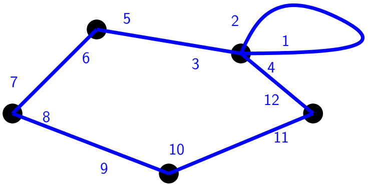
Cartography on unoriented surfaces, with applications to real and quaternionic random matrices

Emily Redelmeier

November 19, 2013

Consider the map:



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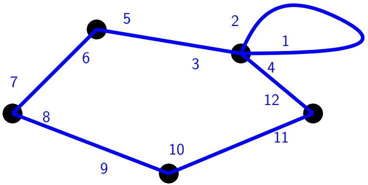

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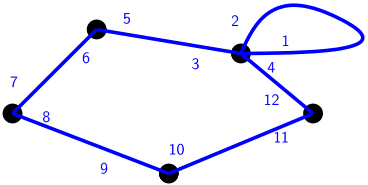

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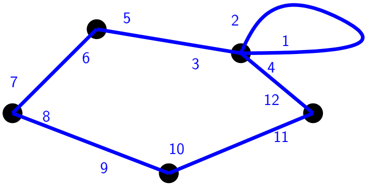
$$\sigma = (1,2,3,4)\,(5,6)\,(7,8)\,(9,10)\,(11,12)\,.$$

The edge information can be encoded in another permutation

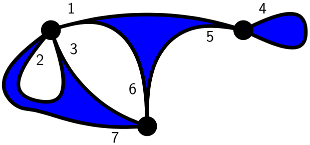
$$\alpha = (1,2)\,(3,5)\,(4,12)\,(6,7)\,(8,9)\,(10,11)\,.$$



The face information is encoded in

$$\varphi := \sigma^{-1}\alpha^{-1} = (1)\,(2,4,11,9,7,5)\,(3,6,8,10,12)\,.$$

This construction works equally well with oriented hypermaps:



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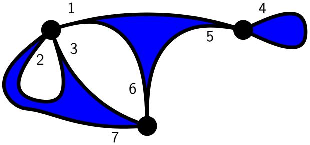

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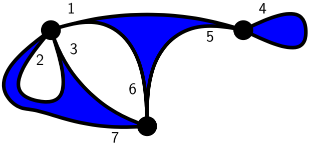

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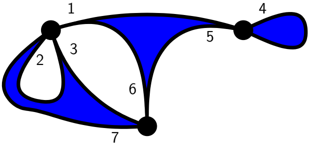

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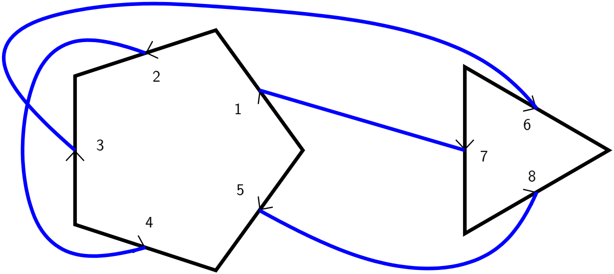
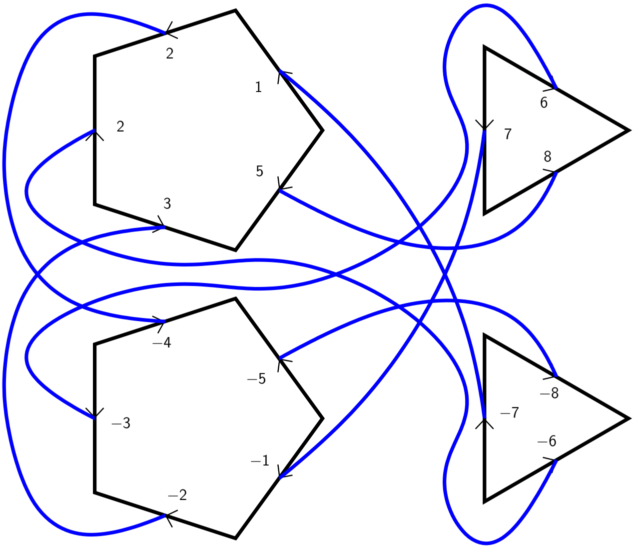
An untwisted edge-identification connects front to front and back to back, while a twisted edge-identification connects front to back and back to front.


1
2
3
4
5
6
7
8


2
1
2
5
3
6
7
8
−4
−5
−3
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−8
−7
−6

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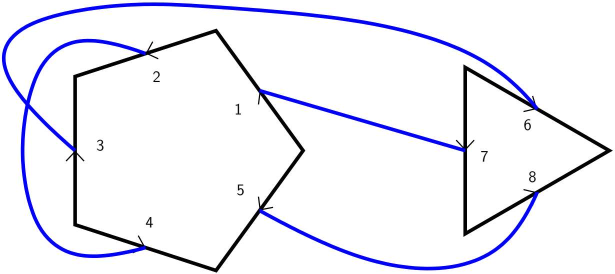
Vertex information is given by $\varphi_+^{-1}\alpha^{-1}\varphi_-$.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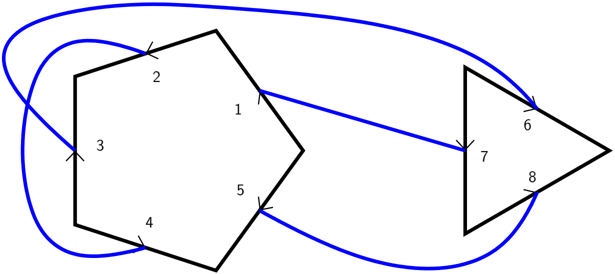
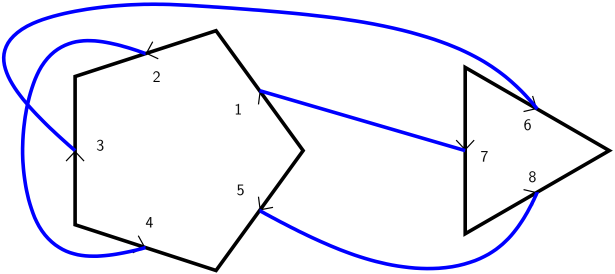


$$\varphi = (1, 2, 3, 4, 5)\,(7, 8, 9)$$



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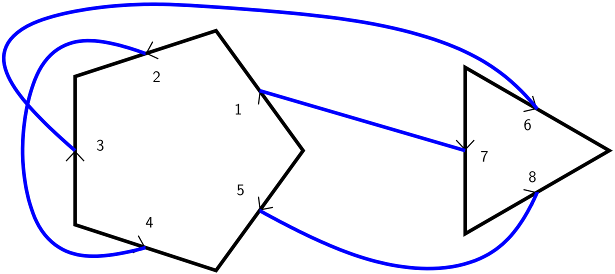
$$\alpha = (1, -7)\,(7, -1)\,(2, -4)\,(4, -2)\,(3, -6)\,(6, -3)\,(5, 8)\,(-8, -5)$$



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$$\sigma = \varphi_+^{-1}\alpha^{-1}\varphi_-\,(1,-3,6,-5,-7)\,(7,5,-6,3,-1)\,(2,-8,-4)\,(4,8,-2)$$

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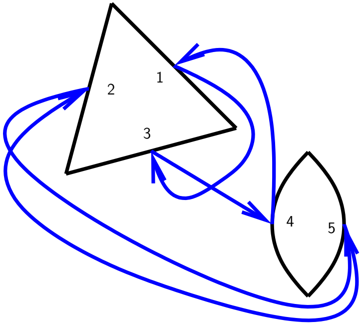

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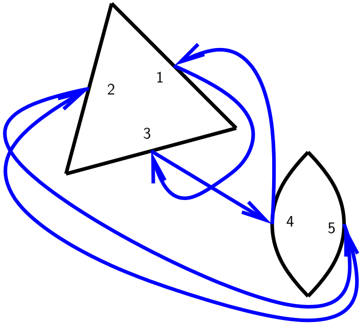

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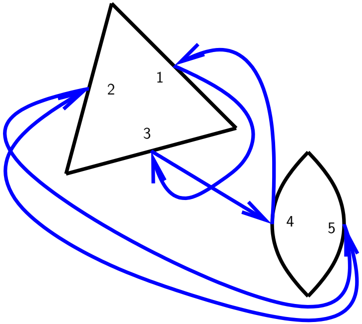

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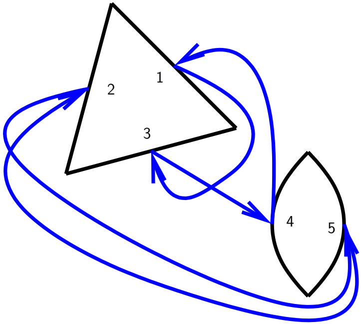

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The covering space of an orientable surface is two copies of the surface.

The covering space of an unorientable surface is the orientable surface with Euler characteristic twice that of the original surface (so the connected sum of *n* projective planes is the ($n - 1$)-holed torus).

Definition

Let I be a finite set of integers which does not contain both k and $-k$ for any k. For a $\gamma \in S(I)$ and a premap $\pi \in PM(\pm I)$, we define

$$\chi(\varphi, \alpha) := \#\left(\varphi_+\varphi_-^{-1}\right)/2 + \#(\alpha)/2 + \#\left(\varphi_+^{-1}\alpha^{-1}\varphi_-\right)/2 - |I|.$$

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If $\pm I_1$ and $\pm I_2$ are disjoint, and $\gamma_i \in S(I_i)$ and $\pi_i \in PM(\pm I_i)$ for $i = 1, 2$, then

$$\chi(\gamma_1, \pi_1) + \chi(\gamma_2, \pi_2) = \chi(\gamma_1\gamma_2, \pi_1\pi_2).$$

Theorem

Let $\pi, \rho \in S(I)$ *for some finite set* I. *Then*

$$\#(\pi) + \#(\pi\rho) + \#(\rho) \leq |I| + 2\#\langle \pi, \rho \rangle.$$

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Let $\varphi \in S_n$, and let $\{V_1, \ldots, V_r\} \in \mathcal{P}(n)$ be the orbits of φ. If $\alpha \in PM(\pm[n])$ connects the blocks of $\{\pm V_1, \ldots, \pm V_r\}$, then $\chi(\gamma, \pi) \leq 2$.

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Surfaces with maximal Euler characteristic are typically associated with noncrossing diagrams.

In terms of indices, traces may be written

$$\mathrm{Tr}\left(X_1 \cdots X_n\right) = \sum_{1 \leq i_1, i_2, \ldots, i_n \leq N} X^{(1)}_{i_1 i_2} X^{(2)}_{i_2 i_3} \cdots X^{(n)}_{i_n i_1}.$$

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We can take traces along the cycles of a permutation $\pi = \left(c_{1,1}, c_{1,2}, \ldots, c_{1,n_1}\right)\left(c_{2,1}, \ldots, c_{2,n_2}\right) \cdots \left(c_{r,1}, \ldots, c_{r,n_r}\right)$:

$$\begin{aligned}\mathrm{Tr}_\pi\left(X_1, \ldots, X_n\right) &= \mathrm{Tr}\left(X_{1,1} \cdots X_{1,n_1}\right) \cdots \mathrm{Tr}\left(X_{c_{r,1}} \cdots X_{c_{r,n_r}}\right) \\ &= \sum_{1 \leq i_1, \ldots, i_n} X^{(1)}_{i_1, i_{\pi(1)}} \cdots X^{(n)}_{i_n, i_{\pi(n)}}.\end{aligned}$$

Say we wish to calculate

$$\mathbb{E}\left(\mathrm{tr}\left(XY_1XY_2X^TY_3XY_4X^TY_5\right)\mathrm{tr}\left(X^TY_6XY_7XY_8\right)\right).$$

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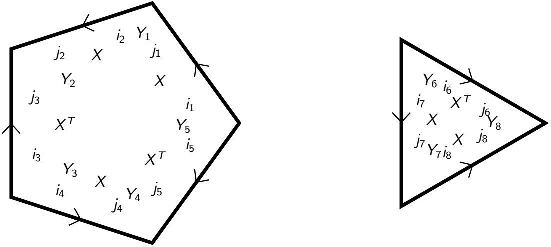
The traces of products are a sum over

$$X_{i_1j_1}Y^{(1)}_{j_1i_2}X_{i_2j_2}Y^{(2)}_{j_2j_3}X^T_{j_3i_3}Y^{(3)}_{i_3i_4}X_{i_4j_4}Y^{(4)}_{j_4j_5}X^T_{j_5i_5}Y^{(5)}_{i_5i_1}X^T_{j_6i_6}Y^{(6)}_{i_6i_7}X_{i_7j_7}Y^{(7)}_{j_7i_8}X_{i_8j_8}Y^{(8)}_{j_8j_6}.$$

We construct the faces:



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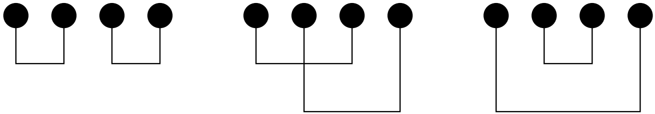

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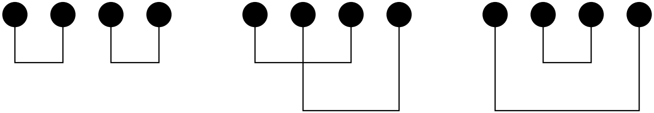

If X_1, X_2, X_3, X_4 are components of a multivariate Gaussian random variable, then

$$\mathbb{E}(X_1X_2X_3X_4) = \mathbb{E}(X_1X_2)\,\mathbb{E}(X_3X_4) + \mathbb{E}(X_1X_3)\,\mathbb{E}(X_2X_4) + \mathbb{E}(X_1X_4)\,\mathbb{E}(X_2X_3).$$

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Theorem

Let $\{f_\lambda : \lambda \in \Lambda\}$, for some index set Λ, be a centred Gaussian family of random variables. Then for $i_1, \ldots, i_n \in \Lambda$,

$$\mathbb{E}\left(f_{i_1} \cdots f_{i_n}\right) = \sum_{\mathcal{P}_2(n)} \prod_{\{k,l\} \in \mathcal{P}_2(n)} \mathbb{E}\left(f_{i_k} f_{i_l}\right).$$

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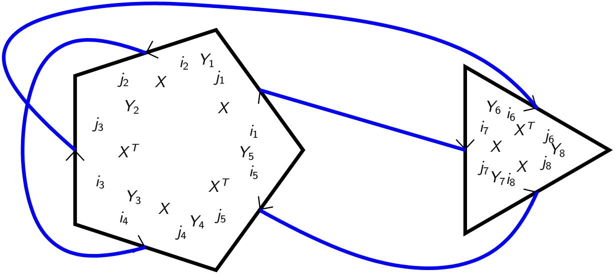
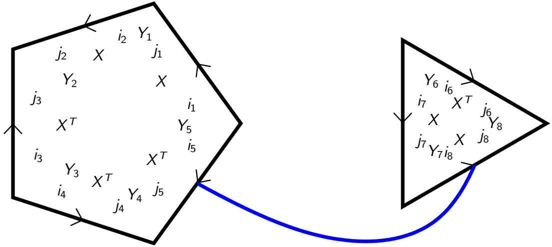
Here, for a pairing $\pi \in \mathcal{P}_2(n)$:

$$\prod_{\{k,l\}} \mathbb{E}(f_{i_k j_k} f_{i_l j_l}) = \begin{cases} 1, & \text{if } i_k = i_l \text{ and } j_k = j_l \text{ for all } \{k,l\} \in \pi \\ 0, & \text{otherwise} \end{cases}.$$

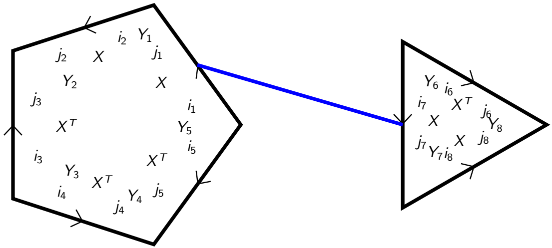
Putting indices which must be equal next to each other, we get a surface gluing:



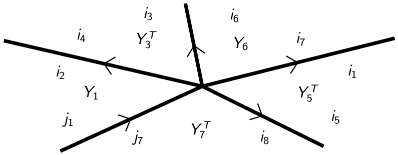
We note that if one term is from X and the other from X^T, the edge identification is untwisted:



If both terms are from X or from X^T, the edge identification is twisted:



The following vertex appears on the surface:



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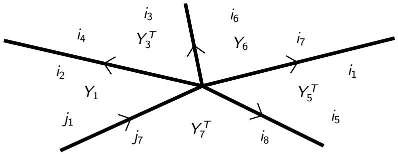

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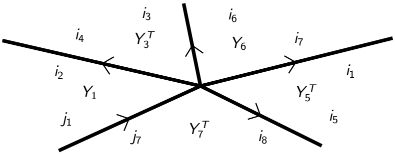

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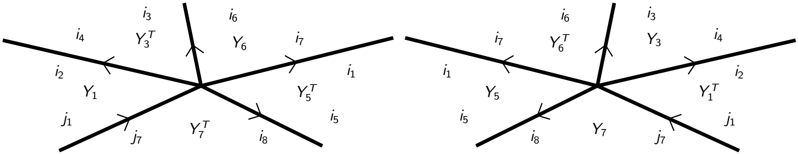
It contributes

$$\mathrm{Tr}\left(Y_1 Y_3^T Y_6 Y_5^T Y_7^T\right).$$

The same vertex viewed from the opposite side contributes the same value:



$$\mathrm{Tr}\left(Y_1 Y_3^T Y_6 Y_5^T Y_7^T\right) = \mathrm{Tr}\left(Y_7 Y_5 Y_6^T Y_3 Y_1^T\right).$$

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Real Wishart matrices are matrices $W := X^T D_k X$ for some deterministic matrix D_k.

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$$= \sum_{\iota:\begin{cases}[n]\to[N]\\-[n]\to[M]\end{cases}} N^{-\#(\varphi)-n}\mathbb{E}\left(f_{\iota_1\iota_{-1}}\cdots f_{\iota_n\iota_{-n}}\right)$$

$$\mathbb{E}\left(Y^{(1)}_{\iota_{-\delta_\varepsilon(1)}\iota_{\delta_\varepsilon\varphi(1)}}\cdots Y^{(n)}_{\iota_{-\delta_\varepsilon(n)}\iota_{\delta_\varepsilon\varphi(n)}}\right).$$

$$\sum_{\iota:\left\{\begin{array}{c}[n]\to[N]\\-[n]\to[M]\end{array}\right.}\ \sum_{\substack{\pi\in\mathcal{P}_2(n)\\ \iota_{\pm k}=\iota_{\pm l}:\{k,l\}\in\pi}} N^{-\#(\varphi)-n}\mathbb{E}\left(Y^{(1)}_{\iota_{-\delta_\varepsilon(1)}\iota_{\delta_\varepsilon\varphi(1)}}\cdots Y^{(n)}_{\iota_{-\delta_\varepsilon(n)}\iota_{\delta_\varepsilon\varphi(n)}}\right)$$

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Reversing the order of summation,

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$$\sum_{\pi\in\mathcal{P}_2(n)} N^{\#\left(\varphi_-^{-1}\delta_\varepsilon\pi\delta\pi\delta_\varepsilon\varphi_+\right)/2-\#(\varphi)-n}\mathbb{E}\left(\mathrm{tr}_{\mathrm{FD}\left(\varphi_-^{-1}\delta_\varepsilon\pi\delta\pi\delta_\varepsilon\varphi_+\right)}\left(Y_1,\ldots,Y_n\right)\right).$$

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This is a sum over all gluings compatible with the edge directions given by the transposes.

If we expand out the GOE matrix $T := \frac{1}{\sqrt{2}}\left(X + X^T\right)$, we get

$$\mathbb{E}\left(\mathrm{tr}_\varphi\left(TY_1, \ldots, TY_n\right)\right) = \sum_{\varepsilon:\{1,\ldots,n\}\to\{1,-1\}} \frac{1}{2^{n/2}} \mathbb{E}\left(\mathrm{tr}_\varphi\left(X^{(\varepsilon(1))}Y_1 \cdots X^{(\varepsilon(n))}Y_n\right)\right).$$

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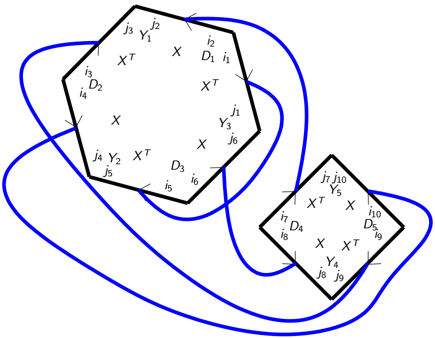
With Wishart matrices $W := X^T D_k X$, we can collapse the edges corresponding to each matrix to a single edge. We can think of the connecting blocks as (possibly twisted) hyperedges.



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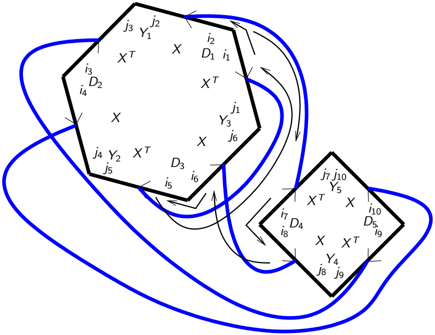

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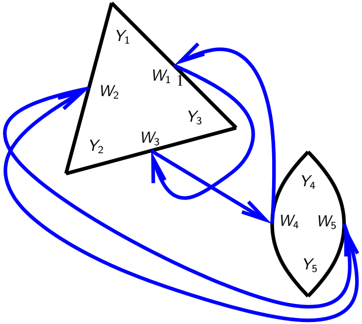

Thus:

$$\mathbb{E}\left(\mathrm{tr}_{\varphi}\left(W_1 Y_1, \cdots, W_n Y_n\right)\right) = \sum_{\alpha \in PM([n])} N^{\chi(\varphi,\alpha)-\#(\varphi)} \mathrm{tr}_{\mathrm{FD}(\alpha^{-1})}\left(D_1, \ldots, D_n\right) \mathbb{E}\left(\mathrm{tr}_{\mathrm{FD}\left(\varphi_-^{-1}\pi\varphi_+\right)}\left(Y_1, \ldots, Y_n\right)\right).$$

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A Haar-distributed orthogonal matrix is a random matrix with left-invariant probability measure on the orthogonal matrices.

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Theorem (Collins, Śniady, 2006)

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- $\mathrm{Wg}\left(\pi_1,\pi_2\right)$ *depends only on the block structure of* $\pi_1 \vee \pi_2$*;*
- *if* $\pi_1 \vee \pi_2$ *has blocks* $2\lambda_1,\ldots,2\lambda_s$*, then*

$$\mathrm{Wg}\left(\pi_1,\pi_2\right) = \left(\prod_{k=1}^{s} (-1)^{\lambda_k-1} C_{\lambda_k-1}\right) N^{-\frac{n}{2}-s} + \mathcal{O}\left(N^{-\frac{n}{2}-s-1}\right).$$

Say we wish to calculate

$$\mathbb{E}\left(\operatorname{tr}\left(OY_1OY_2O^TY_3\right)\operatorname{tr}\left(OY_4O^TY_5O^TY_6OY_7OY_8\right)\right).$$

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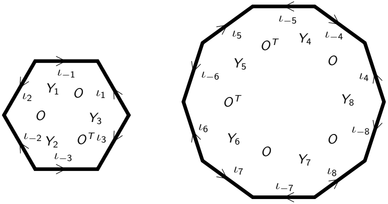
$$= \sum_{\iota:\pm[n]\to[N]} \mathbb{E}\left(O_{\iota_1\iota_{-1}}Y^{(1)}_{\iota_{-1}\iota_2}O_{\iota_2\iota_{-2}}Y^{(2)}_{\iota_{-2}\iota_{-3}}O^T_{\iota_{-3}\iota_3}Y^{(3)}_{\iota_3\iota_1}\right.$$
$$\left.\times O_{\iota_4\iota_{-4}}Y^{(4)}_{\iota_{-4}\iota_{-5}}O^T_{\iota_{-5}\iota_5}Y^{(5)}_{\iota_5\iota_{-6}}O^T_{\iota_{-6}\iota_6}Y^{(6)}_{\iota_6\iota_7}O_{\iota_7\iota_{-7}}Y^{(7)}_{\iota_{-7}\iota_8}O_{\iota_8\iota_{-8}}Y^{(8)}_{\iota_{-8}\iota_4}\right)$$

We construct the faces



$$= \sum_{\iota:\pm[n]\to[N]} \mathbb{E}\left(O_{\iota_1\iota_{-1}}\cdots O_{\iota_8\iota_{-8}}\right)$$

$$\times\, \mathbb{E}\left(Y^{(1)}_{\iota_{-1}\iota_2} Y^{(2)}_{\iota_{-2}\iota_{-3}} Y^{(3)}_{\iota_3\iota_1} Y^{(4)}_{\iota_{-4}\iota_{-5}} Y^{(5)}_{\iota_5\iota_{-6}} Y^{(6)}_{\iota_6\iota_7} Y^{(7)}_{\iota_{-7}\iota_8} Y^{(8)}_{\iota_{-8}\iota_4}\right)$$

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Consider

$$\pi_+ = (1,2)\,(3,5)\,(4,8)\,(6,7)$$

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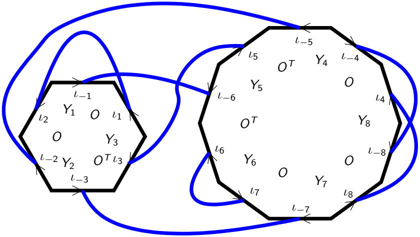

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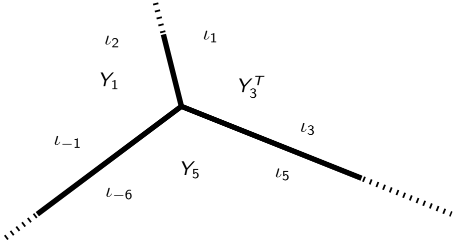

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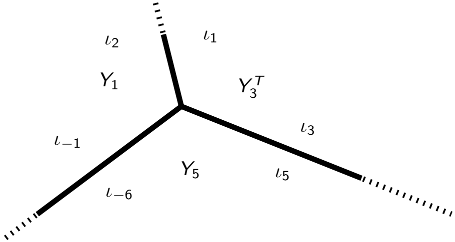
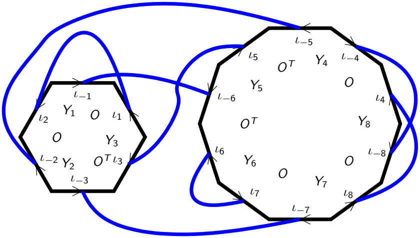


This vertex contributes:

$$\mathrm{Tr}\left(Y_1 Y_3^T Y_5\right).$$

There are also a number of vertices containing the O matrices.



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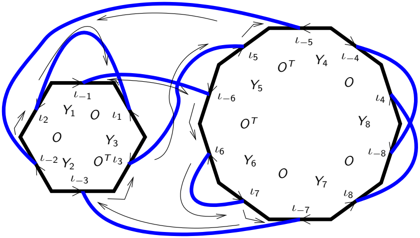

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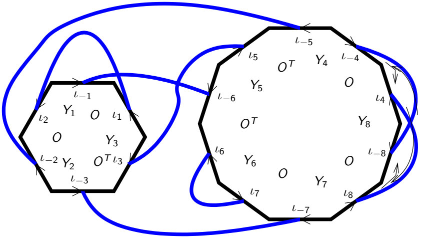

We expect these to contribute:

$$\mathrm{Wg}\left(\frac{v_1}{2}, \frac{v_2}{2}, \ldots, \frac{v_r}{2}\right)$$

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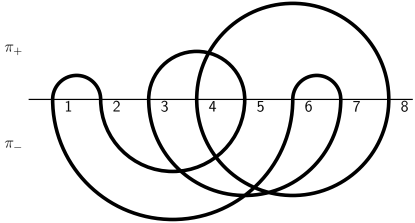
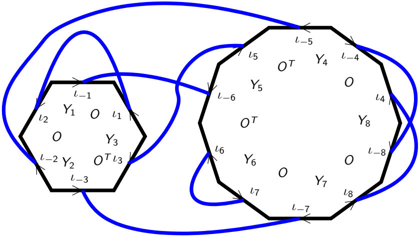
The permutation

$$\pi_-\delta\pi_+ = (1, -2, 5, -3, 7, -6)\,(6, -7, 3, -5, 2, -1)\,(4, -8)\,(8, -4)$$

enumerates the points around the cycles of $\pi_+ \cup \pi_-$:



This suggests another picture, in which $\delta_\varepsilon \pi_- \delta \pi_+ \delta_\varepsilon$ forms a set of hyperedges, and the faces are $\varphi_-^{-1} \delta_\varepsilon \pi_- \delta \pi_+ \delta_\varepsilon \varphi_+$.



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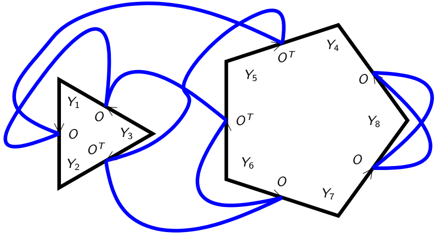

Let $\varphi \in S_n$, let $\varepsilon : [n] \to \{1, -1\}$, and let $Y_1, \ldots, Y_n$ be random matrices independent from O. Then

$$\begin{aligned}
& \mathbb{E}\left(\mathrm{tr}_\varphi\left(O^{\varepsilon(1)}Y_1, \ldots, O^{\varepsilon(n)}Y_n\right)\right) \\
= & \sum_{(\pi_+,\pi_-)\in\mathcal{P}_2(n)^2} N^{\chi(\varphi,\delta_\varepsilon\pi_-\delta\pi_+\delta_\varepsilon)-2\#(\varphi)}\mathrm{wg}\left(\pi_+,\pi_-\right) \\
& \times\mathbb{E}\left(\mathrm{tr}_{\varphi_-^{-1}\delta_\varepsilon\pi_-\delta\pi_+\delta_\varepsilon\varphi_+/2}\left(Y_1, \ldots, Y_n\right)\right) \\
= & \sum_{\alpha\in PM_{\mathrm{alt}}(\pm[n])} N^{\chi(\varphi,\delta_\varepsilon\alpha\delta_\varepsilon)-2\#(\varphi)}\mathrm{wg}\left(\lambda\left(\alpha\right)\right) \\
& \times\mathbb{E}\left(\mathrm{tr}_{\varphi_-^{-1}\delta_\varepsilon\alpha\delta_\varepsilon\varphi_+/2}\left(Y_1, \ldots, Y_n\right)\right).
\end{aligned}$$

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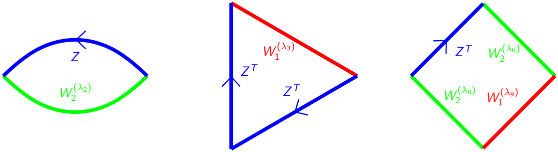

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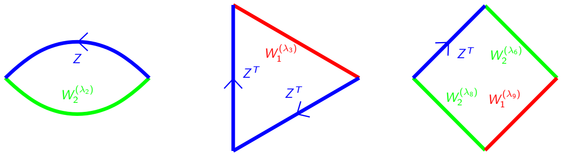
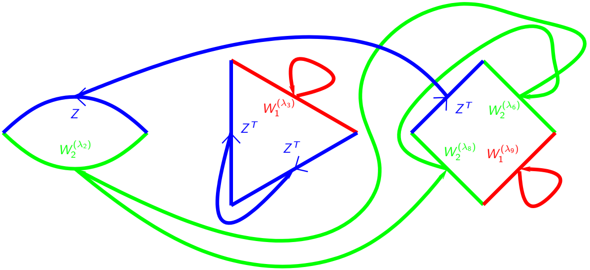


$$\varphi = (1,2)\,(3,4,5)\,(6,7,8,9)$$


Z
$W_2^{(\lambda_2)}$
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Z^T
Z^T
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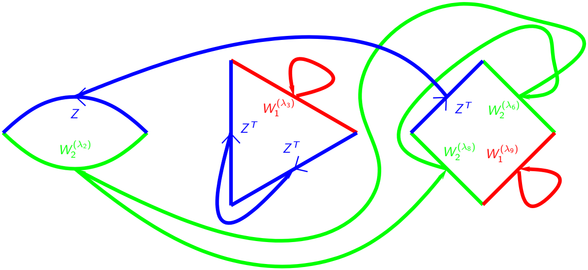
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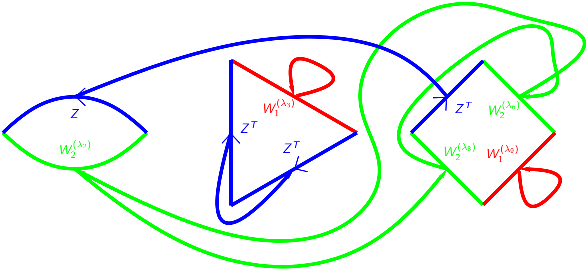
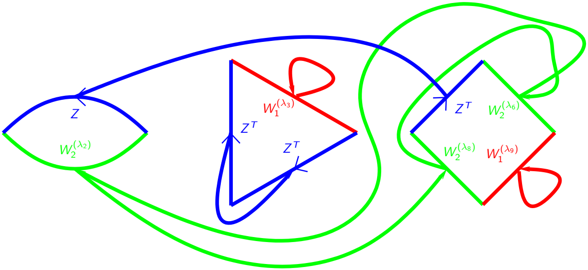
$$\alpha_1 = (3)\,(-3)\,(9)\,(-9)$$



$$\alpha_1 = (3)\,(-3)\,(9)\,(-9)$$

$$\alpha_2 = (2, 8, -6)\,(6, -8, -2)$$



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$$\alpha_3 = (1, -7)\,(-1, 7)\,(4, -5)\,(-4, 5)$$

$$\delta_\varepsilon \alpha \delta_\varepsilon = (1,7)\,(-1,-7)\,(2,8,-6)\,(6,-8,-2)\,(3)\,(-3)\,(4,-5)\,(5,-4)\,(9)\,(-9)$$

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$$\varphi_-^{-1} \delta_\varepsilon \alpha \delta_\varepsilon \varphi_+ = (1,8,9,-7,-2,6)\,(-6,2,7,-9,-8,-1)\,(3,-4,5)\,(-5,4,-3)$$

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$$\mathrm{tr}\left(A_{\lambda_3}\right)\mathrm{tr}\left(A_{\lambda_9}\right)\mathrm{tr}\left(B_{\lambda_2} B_{\lambda_6}^T B_{\lambda_8}\right) N^{-5}$$

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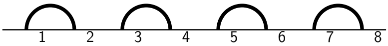
Highest order terms must have a relative orientation of the faces in which none of the edge-identifications are twisted.

The hyperoctahedral group B_n is the stabilizer in S_{2n} of a pairing:



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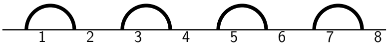
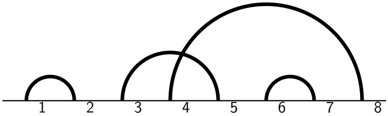
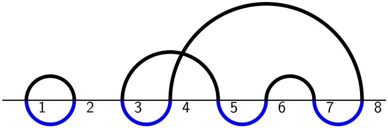


Pairings are in bijection with cosets of the hyperoctahedral group πB_n:



Possible loop structures are in bijection with the double cosets of the hyperoctahedral group $B_n \pi B_n$:



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Up to a normalization convention, the weight of each diagram is a matricial cumulant.

The space of invariant vectors under $O \otimes \cdots \otimes O$ is spanned by the images of

$$\sum_{\iota:[n/2]\to[N]} \left(e_{\iota_1} \otimes e_{\iota_1}\right) \otimes \cdots \otimes \left(e_{\iota_{n/2}} \otimes e_{\iota_{n/2}}\right)$$

under permutations of the tensor factors.

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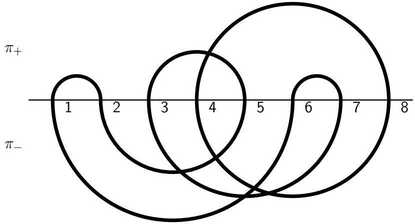
$$\sum_{\iota:[n/2]\to[N]} \left(e_{\iota_1} \otimes e_{\iota_1}\right) \otimes \cdots \otimes \left(e_{\iota_{n/2}} \otimes e_{\iota_{n/2}}\right)$$

under permutations of the tensor factors.

A basis therefore corresponds to cosets of B_n, i.e. to pairings.

The inner product of two basis elements is $N^{\#(\pi_+ \vee \pi_-)}$.



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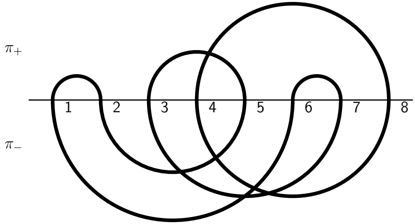

The Weingarten function is the inverse of the matrix of the inner products.

In the quaternionic case, the space of invariant vectors is spanned by the images of

$$\sum_{\substack{\iota:[n/2]\to[N]\\ \eta:[n/2]\to\{1,-1\}}} \eta_1\cdots\eta_{n/2}\left(e_{\iota_1;\eta_1}\otimes e_{\iota_1;-\eta_1}\right)\otimes\cdots\otimes\left(e_{\iota_{n/2};\eta_{n/2}}\otimes e_{\iota_{n/2};-\eta_{n/2}}\right).$$

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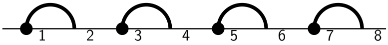
When we act on this vector with an odd permutation from B_n, it reverses the sign.



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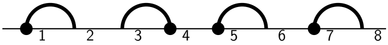

We consider images under even permutations, We consider images under even permutations,

We find that the inner product is

$$(-1)^{n/2}(-2N)^{\#(\pi_1 \vee \pi_2)}.$$

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$$i^2 = j^2 = k^2 = ijk = -1.$$

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A quaternion $a + bi + cj + dk$ may be represented as a 2×2 matrix:

$$\begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix}.$$

$$\overline{a+bi+cj+dk} := a-bi-cj-dk = \begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix}^*$$

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$$Q_{\eta_1\eta_2} = \eta_1\eta_2 Q_{-\eta_2,-\eta_1}$$

We wish to evaluate:

$$\mathbb{E}\left(\mathrm{tr}\left(Y_1 X_1^{(\varepsilon_1)} Y_2 \cdots Y_{n_1-1} X_{n_1-1}^{\left(\varepsilon_{n_1-1}\right)} Y_{n_1}\right)\cdots\right.$$
$$\left.\cdots\mathrm{tr}\left(Y_{n_{r-1}+1} X_{n_{r-1}+1}^{\left(\varepsilon_{n_{r-1}+1}\right)} Y_{n_{r-1}+2} \cdots Y_{n-1} X_{n-1}^{(\varepsilon_{n-1})} Y_n\right)\right)$$

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Because the traces are quaternion-valued, they "see" only one face $\zeta=(1,2,\ldots,n)$, rather than φ.
The asymptotics depend only on the vertices according to the surface constructed from φ, but the vertices of the surface constructed according to ζ will contribute signs and factors of 2.

For each negative ε_k, we get a factor of $\eta_k \eta_{-k}$.

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Regardless of the sign of k, the entry of Y_k may be written

$$Y^{(k)}_{\iota_{\delta\delta_\varepsilon\varphi_-(k)},\iota_{\delta_\varepsilon\varphi_+(k)};\mathrm{sgn}(k)\varepsilon_{\zeta_+^{-1}(k)}\eta_{\delta\delta_\varepsilon\zeta_+^{-1}(k)},\mathrm{sgn}(k)\varepsilon_{\zeta_-^{-1}(k)}\eta_{\delta_\varepsilon\zeta_-^{-1}(k)}}.$$

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For each negative $k \in \mathrm{FD}\left(\zeta_+\delta_\varepsilon\alpha\delta_\varepsilon\zeta_-^{-1}\right)$, we get a factor of $\varepsilon\left(\zeta_+^{-1}\left(k\right)\right)\eta\left(\delta\delta_\varepsilon\zeta_+^{-1}\left(k\right)\right)\varepsilon\left(k\right)\eta\left(\delta_\varepsilon\left(k\right)\right)$.

On a certain island near Haiti, half the inhabitants have been . . . turned into zombies . . . . [T]he zombies . . . always lie and the humans . . . always tell the truth.
The situation is enormously complicated by the fact that . . . . whenever you ask them a yes-no question, they reply "Bal" or "Da"—one of which means *yes* and the other *no* . . . . [W]e do not know which.

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You . . . wish to marry the King's daughter . . . . The test is that you may ask the medicine man any one question . . . . If he answers "Bal" then you may marry the king's daughter; if he answers "Da" then you may not.

Some of the natives answer questions with "Bal" and "Da," but others have broken away . . . and answer with the English words "Yes" and "No." . . . . [A]ny pair of brothers . . . are either both human or both zombies . . . . A native was suspected of high treason.

Question (to A) / Is the defendent innocent?
A's Answer / Bal.
Question (to B) / What does "Bal" mean?
B's Answer / "Bal" means *yes*.
Question (to C) / Are A and B brothers?
C's Answer / No.
Second Question to C / Is the defendent innocent?
C's Answer / Yes.
Is the defendent innocent or guilty?

Traces are taken along the cycles of $\varphi_+ \delta_\varepsilon \alpha \delta_\varepsilon \varphi_-^{-1}$.

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Real parts are taken along the cycles of $\zeta_+\delta_\varepsilon\alpha\delta_\varepsilon\zeta_-^{-1}$.

Definition

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$$\varphi_1(a_1, \ldots, a_p) = 0$$

when the a_i are centred and alternating.

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Definition

Families of matrices are *asymptotically free* if

$$\lim_{N\to\infty} \mathbb{E}\left(\mathrm{tr}\left(\mathring{A}_{1,N} \cdots \mathring{A}_{p,N}\right)\right) = 0$$

when the A_i are from cyclically alternating families

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- $\varphi_2 (1_A, a) = \varphi_2 (a, 1_A) = 0$.

We want to consider covariances of alternating products of centred matrices which are independent and in "general position".

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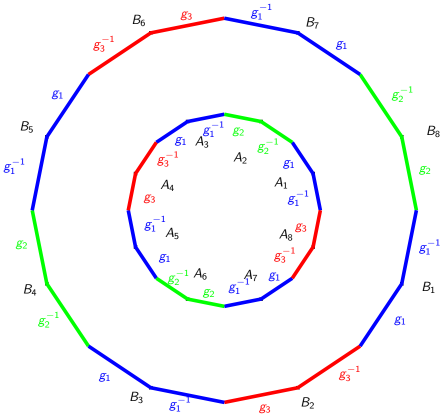
For g a Haar-distributed unitary, orthogonal or symplectic matrix, we consider:

$$\mathrm{cov}\left(\mathrm{Tr}\left(g_{v(1)}^{-1}A_1 g_{v(1)}\cdots g_{v(p)}^{-1}A_p g_{v(p)}\right),\right.$$
$$\left.\mathrm{Tr}\left(g_{w(1)}^{-1}B_1 g_{w(1)}\cdots g_{w(q)}^{-1}B_q g_{w(q)}\right)\right)$$

with $\mathbb{E}\left(\mathrm{tr}\left(A_k\right)\right)=\mathbb{E}\left(\mathrm{tr}\left(B_k\right)\right)=0$ and words v, w alternating.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B_7
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B_5
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g_2
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B_4
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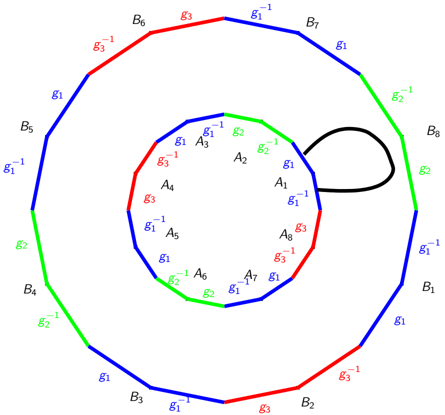
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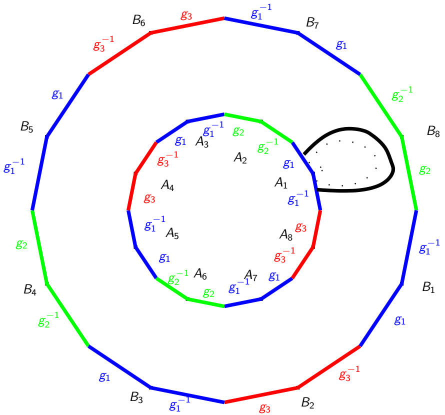
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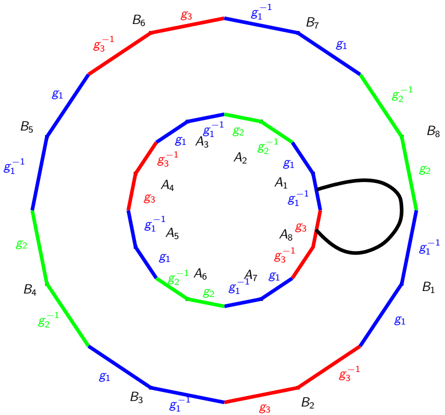
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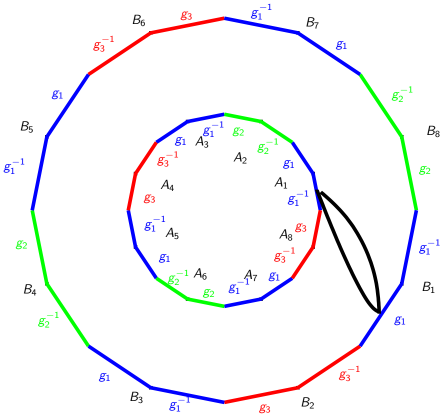
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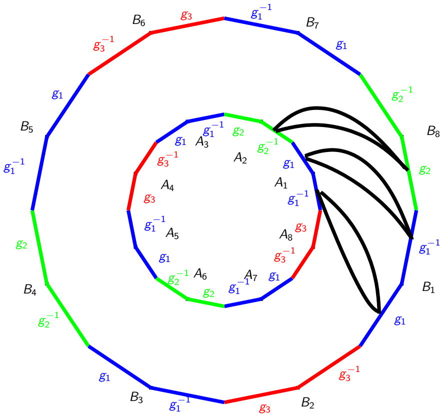
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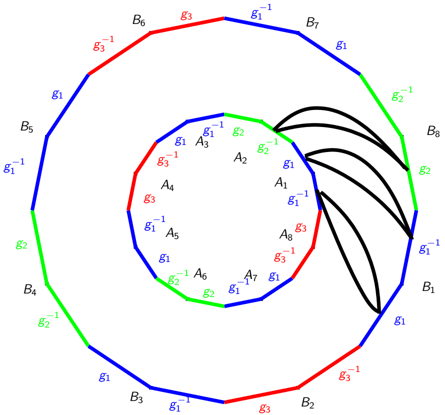

$$\mathbb{E}\left(\mathrm{tr}\left(A_1 B_1\right)\mathrm{tr}\left(A_1 B_8\right)\cdots\mathrm{tr}\left(A_8 B_7\right)\right)$$

Definition (Mingo, Speicher, 2006)

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- and for $p = q$:

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$$= \sum^{p-1} \prod^{p} \left(\lim_{N \to \infty} \left(\mathbb{E} \left(\mathrm{tr} \left(A_i B_{k-i} \right) \right) - \mathbb{E} \left(\mathrm{tr} \left(A_i \right) \right) \mathbb{E} \left(\mathrm{tr} \left(B_{k-i} \right) \right) \right) \right) .$$

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Subalgebras $A_1, \ldots, A_n$ of a second-order noncommutative probability space $(A, \varphi_1, \varphi_2)$ are *complex second-order free* if they are free and for $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ centred and either cyclically alternating or consisting of a single term, we have

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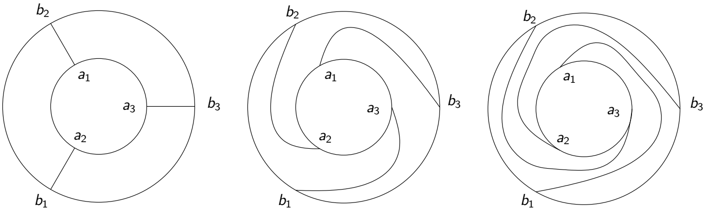
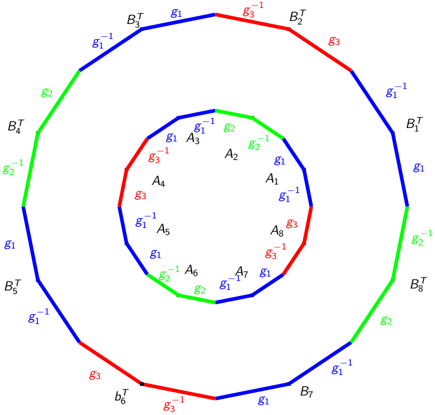
Spoke diagrams:




B_1^T
B_2^T
B_3^T
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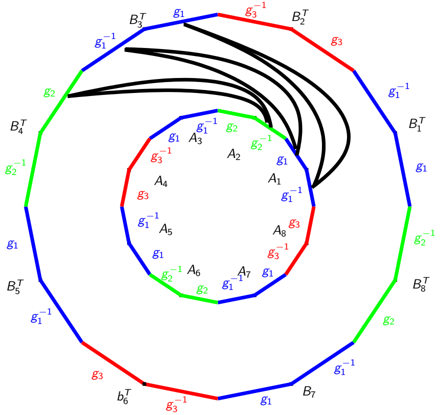


$$\mathbb{E}\left(\operatorname{tr}\left(A_1 B_3^T\right)\operatorname{tr}\left(A_1 B_4^T\right)\cdots\operatorname{tr}\left(A_8 B_2^T\right)\right)$$

Definition

Families of matrices are *asymptotically real second-order free* if they are asymptotically free, have a second-order limit distribution, and for A_i and B_i in algebras generated by cyclically alternating families

$$\lim_{N\to\infty} \mathrm{cov}\left(\mathrm{Tr}\left(\mathring{A}_1\cdots\mathring{A}_p\right), \mathrm{Tr}\left(\mathring{B}_1\cdots\mathring{B}_q\right)\right)$$

vanishes when $p \neq q$, and when $p = q$, is equal to

$$\lim_{N\to\infty} \mathrm{cov}\left(\mathrm{Tr}\left(\mathring{A}_1\cdots\mathring{A}_p\right), \mathrm{Tr}\left(\mathring{B}_1\cdots\mathring{B}_p\right)\right)$$
$$= \sum_{k=0}^{p-1}\prod_{i=1}^{p}\left(\lim_{N\to\infty}\left(\mathbb{E}\left(\mathrm{tr}\left(A_iB_{k-i}\right)\right) - \mathbb{E}\left(\mathrm{tr}\left(A_i\right)\right)\mathbb{E}\left(\mathrm{tr}\left(B_{k-i}\right)\right)\right)\right)$$
$$+\sum_{k=0}^{p-1}\prod_{i=1}^{p}\left(\lim_{N\to\infty}\left(\mathbb{E}\left(\mathrm{tr}\left(A_iB_{k+i}^T\right)\right) - \mathbb{E}\left(\mathrm{tr}\left(A_i\right)\right)\mathbb{E}\left(\mathrm{tr}\left(B_{k+i}^T\right)\right)\right)\right).$$

Definition

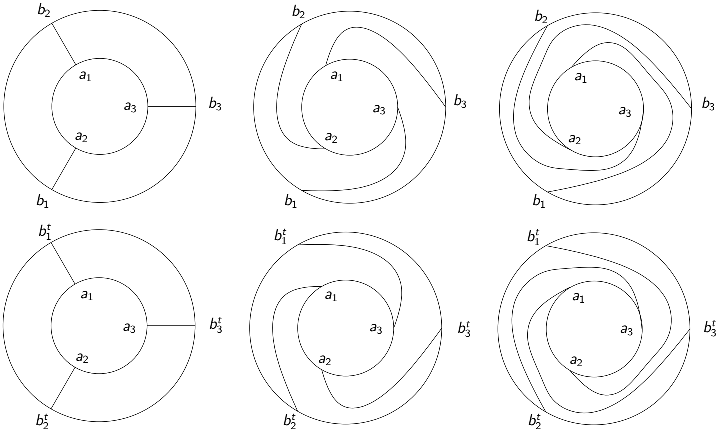
Subalgebras $A_1, \ldots, A_n$ of a second-order noncommutative probability space $(A, \varphi_1, \varphi_2)$ are *real second-order free* if they are free and for $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ centred and either cyclically alternating or consisting of a single term

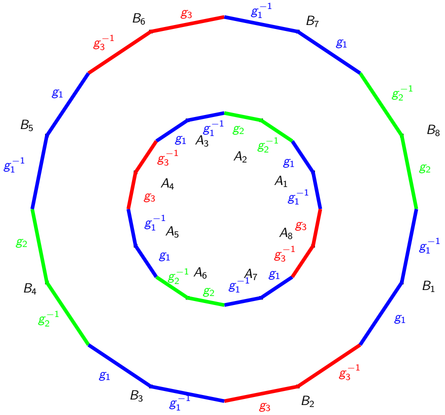
$$\varphi_2 \left(a_1 \cdots a_p, b_1 \cdots b_q \right) = 0$$

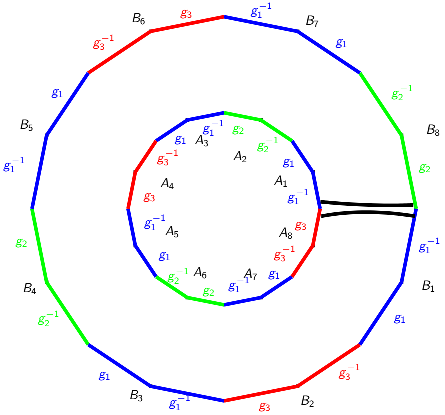
when $p \neq q$ and

$$\varphi_2 \left(a_1 \cdots a_p, b_1 \cdots b_p \right) = \sum_{k=0}^{p-1} \prod_{i=1}^{p} \varphi_1 \left(a_i b_{k-i} \right) + \sum_{k=0}^{p-1} \prod_{i=1}^{p} \varphi_1 \left(a_i b_{k+i}^t \right) .$$

Spoke diagrams for the real case:




B_6
g_3
g_1^{-1}
B_7
g_3^{-1}
g_1
g_1
g_2^{-1}
B_5
B_8
g_1^{-1}
g_2
g_1^{-1}
g_2
g_1
A_3
g_3^{-1}
g_2^{-1}
A_2
g_1
A_1
A_4
g_3
g_1^{-1}
g_1^{-1}
g_3
A_5
A_8
g_3^{-1}
g_1
g_2^{-1}
A_6
A_7
g_1
g_2
g_1^{-1}
g_2
g_1^{-1}
B_4
B_1
g_2^{-1}
g_1
g_1
g_3^{-1}
B_3
g_1^{-1}
g_3
B_2


B_1
B_2
B_3
B_4
B_5
B_6
B_7
B_8
A_1
A_2
A_3
A_4
A_5
A_6
A_7
A_8
g_1
g_1^{-1}
g_2
g_2^{-1}
g_3
g_3^{-1}

Definition

Families of matrices are *asymptotically quaternion second-order free* if they are asymptotically free, have a second-order limit distribution, and for A_i and B_i in algebras generated by cyclically alternating families

$$\lim_{N\to\infty} k_2\left(\mathrm{Tr}\left(\mathring{A}_1\cdots\mathring{A}_p\right),\mathrm{Tr}\left(\mathring{B}_1\cdots\mathring{B}_q\right)\right)$$

vanishes when $p \neq q$,

and when $p = q$, is equal to

$$\lim_{N\to\infty} \operatorname{cov}\left(\operatorname{Tr}\left(\mathring{A}_1 \cdots \mathring{A}_p\right), \operatorname{Tr}\left(\mathring{B}_1 \cdots \mathring{B}_p\right)\right)$$

$$= 4\prod_{i=1}^{p} \operatorname{Re}\left(\lim_{N\to\infty}\left(\mathbb{E}\left(\operatorname{tr}\left(A_i B_{n-i}\right)\right) - \mathbb{E}\left(\operatorname{tr}\left(A_i\right)\right)\mathbb{E}\left(\operatorname{tr}\left(B_{n-i}\right)\right)\right)\right)$$

$$- 2\prod_{i=1}^{p} \operatorname{Re}\left(\lim_{N\to\infty}\left(\mathbb{E}\left(\operatorname{tr}\left(A_i B_i^T\right)\right) - \mathbb{E}\left(\operatorname{tr}\left(A_i\right)\right)\mathbb{E}\left(\operatorname{tr}\left(B_i^T\right)\right)\right)\right)$$

$$+ \sum_{k=1}^{p-1}\prod_{i=1}^{p} \operatorname{Re}\left(\lim_{N\to\infty}\left(\mathbb{E}\left(\operatorname{tr}\left(A_i B_{k-i}\right)\right) - \mathbb{E}\left(\operatorname{tr}\left(A_i\right)\right)\mathbb{E}\left(\operatorname{tr}\left(B_{k-i}\right)\right)\right)\right)$$

$$+ \sum_{k=1}^{p-1}\prod_{i=1}^{p} \operatorname{Re}\left(\lim_{N\to\infty}\left(\mathbb{E}\left(\operatorname{tr}\left(A_i B_{k+i}^T\right)\right) - \mathbb{E}\left(\operatorname{tr}\left(A_i\right)\right)\mathbb{E}\left(\operatorname{tr}\left(B_{k+i}^T\right)\right)\right)\right).$$

Definition

Subalgebras $A_1, \ldots, A_n$ of a second-order noncommutative probability space $(A, \varphi_1, \varphi_2)$ are *quaternion second-order free* if they are free and for $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ centred and either cyclically alternating or consisting of a single term

$$\varphi_2 \left(a_1 \cdots a_p, b_1 \cdots b_q \right) = 0$$

when $p \neq q$ and

$$\begin{aligned}\varphi_2 &\left(a_1 \cdots a_p, b_1 \cdots b_p \right) \\ &= 4 \mathrm{Re} \left(\varphi_1 \left(a_i b_{p-i} \right) \right) - 2 \mathrm{Re} \left(\varphi_1 \left(a_i b_i^t \right) \right) \\ &+ \sum_{k=1}^{p-1} \prod_{i=1}^{p} \mathrm{Re} \left(\varphi_1 \left(a_i b_{k-i} \right) \right) + \sum_{k=1}^{p-1} \prod_{i=1}^{p} \mathrm{Re} \left(\varphi_1 \left(a_i b_{k+i}^t \right) \right) .\end{aligned}$$