Roots $x_k(y)$ of a formal power series $$f(x,y) = \sum_{n=0}^{\infty} a_n(y)\, x^n$$ with applications to graph enumeration and q-series

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Dedicated to the memory of Philippe Flajolet

LECTURE #3

The leading root of the partial theta function

The basic set-up, reviewed

- Start from a formal power series

$$f(x,y) \;=\; \sum_{n=0}^{\infty} a_n(y)\, x^n$$

  where

  (a) $a_0(0) = a_1(0) = 1$
  (b) $a_n(0) = 0$ for $n \ge 2$
  (c) $a_n(y) = O(y^{\nu_n})$ with $\lim\limits_{n\to\infty} \nu_n = \infty$

  and coefficients lie in a commutative ring-with-identity-element R.

- There exists a unique formal power series $x_0(y) \in R[[y]]$ satisfying $f(x_0(y), y) = 0$. We call $x_0(y)$ the **leading root** of f.

- Since $x_0(y)$ has constant term -1, we write $x_0(y) \;=\; -\xi_0(y)$ where $\xi_0(y) = 1 + O(y)$.

- We saw in Lecture #2 that $\xi_0(y)$ can be computed by
  - An elementary method.
  - A method based on the explicit implicit function formula.
  - A method based on the exponential formula.

Method based on the explicit implicit function formula

- In Lecture #2 we derived the formula

$$\frac{\xi_0(y)^\beta - 1}{\beta} = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n_1,\ldots,n_m \ge 0} \binom{\beta - 1 + \sum n_i}{m-1} \prod_{i=1}^{m} (-1)^{n_i} \, \widehat{a}_{n_i}(y)$$

where

$$\widehat{a}_n(y) = \begin{cases} a_n(y) - 1 & \text{for } n = 0, 1 \\ a_n(y) & \text{for } n \ge 2 \end{cases}$$

- Can this formula be used for proofs of nonnegativity???
- Recall the definition: $\xi_0(y) \in \mathcal{S}_\beta$ in case $\dfrac{\xi_0(y)^\beta - 1}{\beta} \succeq 0$ (coefficientwise nonnegativity)
- *Empirically* I know that $\xi_0(y) \in \mathcal{S}_\beta$ when $a_n(y) = \alpha_n \, y^{n(n-1)/2}$ and

  (a) $\beta \ge -2$ with $\alpha_n = 1$ (partial theta function)
  (b) $\beta \ge -1$ with $\alpha_n = 1/n!$ (deformed exponential function)
  (c) $\beta \ge -1$ with $\alpha_n = (1-q)^n/(q;q)_n$ and $q > -1$

- **How can we see these facts from this formula???**
  [open combinatorial problem]
- All these examples have $\widehat{a}_n(y) \succeq 0$. The factors $(-1)^{n_i}$ then seem to cause trouble.

A very simple case: Alternating signs

Proposition. Suppose that

$$(-1)^n \widehat{a}_n(y) \ \succeq \ 0 \qquad \text{for all } n \geq 0$$

where

$$\widehat{a}_n(y) \ = \ \begin{cases} a_n(y) - 1 & \text{for } n = 0, 1 \\ a_n(y) & \text{for } n \geq 2 \end{cases}$$

Then $\xi_0(y) \in \mathcal{S}_\beta$ and in fact

$$\frac{\xi_0(y)^\beta - 1}{\beta} \ \succeq \ \sum_{n=0}^{\infty} (-1)^n \, \widehat{a}_n(y)$$

in the following cases:

(a) $\beta = 1$

(b) $\beta = -1$ whenever $a_0(y) = 1$

(c) $\beta = -3$ whenever $a_0(y) = a_1(y) = 1$

(d) $\beta = -(2k-1)$ whenever $a_0(y) = a_1(y) = 1$ and $a_2(y) = \ldots = a_{k-1}(y) = 0$

Proof. Follows almost immediately from

$$\frac{\xi_0(y)^\beta - 1}{\beta} \ = \ \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n_1,\ldots,n_m \geq 0} \binom{\beta - 1 + \sum n_i}{m-1} \prod_{i=1}^{m} (-1)^{n_i} \, \widehat{a}_{n_i}(y)$$

(a) Set $\beta = 1$. Then the RHS of the Proposition comes from the term $m = 1$. All the other terms are $\succeq 0$ since $(-1)^n \, \widehat{a}_n(y) \succeq 0$ and $\binom{\beta - 1 + \sum n_i}{m-1} \geq 0$.

(b) Set $\beta = -1$ and observe that the sum can be restricted to $n_1, \ldots, n_m \geq 1$. If $m = 1$ we have $\binom{\beta - 1 + \sum n_i}{m-1} = 1$ and we get the RHS of the Proposition. If $m \geq 2$ we have $\sum n_i \geq 2$, so that $\beta - 1 + \sum n_i$ is a nonnegative integer and hence $\binom{\beta - 1 + \sum n_i}{m-1} \geq 0$.

(c) is analogous to (b), but using $\beta = -3$ and observing that the sum can be restricted to $n_1, \ldots, n_m \geq 2$, so that $m \geq 2$ implies $\sum n_i \geq 4$.

(d) is analogous to (b), but using $\beta = -(2k-1)$ and observing that the sum can be restricted to $n_1, \ldots, n_m \geq k$, so that $m \geq 2$ implies $\sum n_i \geq 2k$.

A slight strengthening (by rescaling of f)

Corollary.

(a) If $(-1)^n \dfrac{a_n(y)}{a_1(y)} \succeq 0$ for all $n \neq 1$, then $\xi_0(y) \in \mathcal{S}_1$ and satisfies

$$\xi_0(y) \;\succeq\; \frac{a_0(y)}{a_1(y)} + \sum_{n=2}^{\infty} (-1)^n \frac{a_n(y)}{a_1(y)} \, .$$

(b) If $1 - \dfrac{a_1(y)}{a_0(y)} \succeq 0$ and $(-1)^n \dfrac{a_n(y)}{a_0(y)} \succeq 0$ for all $n \geq 2$, then $\xi_0(y) \in \mathcal{S}_{-1}$ and satisfies

$$\xi_0(y)^{-1} \;\preceq\; \frac{a_1(y)}{a_0(y)} - \sum_{n=2}^{\infty} (-1)^n \frac{a_n(y)}{a_0(y)} \, .$$

Proof.

(a) Apply part (a) of the Proposition to $f(x,y)/a_1(y)$.
(b) Apply part (b) of the Proposition to $f(x,y)/a_0(y)$.

Alternative (elementary) proof of the Corollary

- No need to use explicit implicit function formula. Just bare hands!
- **Proof of part (a):** Start from the equation $f(-\xi_0(y), y) = 0$, divide by $a_1(y)$, and add $\xi_0(y)$ to both sides:

$$\xi_0(y) \;=\; \frac{a_0(y)}{a_1(y)} \;+\; \sum_{n=2}^{\infty} (-1)^n \, \frac{a_n(y)}{a_1(y)} \, \xi_0(y)^n$$

- The unique solution to this equation can be found iteratively as follows: Define a map $\mathcal{F}\colon\; R[[y]] \to R[[y]]$ by

$$(\mathcal{F}\xi)(y) \;=\; \frac{a_0(y)}{a_1(y)} \;+\; \sum_{n=2}^{\infty} (-1)^n \, \frac{a_n(y)}{a_1(y)} \, \xi(y)^n$$

and define a sequence $\xi_0^{(0)}, \xi_0^{(1)}, \ldots \in R[[y]]$ by $\xi_0^{(0)} = 1$ and $\xi_0^{(k+1)} = \mathcal{F}\xi_0^{(k)}$. I then claim that

$$\xi_0^{(0)} \;\preceq\; \xi_0^{(1)} \;\preceq\; \xi_0^{(2)} \;\preceq\; \cdots \;\preceq\; \xi_0$$

and that

$$\xi_0^{(k)}(y) \;=\; \xi_0(y) \;+\; O(y^{k+1}) \; .$$

Proof of claim:

- If $f(y)$ and $g(y)$ are formal power series satisfying $0 \preceq f \preceq g$, then the hypotheses of the Corollary [part (a)] guarantee that $0 \preceq \mathcal{F}f \preceq \mathcal{F}g$.
- Applying this repeatedly to the obvious inequality $0 \preceq \xi_0^{(0)} \preceq \xi_0^{(1)}$, we obtain $\xi_0^{(0)} \preceq \xi_0^{(1)} \preceq \xi_0^{(2)} \preceq \cdots$.

- Likewise, if $f(y)$ and $g(y)$ are formal power series satisfying $f(y)-g(y) = O(y^\ell)$ for some $\ell \ge 0$, then it is easy to see that $(\mathcal{F}f)(y) - (\mathcal{F}g)(y) = O(y^{\ell+1})$ [since $a_n(y)/a_1(y) = O(y)$ for all $n \ge 2$].
- Applying this repeatedly to the obvious fact $\xi_0^{(1)}(y) - \xi_0^{(0)}(y) = O(y)$, we obtain $\xi_0^{(k+1)}(y) - \xi_0^{(k)}(y) = O(y^{k+1})$.
- It follows that $\xi_0^{(k)}(y)$ converges as $k \to \infty$ (in the topology of formal power series) to a limiting series $\xi_0^{(\infty)}(y)$, and that this limiting series satisfies $\mathcal{F}\xi_0^{(\infty)} = \xi_0^{(\infty)}$. But this means that $\xi_0^{(\infty)}(y) = \xi_0(y)$. It also follows that $\xi_0^{(k)}(y) = \xi_0(y) + O(y^{k+1})$. The inequality of the Corollary is precisely the statement $\xi_0 \succeq \xi_0^{(1)}$.

- The proof of part (b) is similar.
- Can parts (c) and (d) of the Proposition be given a similarly elementary proof?
- Can results analogous to the Proposition be proven for the spaces $\mathcal{S}_\beta$ with $\beta \neq 1, -1, -3, -5, \ldots$?

But isn't the case of alternating signs *too trivial*?

- After all, the most interesting examples have *constant signs*.
- Then the irritating factors $(-1)^{n_i}$ cannot be avoided.

The partial theta function $\Theta_0(x, y) = \sum_{n=0}^{\infty} x^n \, y^{n(n-1)/2}$

(which has *constant signs*!)

It seems that $\xi_0(y) \in \mathcal{S}_1$:

$$\begin{aligned}\xi_0(y) &= 1 + y + 2y^2 + 4y^3 + 9y^4 + 21y^5 + 52y^6 + 133y^7 + 351y^8 \\ &\qquad +948y^9 + 2610y^{10} + \ldots + \text{terms through order } y^{6999}\end{aligned}$$

and indeed that $\xi_0(y) \in \mathcal{S}_{-1}$:

$$\begin{aligned}\xi_0(y)^{-1} &= 1 - y - y^2 - y^3 - 2y^4 - 4y^5 - 10y^6 - 25y^7 - 66y^8 \\ &\qquad -178y^9 - 490y^{10} - \ldots - \text{terms through order } y^{6999}\end{aligned}$$

and indeed that $\xi_0(y) \in \mathcal{S}_{-2}$:

$$\begin{aligned}\xi_0(y)^{-2} &= 1 - 2y - y^2 \qquad - y^4 - 2y^5 - 7y^6 - 18y^7 - 50y^8 \\ &\qquad -138y^9 - 386y^{10} - \ldots - \text{terms through order } y^{6999}\end{aligned}$$

Can we prove any of this???

Yes!!!

Proof for the partial theta function

- Use standard notation for q-shifted factorials:

$$
\begin{aligned}
(a;q)_n &= \prod_{j=0}^{n-1} (1-aq^j) \\
(a;q)_\infty &= \prod_{j=0}^{\infty} (1-aq^j) \qquad \text{for } |q|<1
\end{aligned}
$$

- A pair of identities for the partial theta function:

$$
\begin{aligned}
\sum_{n=0}^{\infty} x^n\, y^{n(n-1)/2} &= (y;y)_\infty\, (-x;y)_\infty \sum_{n=0}^{\infty} \frac{y^n}{(y;y)_n\, (-x;y)_n} \\
\sum_{n=0}^{\infty} x^n\, y^{n(n-1)/2} &= (-x;y)_\infty \sum_{n=0}^{\infty} \frac{(-x)^n\, y^{n^2}}{(y;y)_n\, (-x;y)_n}
\end{aligned}
$$

  as formal power series and as analytic functions on $(x,y) \in \mathbb{C} \times \mathbb{D}$

- Rewrite these as

$$
\begin{aligned}
\sum_{n=0}^{\infty} x^n\, y^{n(n-1)/2} &= (y;y)_\infty\, (-xy;y)_\infty \left[1 + x + \sum_{n=1}^{\infty} \frac{y^n}{(y;y)_n\, (-xy;y)_{n-1}} \right] \\
\sum_{n=0}^{\infty} x^n\, y^{n(n-1)/2} &= (-xy;y)_\infty \left[1 + x + \sum_{n=1}^{\infty} \frac{(-x)^n\, y^{n^2}}{(y;y)_n\, (-xy;y)_{n-1}} \right]
\end{aligned}
$$

- Brackets on the RHS (minus the initial $1+x$) have alternating signs in x (i.e. have nonnegative coefficients as a series in $-x$ and y)
- So we have reduced to the easy case of alternating signs!
- The second identity has $a_0(y) = 1$, so we prove also $\xi_0(y) \in \mathcal{S}_{-1}$.
- With a bit more work we can prove $\xi_0(y) \in \mathcal{S}_{-2}$.

The preceding proof, written more explicitly

- Let's say we use the first identity:

$$\Theta_0(x,y) \;=\; (y;y)_\infty\,(-xy;y)_\infty \left[1 + x + \sum_{n=1}^{\infty} \frac{y^n}{(y;y)_n\,(-xy;y)_{n-1}}\right]$$

- So $\Theta_0(x,y) = 0$ is equivalent to "brackets = 0".
- Insert $x = -\xi_0(y)$ and bring $\xi_0(y)$ to the LHS:

$$\xi_0(y) \;=\; 1 \;+\; \sum_{n=1}^{\infty} \frac{y^n}{\prod\limits_{j=1}^{n}(1-y^j)\,\prod\limits_{j=1}^{n-1}[1-y^j\xi_0(y)]}$$

- This formula can be used iteratively to determine $\xi_0(y)$, and in particular to prove the strict positivity of its coefficients:
- Define the map $\mathcal{F}\colon\ \mathbb{Z}[[y]] \to \mathbb{Z}[[y]]$ by

$$(\mathcal{F}\xi)(y) \;=\; 1 \;+\; \sum_{n=1}^{\infty} \frac{y^n}{\prod\limits_{j=1}^{n}(1-y^j)\,\prod\limits_{j=1}^{n-1}[1-y^j\xi(y)]}$$

- Define a sequence $\xi_0^{(0)}, \xi_0^{(1)}, \ldots \in \mathbb{Z}[[y]]$ by $\xi_0^{(0)} = 1$ and $\xi_0^{(k+1)} = \mathcal{F}\xi_0^{(k)}$.
- Then $\xi_0^{(0)} \preceq \xi_0^{(1)} \preceq \ldots \preceq \xi_0$ and $\xi_0^{(k)}(y) \;=\; \xi_0(y) \;+\; O(y^{3k+1})$.
- In particular, $\lim\limits_{k\to\infty} \xi_0^{(k)}(y) = \xi_0(y)$, and $\xi_0(y)$ has strictly positive coefficients.
- Thomas Prellberg has a combinatorial interpretation of $\xi_0(y)$ and $\xi_0^{(k)}(y)$.

Elementary proof of the first identity

- Proof uses nothing more than Euler's first and second identities

$$\begin{aligned}\frac{1}{(t;q)_\infty} &= \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n}\\ (t;q)_\infty &= \sum_{n=0}^{\infty} \frac{(-t)^n\, q^{n(n-1)/2}}{(q;q)_n}\end{aligned}$$

  valid for $(t,q) \in \mathbb{D}\times\mathbb{D}$ and $(t,q)\in\mathbb{C}\times\mathbb{D}$, respectively.

- Write

$$\sum_{n=0}^{\infty} x^n\, y^{n(n-1)/2} = \sum_{n=0}^{\infty} x^n\, y^{n(n-1)/2}\, \frac{(y;y)_\infty}{(y;y)_n\,(y^{n+1};y)_\infty}$$

- Insert Euler's first identity for $1/(y^{n+1};y)_\infty$:

$$\begin{aligned}\Theta_0(x,y) &= (y;y)_\infty \sum_{n=0}^{\infty} \frac{x^n\, y^{n(n-1)/2}}{(y;y)_n} \sum_{k=0}^{\infty} \frac{y^{(n+1)k}}{(y;y)_k}\\ &= (y;y)_\infty \sum_{k=0}^{\infty} \frac{y^k}{(y;y)_k} \sum_{n=0}^{\infty} \frac{(xy^k)^n\, y^{n(n-1)/2}}{(y;y)_n}\\ &= (y;y)_\infty \sum_{k=0}^{\infty} \frac{y^k}{(y;y)_k}\, (-xy^k;y)_\infty \quad \text{by Euler's second identity}\\ &= (y;y)_\infty\, (-x;y)_\infty \sum_{k=0}^{\infty} \frac{y^k}{(y;y)_k\,(-x;y)_k}\end{aligned}$$

- This identity goes back to Heine (1847), but does not seem to be very well known.

- It can be found in Fine (1988) and Andrews and Warnaar (2007).

- **Did anyone know it between 1847 and 1988???**

Proof of the first and second identities

- A simple limiting case of Heine's first and second transformations

$$
\begin{aligned}
{}_2\phi_1(a,b;c;q,z) &= \frac{(b;q)_\infty\,(az;q)_\infty}{(c;q)_\infty\,(z;q)_\infty}\,{}_2\phi_1(c/b,z;az;q,b)\\
{}_2\phi_1(a,b;c;q,z) &= \frac{(c/a;q)_\infty\,(az;q)_\infty}{(c;q)_\infty\,(z;q)_\infty}\,{}_2\phi_1(abz/c,a;az;q,c/a)
\end{aligned}
$$

  for the basic hypergeometric function

$$
{}_2\phi_1(a,b;c;q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n\,(b;q)_n}{(q;q)_n\,(c;q)_n}\,z^n
$$

- Just set $b=q$ and $z=-x/a$, then take $a\to\infty$ and $c\to 0$.
- This is how Heine (1847) proved the first identity.
- Heine didn't know his second transformation, which is apparently due to Rogers (1893).
- **Who first wrote the second identity for the partial theta function???**
- Surely it must have been known before Andrews and Warnaar (2007)!?!

Can any of this be generalized?

- Recall our friend

$$\widetilde{R}(x,y,q) \;=\; \sum_{n=0}^{\infty} \frac{x^n\, y^{n(n-1)/2}}{(1+q)\;\cdots\;(1+q+\ldots+q^{n-1})}$$

- Can this proof be extended to cases $q \neq 0$?
- Here is a general identity:

$$\sum_{n=0}^{\infty} \frac{x^n\, y^{n(n-1)/2}}{(q;q)_n} \;=\; \frac{1}{(q;q)_\infty} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell\, q^{\ell(\ell+1)/2}}{(q;q)_\ell}\, \Theta_0(xq^\ell, y)$$

- Can deduce generalizations of the first and second identities for the partial theta function:

$$\sum_{n=0}^{\infty} \frac{x^n\, y^{n(n-1)/2}}{(q;q)_n} \;=\; \frac{(y;y)_\infty}{(q;q)_\infty} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell\, q^{\ell(\ell+1)/2}}{(q;q)_\ell}\, (-xq^\ell;y)_\infty \sum_{n=0}^{\infty} \frac{y^n}{(y;y)_n\,(-xq^\ell;y)_n}$$

$$\sum_{n=0}^{\infty} \frac{x^n\, y^{n(n-1)/2}}{(q;q)_n} \;=\; \frac{1}{(q;q)_\infty} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell\, q^{\ell(\ell+1)/2}}{(q;q)_\ell}\, (-xq^\ell;y)_\infty \sum_{n=0}^{\infty} \frac{(-xq^\ell)^n\, y^{n^2}}{(y;y)_n\,(-xq^\ell;y)_n}$$

- But I don't know what to do with these formulae, because of the factors $(-1)^\ell$.