Évaluation numérique rigoureuse de fonctions D-finies en SageMath

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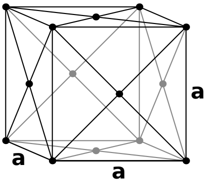
arxiv:1607.01967 [cs.SC]

prepared with GNU TEXMACS

Face-Centered Cubic Lattices

[Koutschan 2013]

dop6 = 410085196915322880 z^{35} + 1129052664742115635 20 z^{34} + 1171669263761496\
1489920 z^{33} + 6908174012870789173632 00 z^{32} + 27204862643846611522761600 z^{31} +
778811406918247228618497600 z^{30} + 1704438412411524078142979 2800 z^{29} + 294245234\
066850000428339092800 z^{28} + 408342458780511706027247612 5800 z^{27} + 459730295491197962\
35386142827300 z^{26} + 41969559889089825320345587674 9930 z^{25} + 306429717617409167171 79\
85958725620 z^{24} + 17169584489259696388755804636033570 z^{23} + 645817719616848100772794\
75394020500 z^{22} + 51714221934272099420476126216766700 z^{21} − 1473967391504437899277 38\
0487903179960 z^{20} − 142375543413213353539202319287238 5940 z^{19} − 832163401343931150 1\
683422098038445434 0 z^{18} − 364019154328107562568847906822488063550 z^{17} − 126157147851\
3401088177035093275526304300 z^{16} − 352834103209889699532343901711795685 6150 z^{15} −
796436951859377802952105607044279446 6900 z^{14} − 142805007261627862547128411638750 01\
728600 z^{13} − 195346531156863425435808319609419789180 00 z^{12} − 1839878333422238008423 8\
012428704731960000 z^{11} − 7553741785990309357234054786177488000000 z^{10} + 888743230941\
952240398397617177569760 0000 z^{9} + 21137039158366320685856256980012112000000 z^{8} +
2268269355393480469044664729550880000000 0 z^{7} + 149381834\
28146261190546354671616000000000 z^{6} + 4690246528584816329940\
199400448000000000 z^{5} − 8728290086347857389261624524800000 0\
0000 z^{4} − 11043279407797458901507731456000000000000 z^{3} − 353898\
70820758085677291929600000000000000 z^{2} − 5207934291077444487 41\
3760000000000000 z − 24287906219318850355200000000000000000 +



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Face-Centered Cubic Lattices

[Koutschan 2013]

$$
\begin{array}{l}
(396415690351478784 0\,z^{36}+1104718489963413534720\,z^{35}+11787108873993035283456 0\,z^{34}+ \\
718328751664447961579520 0\,z^{33}+2931058352189429037818553 60\,z^{32}+870657237873\backslash \\
4984776799502400\,z^{31}+19794977613811586613384925 4880\,z^{30}+355549462414631854804645\backslash \\
3851120\,z^{29}+514578986720138650981112912473 20\,z^{28}+606522834979531840684521625 23\backslash \\
7020\,z^{27}+583536683684602718287692085634895 0\,z^{26}+454552025018263589746062199819 7\backslash \\
4015\,z^{25}+27915340446750206294853155783826075 0\,z^{24}+125227539937283713406150704262\backslash \\
8908795\,z^{23}+29431828029235520381613075847069400 70\,z^{22}-104835131152062893985104 13\backslash \\
216920199750\,z^{21}-1699481829335074791612575655686165307 00\,z^{20}-115496959477627716 0\backslash \\
649077785983820553870\,z^{19}-5548694490781020038019823355124585193590\,z^{18}- \\
20745229517577451272377158241970915439245\,z^{17}-62232963928794638659423069651761\backslash \\
724690290\,z^{16}-150810045901978932864163493046405461262105\,z^{15}-29252862652300566 16\backslash \\
29390236883046859976150\,z^{14}-441395096063183148839008172248580337780300\,z^{13}-48412\backslash \\
35787645370430318612064737152693430 00\,z^{12}-31297658464933476345181066385800419642\backslash \\
0000\,z^{11}+31415133499909950234831915395869293600000\,z^{10}+33073846008767455546849 14\backslash \\
826295584680000 00\,z^{9}+391096978918364972225128472061480072000000\,z^{8}+232460170425\backslash \\
94802734543485030527952000000 0\,z^{7}+18060134934884299834847099345568000000000\,z^{6}- \\
10021340089119210237029332603699200000000 0\,z^{5}-83859064515985136495903099458560\backslash \\
000000000\,z^{4}-29487051768804049058487328972800000000000\,z^{3}-5516482719743042228 47\backslash \\
2840192000000000000\,z^{2}-379553789663505987184558080000000000000\,z+83793276456650\backslash \\
0337254400000000000000)\,\mathrm{Dz}+
\end{array}
$$

Face-Centered Cubic Lattices

[Koutschan 2013]

$$(813335640548723712\,0 z^{37} + 22941317820436643174\,40 z^{36} + 251295328534762193633280\, z^{35} +$$

$$1579545301524081697009152\,0 z^{34} + 666093618246765502077439680\, z^{33} + 2046937571\backslash$$

$$2416843040909376160\, z^{32} + 48181733826733863978332874912\,0 z^{31} + 89679731262120200325\,1\backslash$$

$$7991216960\, z^{30} + 13469691485430453672228186695430\,0 z^{29} + 1651833654984876079820125\,88\backslash$$

$$5678650\, z^{28} + 166074900263434295328115753119492\,30 z^{27} + 13626386945430479914685925\,33\backslash$$

$$46813455\, z^{26} + 89586531932747144763811128987323871\,0 z^{25} + 44897110742643849065299259\backslash$$

$$90254793265\, z^{24} + 144918522834945778266540039325477116\,90 z^{23} - 16850906647119454698\,3\backslash$$

$$174648133542750\, z^{22} - 386265894549826881229123104470731096440\, z^{21} - 3163259131060568\backslash$$

$$5845461133437819875612\,20 z^{20} - 166361820694138211705446840475562205681\,50 z^{19} -$$

$$662467400893936760809815371303780906585\,25 z^{18} - 20908024587285063156631213744961\,95\backslash$$

$$61543730\, z^{17} - 529097465740104776391772834675033946593335\, z^{16} - 106503862075731392\,93\backslash$$

$$916393610323634537509\,30 z^{15} - 165365164468562014216700942212402255522170\,0 z^{14} - 1829\backslash$$

$$38347451397592987402777056329883196780\,0 z^{13} - 108870989683690580666028414927762156\backslash$$

$$8328000\, z^{12} + 4373840673373288869444839633369529040800\,00 z^{11} + 1678380365365432006\,62\backslash$$

$$54514732362690120000\,00 z^{10} + 1564385355592027935922683162898655112000\,000 z^{9} + 30360\backslash$$

$$7398715325954207032303663107840000000\, z^{8} - 929554263384554771136483801584745600\,00\backslash$$

$$0000\, z^{7} - 1334658535726482371536908049179648000000000\, z^{6} - 9419775340065248371828\,79\backslash$$

$$263564800000000000\, z^{5} - 3502976538527876775892750166016000000000000\, z^{4} - 64286241473\backslash$$

$$892148234640584704000000000000\, z^{3} - 37458505751548575098585088000000000000000\, z^{2} +$$

$$66870677798339624740454400000000000000000\, z + 728637186579565510656000000000000000\backslash$$

$$0000)\, \mathrm{Dz}^{2} +$$

Face-Centered Cubic Lattices

[Koutschan 2013]

$(6219625486549063680\,z^{38} + 1775531336308022522880\,z^{37} + 199409996635132589752320\,z^{36} +$
$12904862497592448920163840\,z^{35} + 561222248755128125708191680\,z^{34} + 17798695421\backslash$
$072697669468739680\,z^{33} + 432530168604725658189210596640\,z^{32} + 8315189920333341531658\backslash$
$617695280\,z^{31} + 129103723904595771409928232487740\,z^{30} + 1639190738531986170699647097\backslash$
$803790\,z^{29} + 17111040709840823035760757618682440\,z^{28} + 14651890144386180365877132986\backslash$
$6897880\,z^{27} + 1015534278806669843159745327151252620\,z^{26} + 54963382760530760750687544\backslash$
$67310102760\,z^{25} + 20890574209714927539267068315744951640\,z^{24} + 301646099705919471898\backslash$
$27076234922007050\,z^{23} - 289127416281529376142095631015519267120\,z^{22} - 30536692798730\backslash$
$63346793150591937974700130\,z^{21} - 17566486109105161467894504789161406270600\,z^{20} -$
$73673650638461574538679743097050051115220\,z^{19} - 24040743896755739891331729697533\backslash$
$6574702980\,z^{18} - 619168293687639511251067273975020197114100\,z^{17} - 124122530329046062\backslash$
$3795990859905959226579320\,z^{16} - 1835553795134837646262350779261931882894750\,z^{15} -$
$1670314602837141110845640706031012073555700\,z^{14} + 3066874861862296186275087607527\backslash$
$5710932000\,z^{13} + 2931594155313390328935716187001614568260000\,z^{12} + 49194564588996664\backslash$
$98684069708388548285600000\,z^{11} + 3777365646243762653104795884206143332000000\,z^{10} +$
$68195639154415674514017863641593600000000\,z^{9} - 311881093752325372623566678266609\backslash$
$6800000000\,z^{8} - 3771833787399616704258908808294288000000000\,z^{7} - 248844483193099682\backslash$
$4908954989144320000000000\,z^{6} - 893959103422093803323305262745600000000000\,z^{5} - 1000\backslash$
$83705719332806676962561024000000000000\,z^{4} + 2564203084567737476418017280000000000\backslash$
$0000\,z^{3} + 1028076103183337304001413120000000000000000\,z^{2} + 10337540084597585682432000\backslash$
$00000000000000\,z)\,\mathrm{Dz}^{3} +$

Face-Centered Cubic Lattices

[Koutschan 2013]

$$
\begin{array}{l}
(2192816677949990400\, z^{39} + 6334902134773087680000\, z^{38} + 72864986011484455353600\, z^{37} + \\
484748686979553726053280 0\, z^{36} + 21701401704876164561481600 0\, z^{35} + 708801699580\backslash \\
0124707996090560\, z^{34} + 177409657131610270482016190640\, z^{33} + 3513089912736584156549 23\backslash \\
8676620\, z^{32} + 56201587732740449959670675451690\, z^{31} + 735847759326730529024504240 98\backslash \\
8015\, z^{30} + 7934411063073314432988482485900500\, z^{29} + 704051719126302863599455718967 7\backslash \\
4110\, z^{28} + 50888281392085061069923563367732422 0\, z^{27} + 2913386772290646501282812655 54\backslash \\
6011475\, z^{26} + 1223798977496446338506289092696321595 0\, z^{25} + 2730316387455561605215589\backslash \\
8475524386210\, z^{24} - 8440072427260140506527177326439720953 0\, z^{23} - 1309329548085768562\backslash \\
97312907253772416495 5\, z^{22} - 822926919906244244426426023497784780536 0\, z^{21} - \\
35948740918844475140318574840001115213670\, z^{20} - 1192466813203331345939144884033\backslash \\
28142970080\, z^{19} - 304273308297438630162837099041285050546455\, z^{18} - 5771450308959017 9\backslash \\
06973111263868960367674 90\, z^{17} - 707778167790136602728038144967670916837350\, z^{16} - \\
153341406553907334245470038125935935813900\, z^{15} + 1611202920628825942940219406515 8\backslash \\
76419542000\, z^{14} + 41889936162050170468997391242115445434 60000\, z^{13} + 5699626392082018\backslash \\
03745325939619490638800000 0\, z^{12} + 414514018720330983618331139825246996400000 0\, z^{11} + \\
950688923971337731996303622505069600000 00\, z^{10} - 3224476947136810362409243540972 02\backslash \\
9600000000\, z^{9} - 36368355281389283027676649873995360000000 00\, z^{8} - 210391871123659328\backslash \\
537319611153280000000000 0\, z^{7} - 551590602425414384164117858252800000000000 0\, z^{6} + 1162\backslash \\
1592569441090242089817804800000000000000\, z^{5} + 1128385276844823710179815628800000000 0\backslash \\
0000\, z^{4} + 2664044091169081397355479040000000000000000\, z^{3} + 242727262779317760737 28000\backslash \\
00000000000000\, z^{2})\, \mathrm{Dz}^{4} +
\end{array}
$$

Face-Centered Cubic Lattices

[Koutschan 2013]

$$
\begin{gathered}
(390720062616543744\,z^{40} + 114216661424360307456\,z^{39} + 13440822351615963069696\,z^{38} + \\
917965180366474611870720\,z^{37} + 42237673932263775988570560\,z^{36} + 14182188393104\backslash \\
81932976078400\,z^{35} + 3648720683640800119768991 0640\,z^{34} + 742504152062765602237759 45\backslash \\
2720\,z^{33} + 122056946669190116424625606509 30\,z^{32} + 164251022394443778986763736 53\backslash \\
9405\,z^{31} + 182176711561283643405373740475 5054\,z^{30} + 166579383028240072672437243 6543\backslash \\
4191\,z^{29} + 124533460849620200009711445328 730256\,z^{28} + 743593796442908540070532245 488\backslash \\
378205\,z^{27} + 333600460708810753163436188906 1221370\,z^{26} + 902022216185473608420262982\backslash \\
4390547047\,z^{25} - 919882472294340420544742129940 4277112\,z^{24} - 279449868802402514175 67\backslash \\
7041789492570017\,z^{23} - 190786342766193988557672312 6598906643790\,z^{22} - 85740836464757\backslash \\
100507575655426729796745 55\,z^{21} - 284055872968472310701836068565 83770811720\,z^{20} - \\
695742581753129555144407139736536164 28745\,z^{19} - 114991436892487711669937849824 91\backslash \\
2517430330\,z^{18} - 703780172015798633644954321671827 25333675\,z^{17} + 261641966501089 1478\backslash \\
43656083216157842879550\,z^{16} + 104941082479513638483746720981002553 9400000\,z^{15} + 2089\backslash \\
663780964272997600159898811800513390000\,z^{14} + 259786002679636380331331345092974504\backslash \\
0000000\,z^{13} + 175913683458515608543211372007264726 6000000\,z^{12} - 134675614991223713 10\backslash \\
874092829081128 0000000\,z^{11} - 157899609879137074628470745343916920 0000000\,z^{10} - 15568\backslash \\
116813227200258945319559980 40000000000\,z^{9} - 678908917613499441761342434795200000 00\backslash \\
0000\,z^{8} + 314922159293504699010388187520 0000000000\,z^{7} + 192407344459752425261121833\backslash \\
472000000000000\,z^{6} + 104126840210444782060628951040000000000000\,z^{5} + 20982276643045\backslash \\
393599012864000000000000000\,z^{4} + 178743809832799664332800000000000000000000\,z^{3})\,\mathrm{Dz}^{5} +
\end{gathered}
$$

Face-Centered Cubic Lattices

[Koutschan 2013]

$$\begin{array}{l}
(35882454730090752\,z^{41}+1061260405161448665 6\,z^{40}+127653260094221277516 8\,z^{39}+\\
8939398012943303209632 0\,z^{38}+4221606838983473228197008\,z^{37}+145494567985766484\backslash\\
898923048\,z^{36}+38408280044909200609509694 80\,z^{35}+8016006238826772717221198508 0\,z^{34}+\\
1350855094398006902682870922050\,z^{33}+1863108289263053682422294940958 5\,z^{32}+2118\backslash\\
15796834464054711973645322142\,z^{31}+19867083220856675726655250160374 11\,z^{30}+1526308\backslash\\
2383031406770429022758762048\,z^{29}+94068732852089205756130773605094705\,z^{28}+4410553\backslash\\
7622909592151335713091881133 8\,z^{27}+13196369454987612649737442242823787 79\,z^{26}-1376\backslash\\
268096732267953995912640790411 12\,z^{25}-3107200173797029922140553319870630314 1\,z^{24}-\\
2268861766669185609872402007686316931 50\,z^{23}-10339540172663822489847675868520 72\backslash\\
344191\,z^{22}-3356732946224373601649087937349109785896\,z^{21}-757312621278500761889122\backslash\\
5542456994124245\,z^{20}-9076459539413303184641722134776573895810\,z^{19}+10278671248090\backslash\\
3353774089183588154087884 25\,z^{18}+8514927435704329238592503365329429185355 0\,z^{17}+\\
240689360358498296007939096187740586134000\,z^{16}+429409878921957648790555775268 2\backslash\\
42743350000\,z^{15}+495779225046771906420255540348281344800000\,z^{14}+28712136337931261\backslash\\
6871562346484465378000000\,z^{13}-119682652007548350954457856750250720000000\,z^{12}-\\
39568346559268086740129348061619800000000\,z^{11}-327383462755042385949747691240 82\backslash\\
4000000000\,z^{10}-86642575450501391066787202019520000000000\,z^{9}+597046839721706795 4\backslash\\
8931977222400000000000\,z^{8}+72511610277412390990839363072000000000000\,z^{7}+33882896\backslash\\
755872071956886261760000000000000\,z^{6}+6311156771304917325766656000000000000000\,z^{5}+\\
512323021813756999680000000000000000000\,z^{4})\,\mathrm{Dz}^{6}+
\end{array}$$

Face-Centered Cubic Lattices

[Koutschan 2013]

$$
\begin{aligned}
&(1600200173148416\, z^{42} + 478782978712278912\, z^{41} + 5881538078613556 7104\, z^{40} + 4218590\backslash \\
&040421804170816\, z^{39} + 2042164444698164466 53424\, z^{38} + 7214118624119937529541160\, z^{37} + \\
&19510607071245354750679 8168\, z^{36} + 416887031952436819753395 9000\, z^{35} + 7187441279\backslash \\
&5312940511795668940\, z^{34} + 10135280399132492073678 42378270\, z^{33} + 117759241810488 93848\backslash \\
&357395670676\, z^{32} + 112862055818213392356279768225402\, z^{31} + 886340494836475569 8667411\backslash \\
&39358344\, z^{30} + 5592675973186567437279733685351646\, z^{29} + 269836357593337112434 2782835\backslash \\
&4079724\, z^{28} + 8505938846326414231366252654242 0618\, z^{27} + 248169568331816444804 0331315\backslash \\
&0735864\, z^{26} - 17397315299235032959847968065267 52758\, z^{25} - 1321568542142315740 1833903\backslash \\
&137021991092\, z^{24} - 601015147325177793295427498988934 53858\, z^{23} - 187406121933740 01721\backslash \\
&274116747818513 7320\, z^{22} - 367088786736715063908412462166156515566\, z^{21} - 136331238303\backslash \\
&9883490014154141815321 46340\, z^{20} + 2052937632229799753666758504303681446150\, z^{19} + \\
&894222086471130209202395016834853485 6300\, z^{18} + 2211277908345604739979169031967 3356\backslash \\
&808000\, z^{17} + 3666229983096485354830089546872350248 0000\, z^{16} + 386639362090547399 55701\backslash \\
&6490767847084 00000\, z^{15} + 155758412096326841847250746805511760 00000\, z^{14} - 23399775927\backslash \\
&110778754739301057544560000000\, z^{13} - 459575818445550681083386929618072000 00000\, z^{12} - \\
&3252500528545981111206628950523200000000 0\, z^{11} - 366121829233752392904666430464000\backslash \\
&0000000\, z^{10} + 1097039530150661129281453716480000000000 0\, z^{9} + 971311240519793594 2595\backslash \\
&533824000000000000\, z^{8} + 426097871942078389250377728000000000000000\, z^{7} + 7564670791225\backslash \\
&350266880000000000000000000\, z^{6} + 59201771409589697740800000000000000000\, z^{5})\, \mathrm{Dz}^{7} +
\end{aligned}
$$

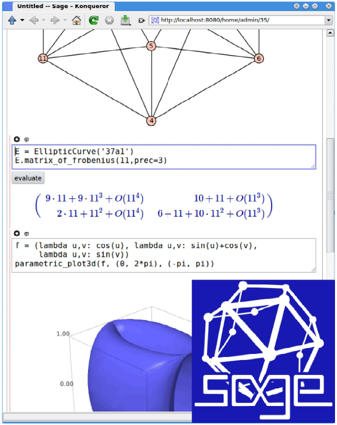
Face-Centered Cubic Lattices

[Koutschan 2013]

$$
\begin{array}{l}
(27122036833024\, z^{43} + 8208413201024064\, z^{42} + 1028987679702510976\, z^{41} + 75518451\backslash \\
137118783792\, z^{40} + 3743195619381989907184\, z^{39} + 135369638077546936261428\, z^{38} + 374561\backslash \\
5314367420203992832\, z^{37} + 818116193678600490459846 75\, z^{36} + 1440466637248203913774 33\backslash \\
4250\, z^{35} + 2072433111304027502371917 2850\, z^{34} + 2454466275416520460977927 68214\, z^{33} + \\
2395828801191215780780578117794\, z^{32} + 1914740747067311123186224941816 6\, z^{31} + 1228\backslash \\
639636214967463701886596967 02\, z^{30} + 6026212556484859243787006723310 54\, z^{29} + 19351926\backslash \\
64301617476137337671088360\, z^{28} + 694152712036783264243644290673234\, z^{27} - 39030042885\backslash \\
8189354559012891338726 22\, z^{26} - 29764596280393319656487373367019 1774\, z^{25} - 1329742929\backslash \\
72800721570400254928159 1538\, z^{24} - 3903989614825648819224432657208727646\, z^{23} - 603801\backslash \\
5534019664017777438417359311914\, z^{22} + 756528095115600975099282347955069 4170\, z^{21} + \\
83328126336960183101771239549883786325\, z^{20} + 297859836972471180382017327162905 95\backslash \\
5900\, z^{19} + 68122669439368525201713007390832584 0500\, z^{18} + 1055504316932586226613 39004\backslash \\
4310017920000\, z^{17} + 97498262514411065483466068899043460 0000\, z^{16} + 809214817274247946\backslash \\
2313593062347200 0000\, z^{15} - 1245692778975371208980497936649580000000\, z^{14} - 1877612972\backslash \\
16604654284152589154800000000 0\, z^{13} - 1186201691981014544058180007080000000000\, z^{12} - \\
44241392637013980291878304000000000000\, z^{11} + 5265884988550235591341773120000000 0\backslash \\
0000\, z^{10} + 41099923473883401024746956800000000000 0\, z^{9} + 174333012213810958051184640\backslash \\
000000000000\, z^{8} + 29950530354743793153638400000000000000 0\, z^{7} + 2276991208061142220 80\backslash \\
0000000000000000\, z^{6})\ \mathrm{Dz}^{8}
\end{array}
$$

order 8, degree 43, 43-digit coefficients

What it Is: A SageMath Implementation



```
E = EllipticCurve('37a1')
E.matrix_of_frobenius(11,prec=3)
```

$$\begin{pmatrix} 9\cdot 11+9\cdot 11^3+O(11^4) & 10+11+O(11^3) \\ 2\cdot 11+11^2+O(11^4) & 6+11+10\cdot 11^2+O(11^3) \end{pmatrix}$$

```
f = (lambda u,v: cos(u), lambda u,v: sin(u)+cos(v),
     lambda u,v: sin(v))
parametric_plot3d(f, (0, 2*pi), (-pi, pi))
```

- ▷ Python library
- ▷ "A viable alternative to Magma, Maple, Mathematica and Matlab"

```
sage: Pols.<z> =
      PolynomialRing(QQ)
sage: (z + 1)*(z-1)
z^2 - 1
```

http://sagemath.org/

GNU GPL v2+

What it Is: Based on ore_algebra

[Kauers, Jaroschek, Johansson, 2013–]

```
sage: from ore_algebra import OreAlgebra
sage: DiffOps.<Dz> = OreAlgebra(Pols)
sage: DiffOps
Univariate Ore algebra in Dz over Univariate Polynomial Ring
in z over Rational Field
sage: Dz*z
z*Dz + 1
```

Features: Euclidean arithmetic, closure properties, formal solutions, desingularization, first-order factors, guessing...

http://www.risc.jku.at/research/combinat/software/ore_algebra/

What it Is: The `-analytic` Branch

- **Symbolic-numeric** extensions for ore_algebra

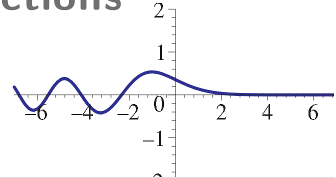
- Real & complex arithmetic based on **Arb** [Johansson 2012–]
  (`{Real,Complex}BallField` in Sage)

- Both for “end users” and for prototyping algorithms

- Development branch, not (yet) integrated into any release of ore_algebra

`http://marc.mezzarobba.net/code/ore_algebra-analytic`
GNU GPL v2+

What it Does: Special Functions



```
sage: diffop = Dz^2 - z
```

```
sage: diffop.numerical_solution(
        [1/(gamma(2/3)*3^(2/3)), -1/(gamma(1/3)*3^(1/3))],
        [0, i], 1e-40)
```

```
[0.3314933054321411889845293326171343458866 +/- 5.51e-41] +
[-0.31744985896844377347764292790925852645896 +/- 7.23e-42]*I
```

```
sage: ComplexBallField(138)(i).airy_ai()
```

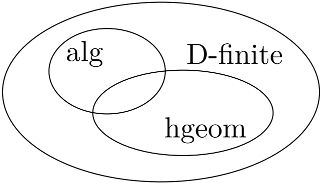
```
[0.33149330543214118898452933261713434588655 +/-  5.25e-42] +
[-0.31744985896844377347764292790925852645896 +/- 1.59e-42]*I
```

What it Does: D-Finite Functions

[Stanley, Zeilberger... 1980–]

An analytic function $y\colon \mathbb{C} \to \mathbb{C}$ is **D-finite** (holonomic) iff it satisfies a linear homogeneous ODE with polynomial coefficients:

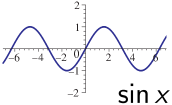
$$a_r(z)\, y^{(r)}(z) + \cdots + a_1(z) y'(z) + a_0(z)\, y(z) = 0, \qquad a_j \in \mathbb{K}[z]$$



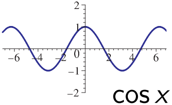
Philosophy:

Provide **general algorithms** for D-finite functions, using { ODE + initial values } as a data structure.

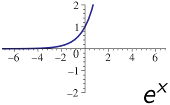
What it Does: D-Finite Functions



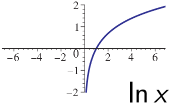
$\sin x$ ✓



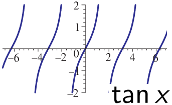
$\cos x$ ✓



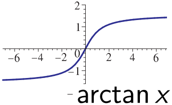
e^x ✓



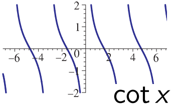
$\ln x$ ✓



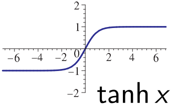
$\tan x$ ✗



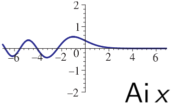
$\arctan x$ ✓



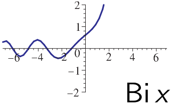
$\cot x$ ✗



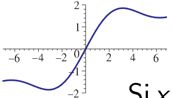
$\tanh x$ ✓



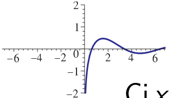
$\mathrm{Ai}\, x$ ✓



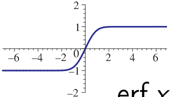
$\mathrm{Bi}\, x$ ✓



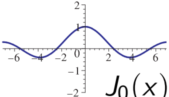
$\mathrm{Si}\, x$ ✓



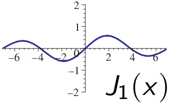
$\mathrm{Ci}\, x$ ✓



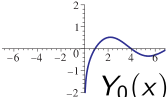
$\mathrm{erf}\, x$ ✓



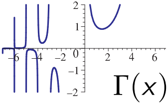
$J_0(x)$ ✓



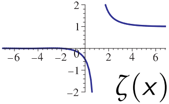
$J_1(x)$ ✓



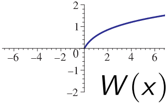
$Y_0(x)$ ✓



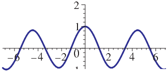
$\Gamma(x)$ ✗



$\zeta(x)$ ✗



$W(x)$ ✗



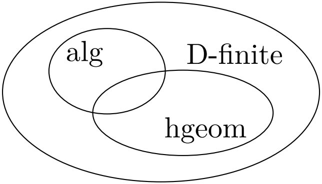
$C_{2,1/5}(x)$ ✗

What it Does: D-Finite Functions

[Stanley, Zeilberger... 1980–]

An analytic function $y\colon \mathbb{C} \to \mathbb{C}$ is **D-finite** (holonomic) iff it satisfies a linear homogeneous ODE with polynomial coefficients:

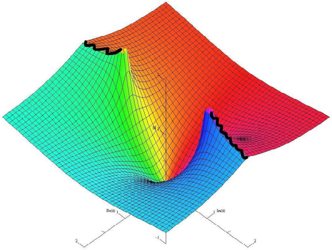
$$a_r(z)\, y^{(r)}(z) + \cdots + a_1(z) y'(z) + a_0(z)\, y(z) = 0, \qquad a_j \in \mathbb{K}[z]$$



Philosophy:

Provide **general algorithms** for D-finite functions, using { ODE + initial values } as a data structure.

What is Does: Analytic Continuation



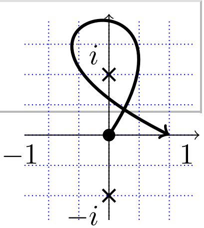
```
sage: dop = (z^2+1)*Dz^2 + 2*z*Dz
```

```
sage: dop.numerical_solution(
          ini=[0,1], path=[0,1])
```

```
[0.78539816339744831 +/- 1.08e-18]
```

```
sage: dop.numerical_solution(
          ini=[0,1],
          path=[0,i+1,2*i,i-1,0,1])
```

```
[3.9269908169872415
      +/- 4.81e-17]
+ [+/- 4.63e-21]*I
```



$$y(z) = \arctan(z)$$

$$(z^2+1)\,y''(z) + 2\,z\,y'(z) = 0$$

What it Does: Transition Matrices

$$(z^2+1)\,y''(z)+2\,z\,y'(z)=0 \qquad \Rightarrow \qquad \begin{bmatrix} y(1) \\ y'(1) \end{bmatrix} = \underbrace{\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}}_{\nearrow} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

“transition matrix”

```
sage: dop.numerical_transition_matrix([0,1])
```

```
[ 1.000000000000000 [0.7853981633974483 +/- 1.46e-17]]
[                 0 [0.5000000000000000 +/- 1.03e-17]]
```

```
sage: n(pi/4)
```

```
0.785398163397448
```

$$\begin{aligned} f(z) &= 1 & &= 1+0\cdot z+O(z^2) \\ g(z) &= \arctan(z) & &= 0+1\cdot z+O(z^2) \end{aligned} \qquad \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} = \begin{bmatrix} f(1) & g(1) \\ f'(1) & g'(1) \end{bmatrix}$$

What it Does: Regular Singular Points

The previous examples only involved **ordinary** (= non-singular) points.

$$\boldsymbol{z}^2\, y''(z) + z\, y'(z) + (z^2 - \nu^2)\, y(z) = 0 \qquad \text{(Bessel eq.)}$$

$\nwarrow$

singular point at 0
regular (≈ tame) in this case

✓ $z^{-3/2} \log z$ ✓ $z^{i\sqrt{2}}$ × $e^{\pm 1/z}$

Theorem **[Fuchs, 1866]**

Assume that 0 is a regular singular point. Then, for some $D \ni 0$, there exists a basis of solutions defined on $D \backslash \{0\}$ of the form

$$z^{\lambda}\,(y_0(z) + y_1(z) \log z + \cdots + y_t(z) \log^t z), \qquad \lambda \in \bar{\mathbb{Q}}, \qquad y_i \textbf{ analytic} \text{ on } D.$$

Regular Singular Connection Problems

$$z^2\,y''(z) + z\,y'(z) + z^2\,y(z) = 0 \quad \Rightarrow \quad \begin{bmatrix} y(1) \\ y'(1) \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

where $y(z) = a \cdot \mathbf{log\,z} + b \cdot \mathbf{1} + O(z)$

```
sage: dop = z*Dz^2 + Dz + z^2
```

```
sage: dop.local_basis_monomials(0)
```

```
 [log(z), 1]
```

```
sage: dop.numerical_transition_matrix([0, 1], 1e-10)
```

```
[[0.071033860082 +/- 1.75e-13]  [0.89193746813 +/- 2.73e-12]]
[  [1.0960559318 +/- 2.73e-11] [-0.31515459259 +/- 4.76e-13]]
```

Applications: special functions, analytic combinatorics, resummation...

Asymptotics of Apéry Numbers

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \qquad b_n = \sum_{k=1}^{n} \left(\frac{a_n}{k^3} - \sum_{m=1}^{k} \frac{(-1)^m \binom{n}{k}^2 \binom{n+k}{k}^2}{2\, m^3 \binom{n}{m} \binom{n+m}{m}} \right)$$

(1, 5, 73, 1445, 33001...) (0, 6, 351/4, 62531/36, ...)

- ▷ The OGS $a(z)$ and $b(z)$ are solutions of $L = \boldsymbol{z^2\,(z^2 - 34\,z + 1)}\, D_x^4 + \cdots$
- ▷ Singular points: 0, $\boldsymbol{\alpha} = \left(\sqrt{2}+1\right)^4 \approx 33.9$, $\boldsymbol{\alpha^{-1}} = \left(\sqrt{2}-1\right)^4 \approx 0.0294$
- ▷ Prove: $a_n, b_n = \alpha^{n+o(n)}$ $b_n - \zeta(3)\, a_n = \alpha^{-n+o(n)}$
- ▷ Local expansion at $\boldsymbol{\alpha^{-1}}$: $a(z) = c_0\, f_0(z) + \boldsymbol{c_1}\, f_1(z) + c_2\, f_2(z) + c_3\, f_3(z)$

  where $f_0(\alpha^{-1}+t) = 1 + \mathrm{O}(t^3)$ $f_3(\alpha^{-1}+t) = t + \mathrm{O}(t^3)$

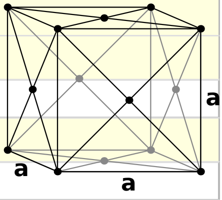
  $f_1(\alpha^{-1}+t) = \sqrt{\boldsymbol{t}} + \mathrm{O}(t^3)$ $f_4(\alpha^{-1}+t) = t + \mathrm{O}(t^3)$
- ▷ Singularity analysis: $(c_1 \neq 0)$

$$a(z) \sim \boldsymbol{c_1}\, \sqrt{z - \alpha^{-1}} \qquad \Rightarrow \qquad a_n \sim \boldsymbol{c_1}\, [z^n] \sqrt{z - \alpha^{-1}} \sim \frac{\boldsymbol{c_1}\, i}{2\sqrt{\alpha\,\pi}}\, \alpha^n\, n^{-3/2}$$

Face-Centered Cubic Lattices

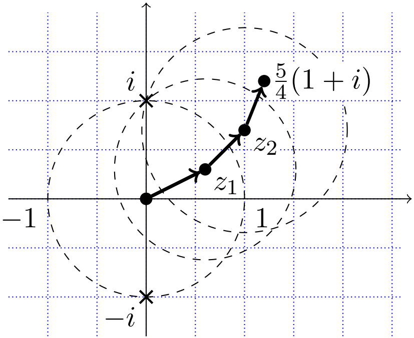
[Koutschan 2013]

```
sage: dop4 = ((-1 + z)*z^3*(2 + z)*(3 + z)*(6 + z)*(8
      + z)*(4 + 3*z)^2*Dz^4 + 2*z^2*(4 + 3*z)*(-3456 -
      2304*z + 3676*z^2 + 4920*z^3 + 2079*z^4 + 356*z^5
      + 21*z^6)*Dz^3 + 6*z*(-5376 - 5248*z + 11080*z^2
      + 25286*z^3 + 19898*z^4 + 7432*z^5 + 1286*z^6 +
      81*z^7)*Dz^2 + 12*(-384 + 224*z + 3716*z^2 + 7633*z^3
      + 6734*z^4 + 2939*z^5 + 604*z^6 + 45*z^7)*Dz +
      12*z*(256 + 632*z + 702*z^2 + 382*z^3 + 98*z^4 +
      9*z^5))
sage: dop4.local_basis_monomials(0)
 [1/6*log(z)^3, 1/2*log(z)^2, log(z), 1]
sage: dop4.local_basis_monomials(1)
 [1, (z - 1)*log(z - 1), z - 1, (z - 1)^2]
sage: dop4.numerical_transition_matrix([0,1])[0,-1]
[1.1058437979212048 +/- 3.99e-17] + [+/- 6.96e-26]*I
```



© User:Baszoetekouw

How it Works: A Taylor Series Method



$\arctan\left(\frac{5}{4}(1+i)\right)$?

$$\begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix} = \begin{bmatrix} 1 & 0.57\ldots + 0.22\ldots \\ 0 & 0.72\ldots - 0.20\ldots \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

$$\begin{bmatrix} y(z_2) \\ y'(z_2) \end{bmatrix} = \begin{bmatrix} 1 & 0.36\ldots + 0.32\ldots \\ 0 & 0.75\ldots - 0.07\ldots \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

…

At each step, compute the sum of the power series expansion of each entry of the transition matrix.

The idea extends to the regular singular case.

How it Works: Recurrences

The **Taylor coefficients** of a D-finite function $y(z) = \sum_{n=0}^{\infty} y_n z^n$ obey a linear **recurrence relation** with polynomial coefficients:

$$b_s(n)\, y_{n+s} + \cdots + b_1(n)\, y_{n+1} + b_0(n)\, y_n = 0.$$

(And conversely, for D-finite formal power series.)

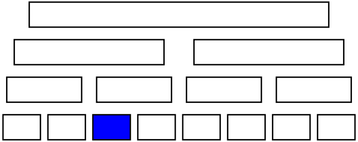
Leads to **fast algorithms** (not fully implemented yet)

[Schroeppel 1972; Brent 1976; Chudnovsky & Chudnovsky 1988; van der Hoeven 1999, 2001; M. 2010, 2012; Johansson 2014]

Best complexity:

time $\boldsymbol{O(M(n \log^2 n))}$, space $\boldsymbol{O(n)}$

for fixed z and $\varepsilon = 2^{-n}$



How it Works: Truncation Errors

Problem

$$\sum_{n=0}^{\infty} u_n z^n = \underbrace{\sum_{n=0}^{N-1} u_n z^n}_{\text{known}} + \underbrace{\sum_{n=N}^{\infty} u_n z^n}_{|\cdot| \leqslant \mathbf{?}}$$

"Adaptive" bounds:

$$A x = b \qquad \|A^{-1}\| \leqslant \boldsymbol{M}$$

$$A \in \mathrm{GL}_n(\mathbb{C})$$

$$A \tilde{x} = \tilde{b} \qquad \Rightarrow \qquad \|x - \tilde{x}\| \leqslant \boldsymbol{M} \cdot \underbrace{\|b - \tilde{b}\|}_{\text{known}}$$

Compute a bound similar to $\boldsymbol{M}$ when A replaced with a differential operator?

How it Works: Majorant Series

[Cauchy; …; van der Hoeven 2001; M. & Salvy 2010]

- ▷ Instead of directly bounding $\left|\sum_{n \geqslant N} u_n z^n\right|$, compute a **majorant series**:

$$\sum v_n z^n \in \mathbb{R}_{\geqslant 0}[[z]] \qquad \text{s.t.} \qquad \forall n, \quad |u_n| \leqslant v_n$$

- ▷ Bound the differential equation with a simple "**model equation**":

$$L(z, D_z) \cdot u = 0 \qquad \lll \qquad v'(z) - \frac{1}{(1 - \alpha z)} v(z) = 0$$

for us: always 1st order

- ▷ Solve the model equation and study the solutions:

$$v(z) = \exp \int^z \frac{\mathrm{d}t}{1 - \alpha t} \qquad \left|\sum_{n=N}^{+\infty} u_n z^n\right| \leqslant \sum_{n=N}^{+\infty} v_n |z|^n \leqslant ?$$

How it Works: Adaptive Majorants

$$L(z,D_z)\cdot u=0 \qquad\qquad u(z)=\sum_{n=0}^{\infty}u_n\,z^n=\underbrace{\sum_{n=0}^{N-1}u_n\,z^n}_{\tilde{u}(z)}+\sum_{n=N}^{\infty}u_n\,z^n$$

Residual: $q(z):=L(z,D_z)\cdot\tilde{u}$

- ▷ Model equation

$$\begin{aligned} q(z) &\quad \prec\!\!\prec \quad \hat{q}(z)\\ L(z,D_z)\cdot(\tilde{u}-u)=q &\quad \prec\!\!\prec \quad \hat{L}(z,D_z)\cdot v=\hat{q}\end{aligned}$$

- ▷ Majorant property:

$$(\forall n\leqslant n_0)\quad |u_n|\leqslant v_n \qquad\Rightarrow\qquad (\forall n)\quad |u_n|\leqslant v_n$$

- ▷ Solve the model equation

$$v(z)=h(z)\left(\underset{\substack{\nearrow\\ \text{choose cst}=0}}{\text{cst}}+\int^z\underset{\substack{\nwarrow\\ =\mathrm{O}(z^N)}}{\frac{t^{-1}\,\hat{q}(t)}{h(t)}}\,\mathrm{d}t\right) \qquad \text{where } h(z)=\exp\int^z t^{-1}\,\hat{a}(t)\,\mathrm{d}t$$

Summary

What it is: an extension of ore_algebra written in/for SageMath

What it does: numerical analytic continuation & singular connection, for arbitrary D-finite functions, with rigorous error bounds

How it works: Taylor series, analytic continuation, recurrences, majorants, ball arithmetic...

Code available at

`http://marc.mezzarobba.net/code/ore_algebra-analytic`

Perspectives

Fast algorithms, lower-level code,

evaluation on intervals, D-finite functions as objects,

irregular singular connection problems [van der Hoeven 2006]...

Comments, bug reports, feature requests, examples welcome!

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