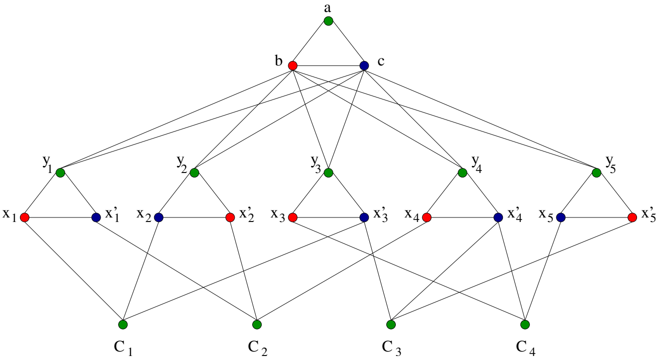

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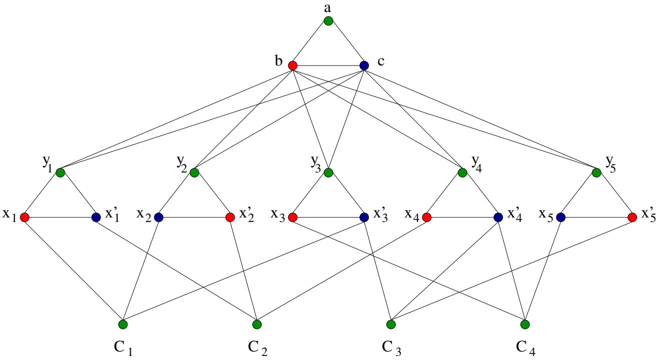

Problems IP and IkP are NP-complete [Dunbar et al.,00].

Graph Products

- The *direct product* $G \times H$ of two graphs G and H is defined by $V(G \times H) = V(G) \times V(H)$, and where two vertices $(u_1, u_2), (v_1, v_2)$ are joined by an edge in $E(G \times H)$ if $\{u_1, v_1\} \in E(G)$ and $\{u_2, v_2\} \in E(H)$.
- Let G and H be two graphs. An homomorphism ψ from G to H is an application from $V(G)$ to $V(H)$ which preserves adjacencies.
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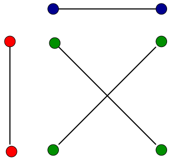

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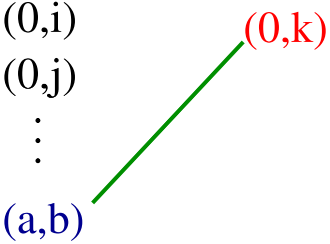

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[Dunbar et al., 00] For any integers $m, n \geq 2$, $K_m \times K_n$ has only idomatic k-partitions, where $k \in \{m, n\}$.

Idomatic partitions of $\times_{i=0}^{2} K_{n_i}$

- **Ex.** The graph $K_2 \times K_3 \times K_4$ has an idomatic 6-partition.

(0,0,0)	(0,1,0)	(0,2,0)	(0,0,2)	(0,1,2)	(0,2,2)
(0,1,1)	(0,2,1)	(0,0,1)	(0,1,3)	(0,2,3)	(0,0,3)
(1,0,1)	(1,1,1)	(1,2,1)	(1,0,3)	(1,1,3)	(1,2,3)
(1,1,0)	(1,2,0)	(1,0,0)	(1,1,2)	(1,2,2)	(1,0,2)

- **Question.** For which values of k there exists an idomatic k-partition of the direct product of three or more complete graphs ?

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(1,0,1)	(1,1,1)	(1,2,1)	(1,0,3)	(1,1,3)	(1,2,3)
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(0,1,1)	(0,2,1)	(0,0,1)	(0,1,3)	(0,2,3)	(0,0,3)
(1,0,1)	(1,1,1)	(1,2,1)	(1,0,3)	(1,1,3)	(1,2,3)
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- **Question.** For which values of k there exists an idomatic k-partition of the direct product of three or more complete graphs ?

Idomatic sets of $K_{n_0} \times K_{n_1} \times K_{n_2}$

- Let Γ be a group and C a subset of Γ (i.e. *the connector set*) closed under inverses and identity free. The *Cayley graph* $\mathrm{Cay}(\Gamma, C)$ is the graph with Γ as its vertex set, two vertices u and v being joined by an edge if and only if $u^{-1}v \in C$. Ex. cycles, complete graphs, etc. Cayley graphs constitute a rich class of vertex-transitive graphs.
- Let $t \geq 1$ be an integer and let $n_1, n_2, \ldots, n_t$ be positive integers. The graph $G = K_{n_1} \times K_{n_2} \times \ldots \times K_{n_t}$ can be seen as the Cayley graph of the direct product group $\mathcal{G} = Z_{n_1} \times Z_{n_2} \times \ldots \times Z_{n_t}$ with connector set $[n_1] \setminus \{0\} \times \ldots \times [n_t] \setminus \{0\}$, where Z_{n_i} denotes the additive cyclic group of integers modulo n_i.

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- H_1: Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$, and let I be an independent dominating set in G. If the set I contains at least two vertices of G agreeing in exactly two positions, then I is equal to the set $[n_s] \times \{i\} \times [n_t]$ for some $i \in [n_p]$, with $s, t, p \in [3]$ and s, t and p pairwise different.
- H_2: Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$, and let I be an independent set of G such that no two vertices in it agreeing in exactly two positions. Thus, the set I is a dominating set of G if and only if

$$I = \{(\alpha_0, \alpha_1, \alpha_2), (\alpha_0, \beta_1, \beta_2), (\beta_0, \alpha_1, \beta_2), (\beta_0, \beta_1, \alpha_2)\},$$

for some $\alpha_i, \beta_i \in [n_i]$, with $\alpha_i \neq \beta_i$ and $i \in [3]$.

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Idomatic sets of $K_{n_0} \times K_{n_1} \times K_{n_2}$

- **Def.** Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$, and let I be an independent dominating set in G. The set I is said to be of **Type A** if it verifies the hypothesis H_1 and it is said to be of **Type B** if it verifies the hypothesis H_2.
- Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$, and let I be an independent set in G. Then, I is also a dominating set in G if and only if it is of Type A or Type B.

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Idomatic partitions of $K_{n_0} \times K_{n_1} \times K_{n_2}$

- **Def.:** Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$, and let $G_1, G_2, \ldots, G_t$ be an idomatic t-partition of G, with $t > 1$. Such an idomatic partition is called
  - of **Type A**: If all independent dominating sets G_i are of Type A.
  - of **Type B**: If all independent dominating sets G_i are of Type B.
  - of **Type C**: If there is at least one independent dominating set G_i of Type A, and at least one independent dominating set G_j of Type B, with $i \neq j$.

(0,0,0)	(0,0,1)	(0,0,2)	(0,0,3)
(0,1,1)	(0,1,2)	(0,1,3)	(0,1,0)
(1,0,1)	(1,0,2)	(1,0,3)	(1,0,0)
(1,1,0)	(1,1,1)	(1,1,2)	(1,1,3)

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(0,0,0)	(0,0,1)	(0,0,2)	(0,0,3)
(0,1,1)	(0,1,2)	(0,1,3)	(0,1,0)
(1,0,1)	(1,0,2)	(1,0,3)	(1,0,0)
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- Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$. Then, G has an idomatic n_i-partition of Type A for each $i \in [3]$. Moreover, such partitions are the only idomatic partitions of Type A of G.
- Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$. If G has an idomatic partition of Type B then there exist $j, k \in [3]$, with $j \neq k$, such that n_j and n_k are both even.
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Idomatic partitions of Type B for $\times_{i=0}^{2} K_{n_i}$

- Let n_1, n_2 be even and let $\mathcal{G} = \mathbb{Z}_{n_0} \times \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ be a group.
- Let $< a_i >$ be a cyclic subgroup of order $n_i/2$ in $\mathbb{Z}_{n_i}$, for $i = 1, 2$.
- Let $\mathcal{P} =< (1,0,0) > . < (0,a_1,0) > . < (0,0,a_2) >$ be the subgroup of $\mathcal{G}$ induced by the join of the cyclic subgroups $< (1,0,0) >$, $< (0,a_1,0) >$ and $< (0,0,a_2) >$ of $\mathcal{G}$.
- Let $\mathcal{P} = \{p_1, \ldots, p_r\}$, with $p_1 = (0,0,0)$ and $r = \prod n_i/4$. Then, $\mathcal{P}$, $\mathcal{P} + (0,1,1)$, $\mathcal{P} + (1,0,1)$, and $\mathcal{P} + (1,1,0)$ is a partition of $\mathcal{G}$ into cosets of $\mathcal{P}$.
- $\times K_{n_i} \cong \text{Cay}(\times \mathbb{Z}_{n_i}, \times([n_i] \setminus \{0\}))$. Indeed, for any vertices $a, b, c \in \times K_{n_i}$, we have that that $a + b \sim a + c$ iff $b \sim c$.

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Idomatic partitions of $K_{n_0} \times K_{n_1} \times K_{n_2}$

- Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$. Then, G has an idomatic partition of Type B if and only if there exist $j, k \in [3]$, with $j \neq k$, such that n_j and n_k are both even.
- Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$, and let q_1, q_2 be two positive integers. Then, G has an idomatic $(q_1 + q_2)$-partition of Type C if and only if there exists $i \in [3]$ such that $n_i - q_1 > 1$ and $K_{n_j} \times K_{n_k} \times K_{n_i - q_1}$ admits an idomatic q_2-partition of Type B, with $j, k, i \in [3]$ and j, k, i pairwise different.
- Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$. If $\mathcal{I}$ is an idomatic partition of G, then $\mathcal{I}$ must be of Type A, B or C.

Idomatic partitions of $K_{n_0} \times K_{n_1} \times K_{n_2}$

- Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$. Then, G has an idomatic partition of Type B if and only if there exist $j, k \in [3]$, with $j \neq k$, such that n_j and n_k are both even.
- Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$, and let q_1, q_2 be two positive integers. Then, G has an idomatic $(q_1 + q_2)$-partition of Type C if and only if there exists $i \in [3]$ such that $n_i - q_1 > 1$ and $K_{n_j} \times K_{n_k} \times K_{n_i - q_1}$ admits an idomatic q_2-partition of Type B, with $j, k, i \in [3]$ and j, k, i pairwise different.
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Idomatic number of $\times_{i=0}^{2} k_{n_i}$

- Let $G = \times_{i=0}^{2} k_{n_i}$, with $n_i \geq 2$, and let $id(G)$ denote the idomatic number of graph G. Let $t = \max\{n_0, n_1, n_2\}$. Then,
  1. If n_i is an odd integer for all $i \in [3]$, then $id(G) = t$.
  2. If n_i is an even integer and $n_j \leq n_k$ are odd integers, with $i, j, k \in [3]$ and i, j and k pairwise different, then $id(G) = \max\{t, \frac{n_i.n_j.(n_k-1)}{4} + 1\}$.
  3. If n_i and n_j are even integers, with $i, j \in [3]$ and $i \neq j$, then $id(G) = \frac{n_i.n_j.n_k}{4}$.

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Open Problems

- Let $G = \times_{i=1}^{k} K_{n_i}$ and let $u = (u_1, \ldots, u_k)$ and $v = (v_1, \ldots, v_k)$ be vertices of G. Then let $e(u,v) = |\{i : u_i = v_i\}|$. Thus $u \sim v$ iff $e(u,v) = 0$.
- Let $X \subset V(G)$ and let $\{e(u,v) : u, v \in X, u \neq v\} = \{j_1, \ldots, j_r\}$. Then, we say that X is a $T_{\{j_1,\ldots,j_r\}}$-set.
- [Klavzar et al.,10] if I is an idomatic set of $\times_{i=0}^{3} K_{n_i}$ then, I is either a $T_{\{1\}}$ or $T_{\{1,2\}}$ or $T_{\{1,2,3\}}$-set. Indeed, for each one of these T sets, there exists an idomatic partition of G composed of such T sets.
- [Conjecture 1] For $k > 3$, if I is an idomatic set of $G = \times_{i=0}^{k} K_{n_i}$ then, I is a $T_{\{1,\ldots,i\}}$ for some $1 \leq i < k$. Indeed, for each i, there exists an idomatic $T_{\{1,\ldots,i\}}$-set and there exists an idomatic partition of G composed of such T sets.

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Relaxed Idomatic partitions : b-colorings

- Observation Let ϕ be a proper coloring of $G = K_n \times k_m$, with $m, n \geq 2$. Then, ϕ is a b-coloring of G iff ϕ is an idomatic partition of G.
- Forbidden Configurations for b-colorings:

- [Problem] Let $G = \times_{i=1}^{k} K_{n_i}$, with $k > 2$ and $n_i \geq 2$. Is it any b-coloring of G an idomatic partition of G ?

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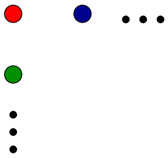
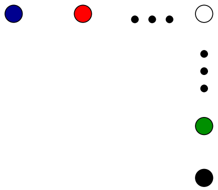


Configuration A



Configuration B

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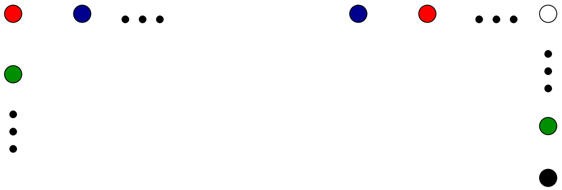

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Thank You !