Open arc diagrams and plane walks

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École polytechnique

Joint work with:

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- Éric Fusy
- Marni Mishna

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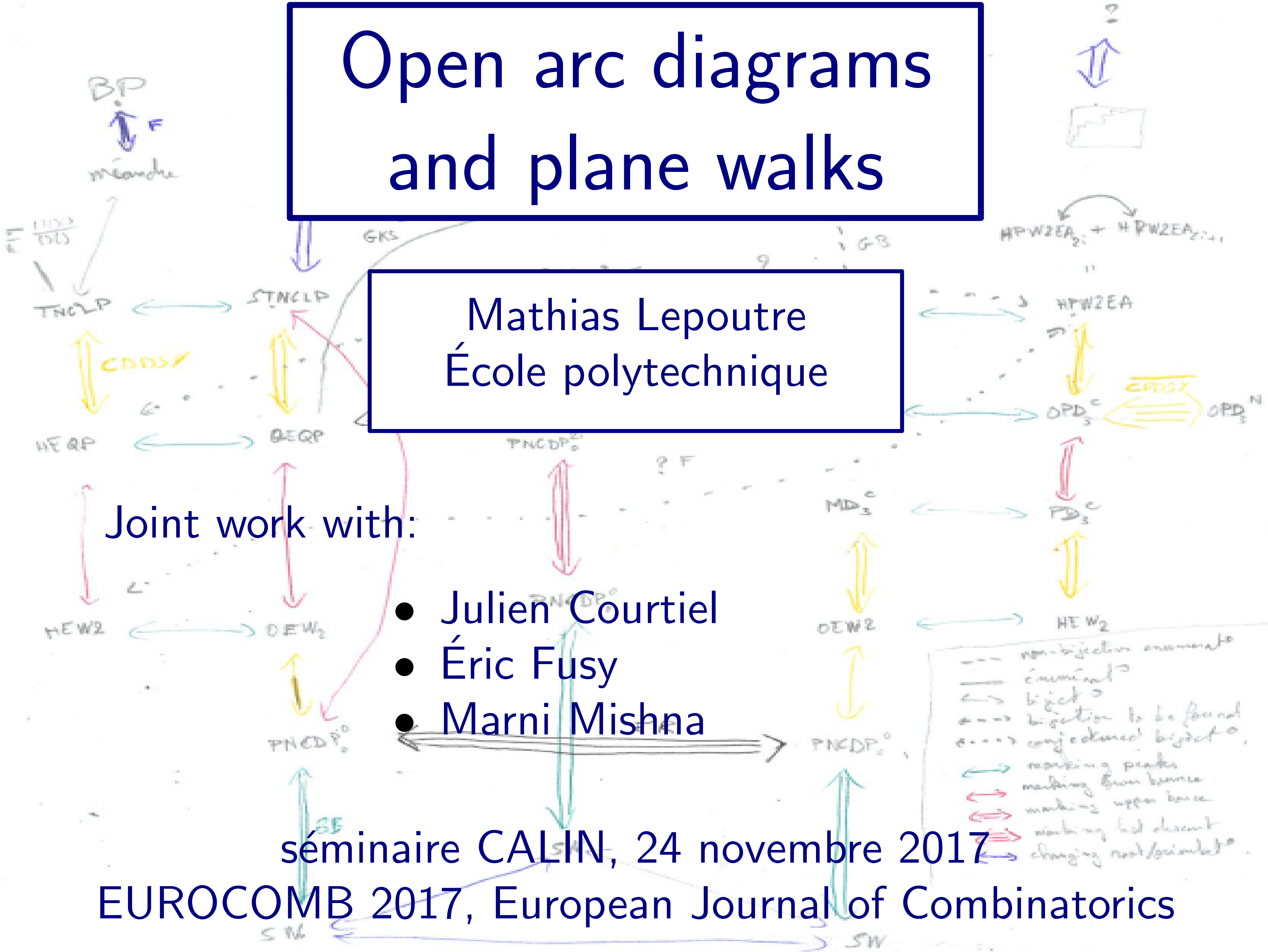

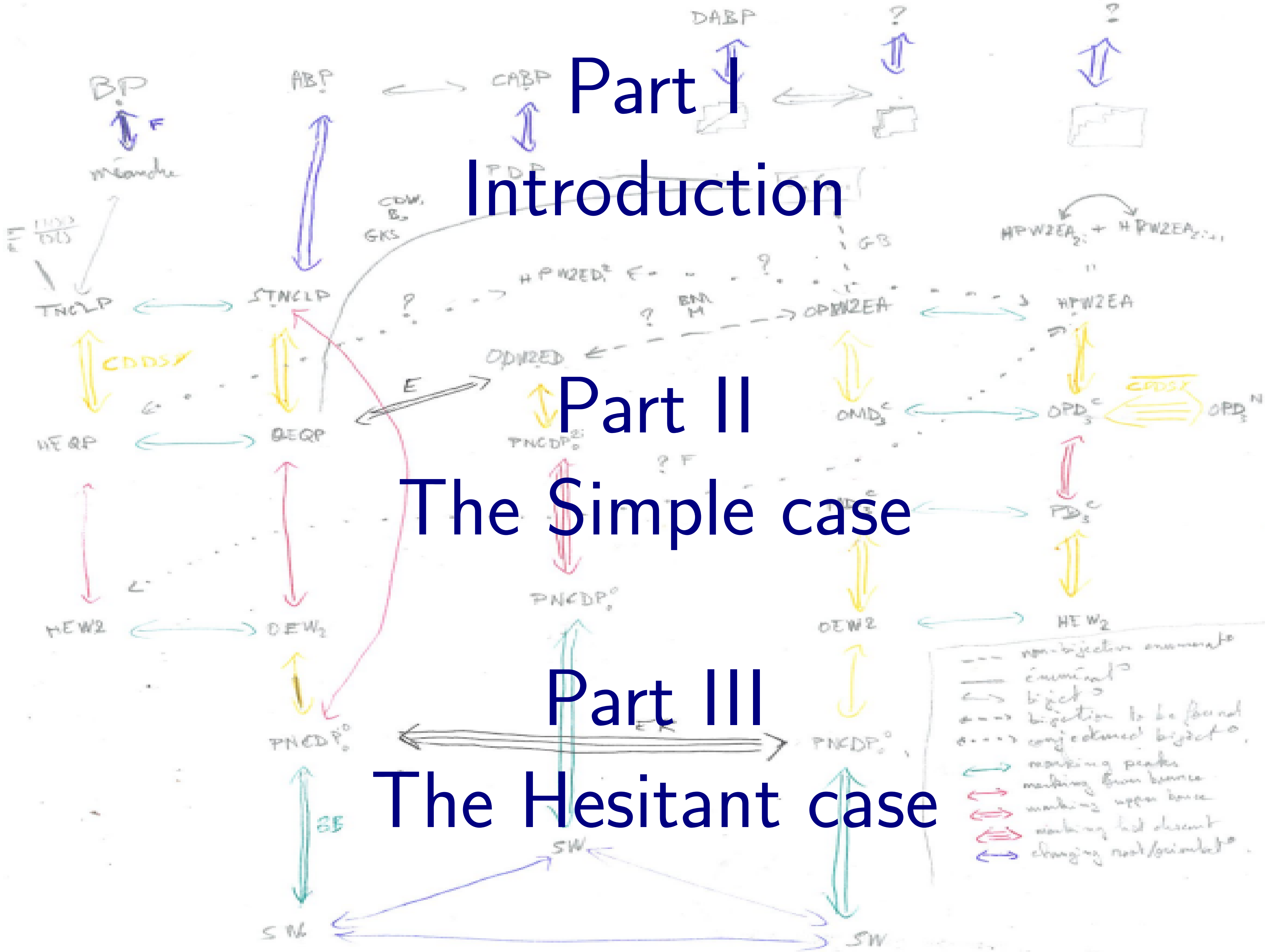

Part I
Introduction

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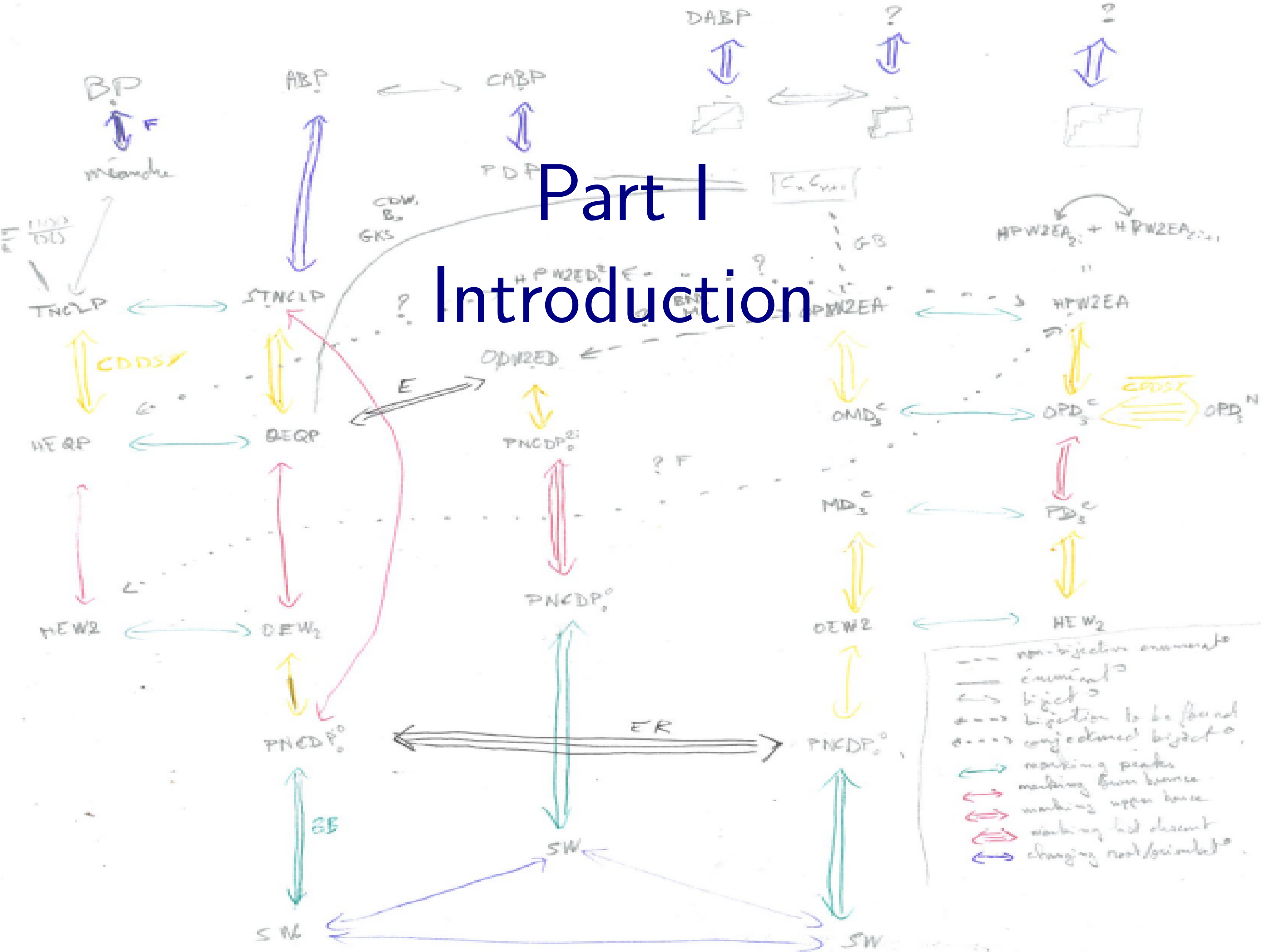
The Simple case

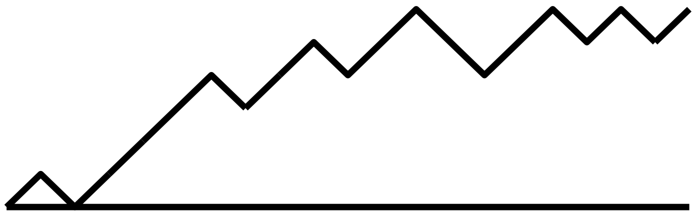
Part III
The Hesitant case

Part I
Introduction

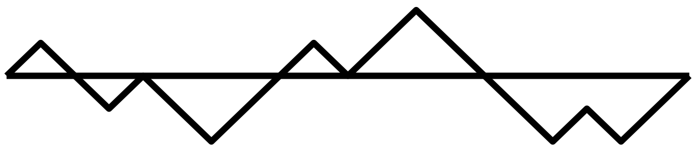


Domain constraint, marking, ending constraint

meander

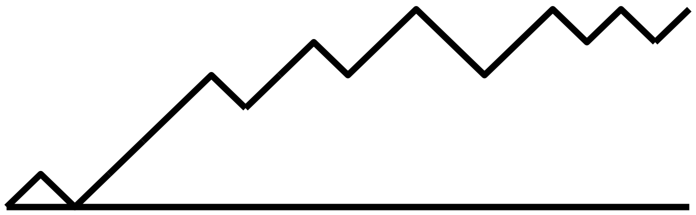


bridge

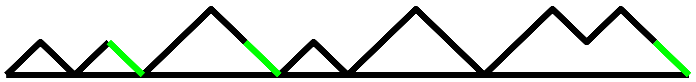


Domain constraint, marking, ending constraint

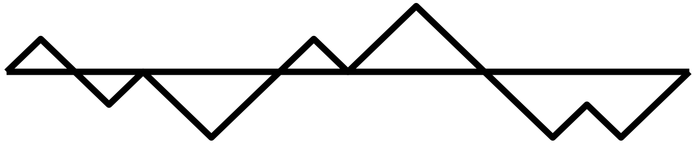
meander



Dyck path with marked steps from 1 to 0

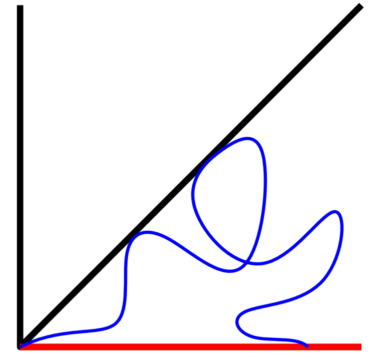


bridge

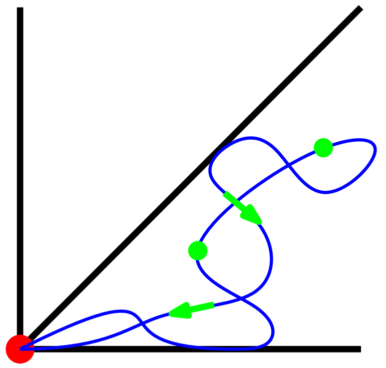


Domain constraint, marking, ending constraint

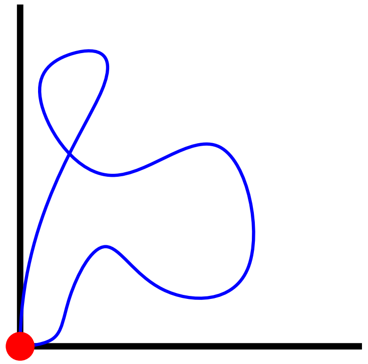
Axis-walk in the octant



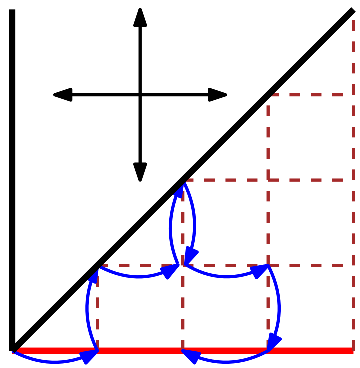
Excursion in the octant
with marking



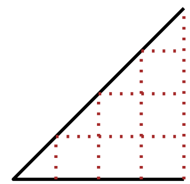
Excursion in the
quarter-plane



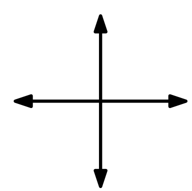
Walks, Tableaux, Diagrams
The Simple case



Walk



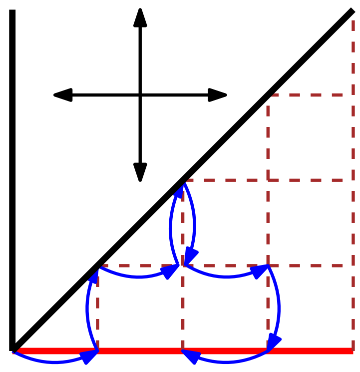
Walk in the (2-dimensional) octant



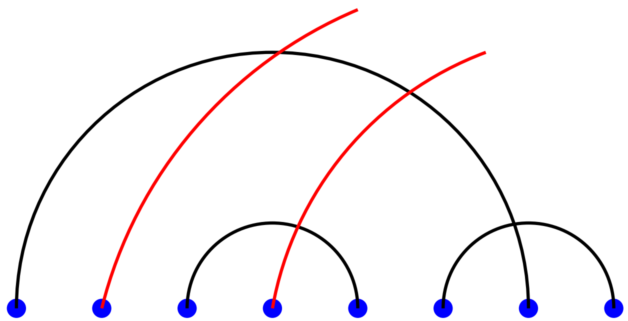
n step of type N, S, E, O

ending on the axis at $(i, 0)$

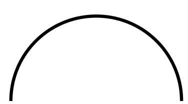
Walks, Tableaux, Diagrams
The Simple case



Walk

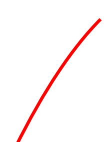


Diagram

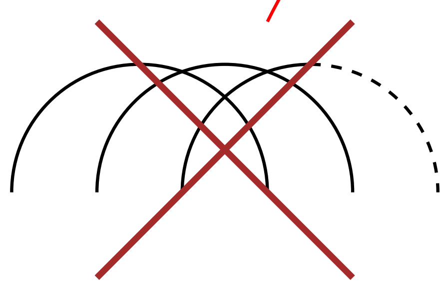


matching diagram

of length n

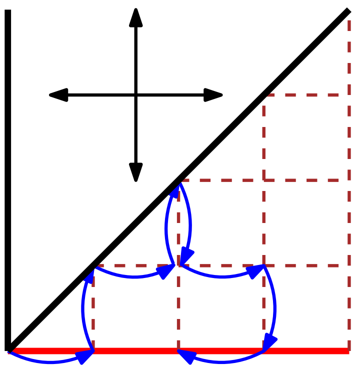


with i open arcs

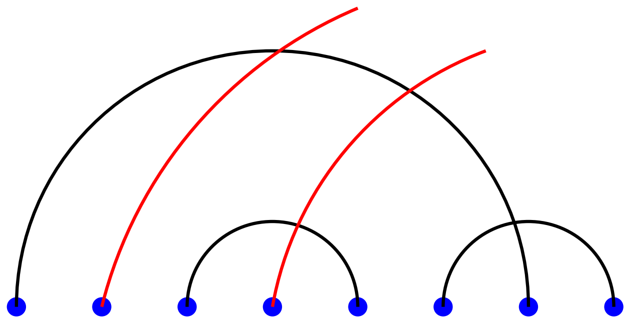


without 3-crossing

Walks, Tableaux, Diagrams
The Simple case



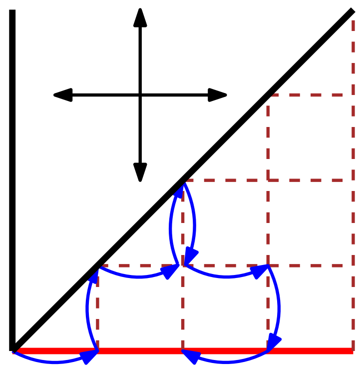
Walk

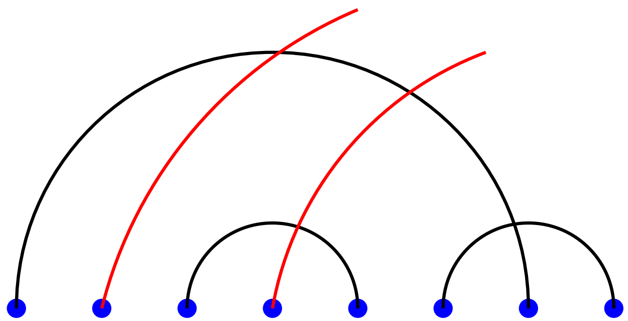


Diagram

Respective advantages :

Walks, Tableaux, Diagrams
The Simple case

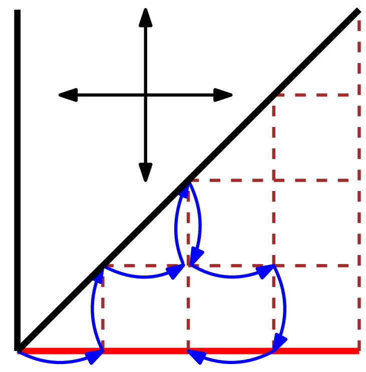




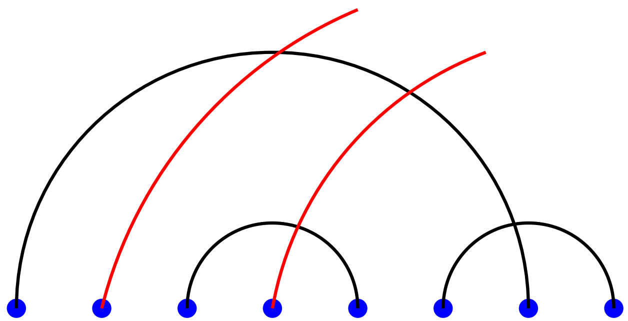
Respective advantages :

- well-known objects
- easy reccurence relations for generating series
- a more natural phrasing of problems

Walks, Tableaux, Diagrams
The Simple case



Walk



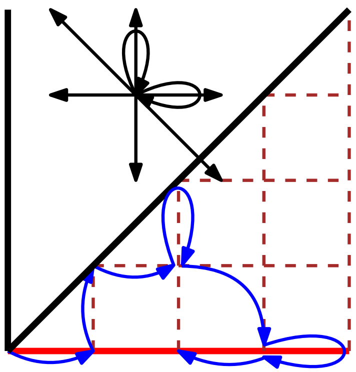
Diagram

Respective advantages :

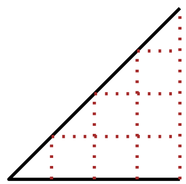
- well-known objects
- easy reccurence relations for generating series
- a more natural phrasing of problems

- new generating trees
- easily-removable open arcs

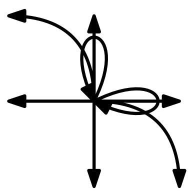
Walks, Tableaux, Diagrams
The Hesitating case



Walk



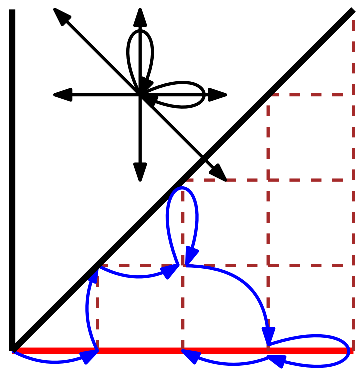
Walk in the $(2$-dimensional$)$ octant



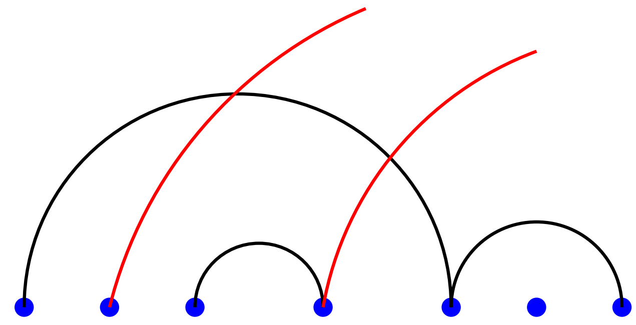
n steps of type N, S, E, O, NE, NS, EO, ES

enging on the axis at $(i, 0)$

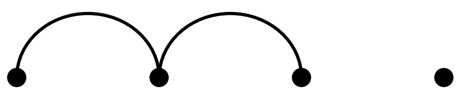
Walks, Tableaux, Diagrams
The Hesitating case



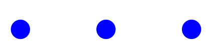
Walk



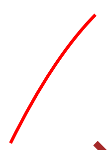
Diagram



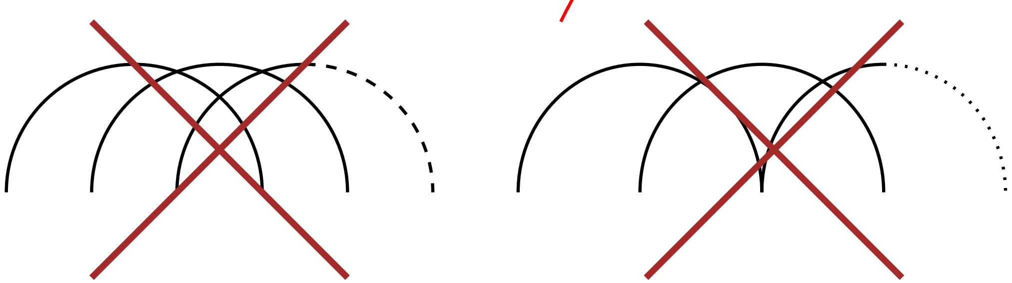
partition diagram



of length n



with i open arcs



without enhanced 3-crossing

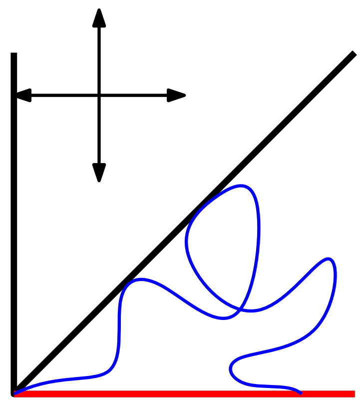
Domain constraint $\leftrightarrow$ ending constraint

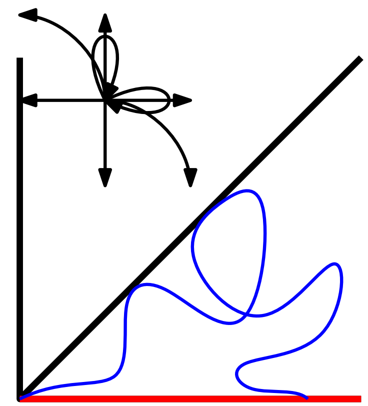
Domain constraint $\leftrightarrow$ ending constraint

The simple case

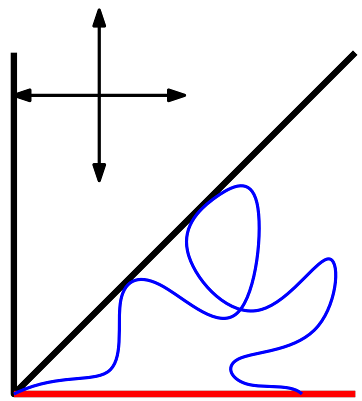
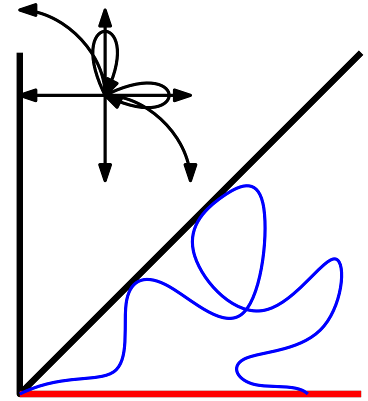
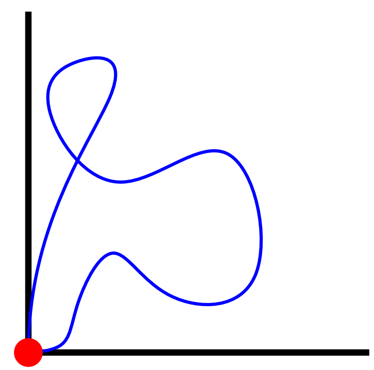
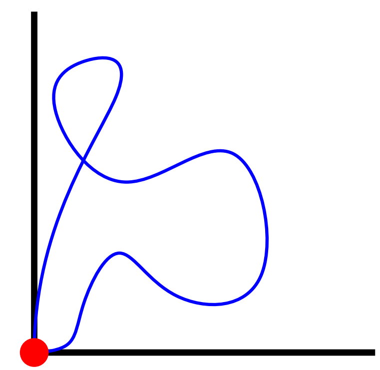
The Hesitating case

Axis-walk in the octant

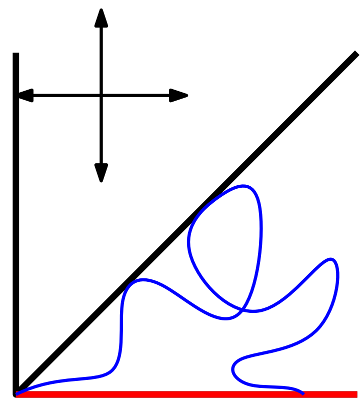
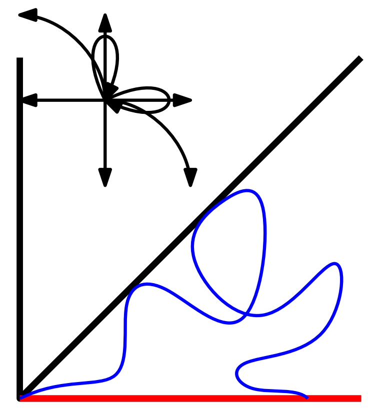
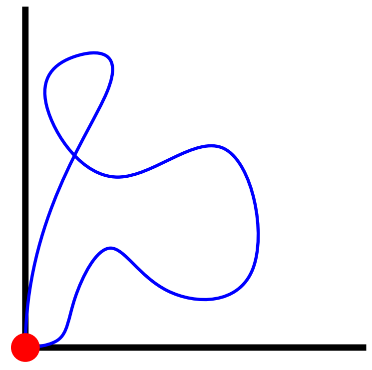
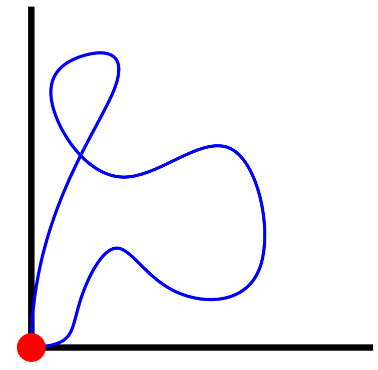




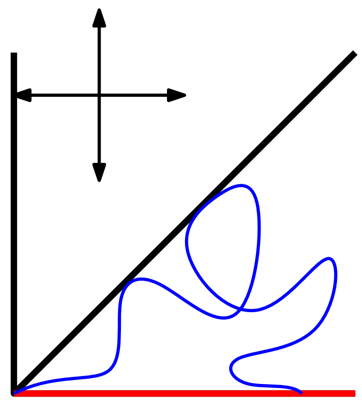
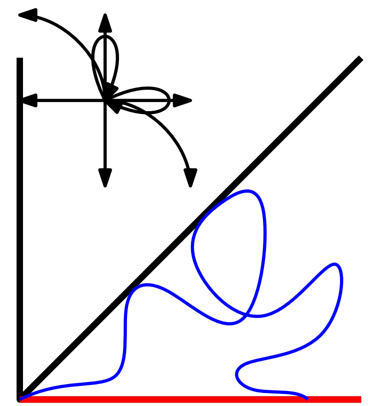
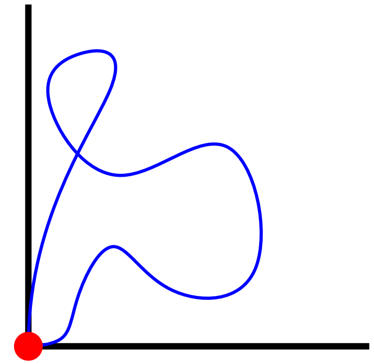
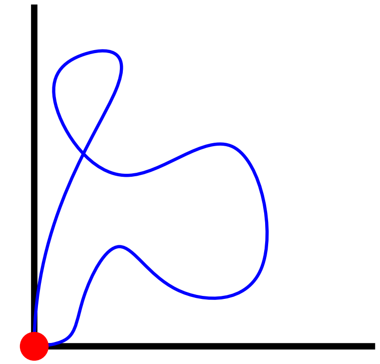
Domain constraint $\leftrightarrow$ ending constraint

	The simple case	The Hesitating case
Axis-walk in the octant		
Excursion in the quarter-plane		

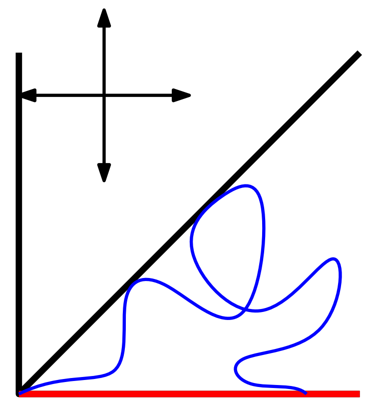
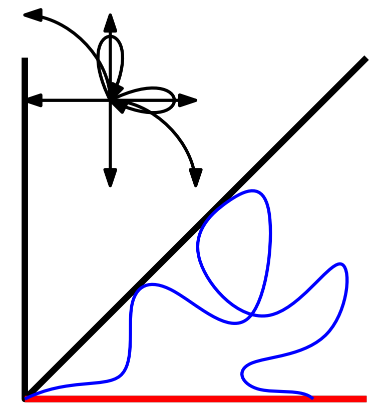
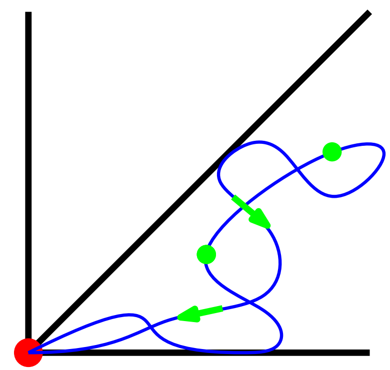
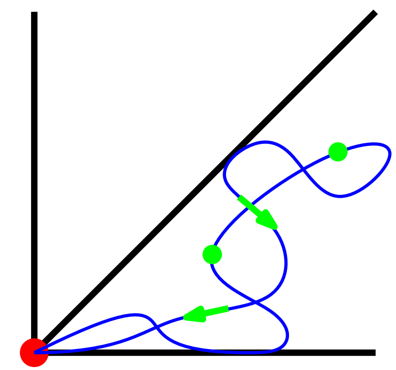
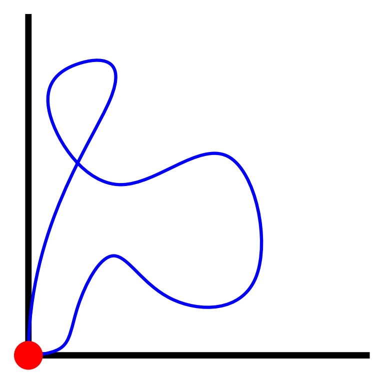
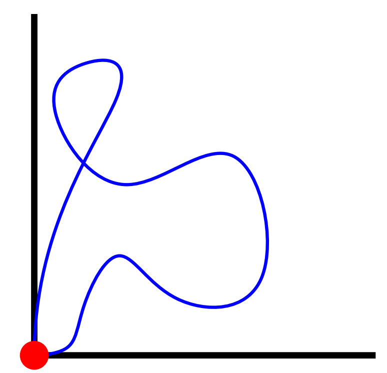
Domain constraint $\leftrightarrow$ ending constraint

	The simple case	The Hesitating case
Axis-walk in the octant		
Excursion in the quarter-plane		
Cardinality	$\mathcal{C}_n \cdot \mathcal{C}_{n+1}$	$\mathcal{B}_{n+1}$

Domain constraint $\leftrightarrow$ ending constraint

	The simple case	The Hesitating case
Axis-walk in the octant		
Excursion in the quarter-plane		

Domain constraint $\leftrightarrow$ ending constraint

	The simple case	The Hesitating case
Axis-walk in the octant		
Excursion in the octant with marking		
Excursion in the quarter-plane		

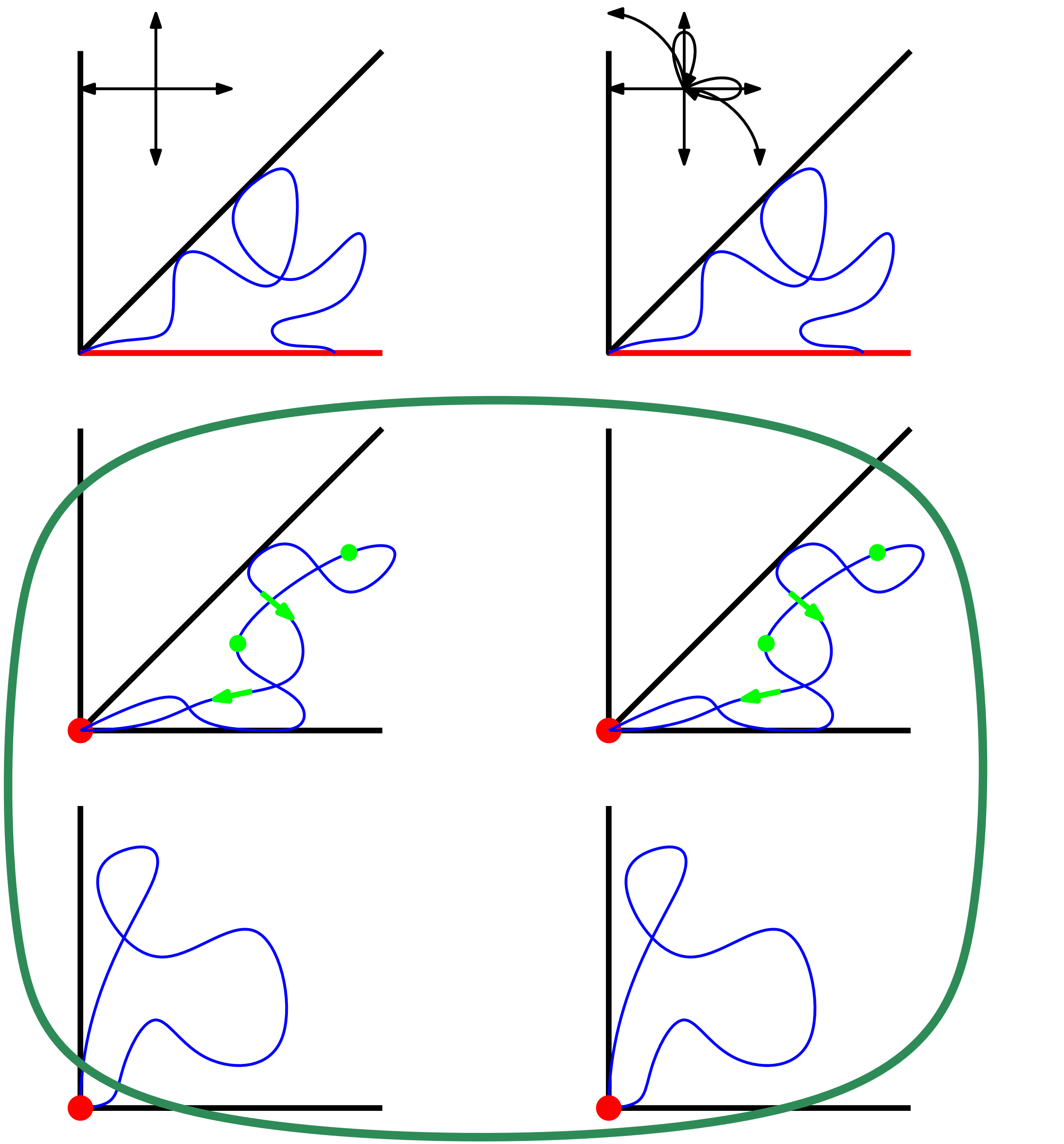
Domain constraint $\leftrightarrow$ ending constraint

The simple case | The Hesitating case

Axis-walk in the octant

Excursion in the octant with marking

Excursion in the quarter-plane



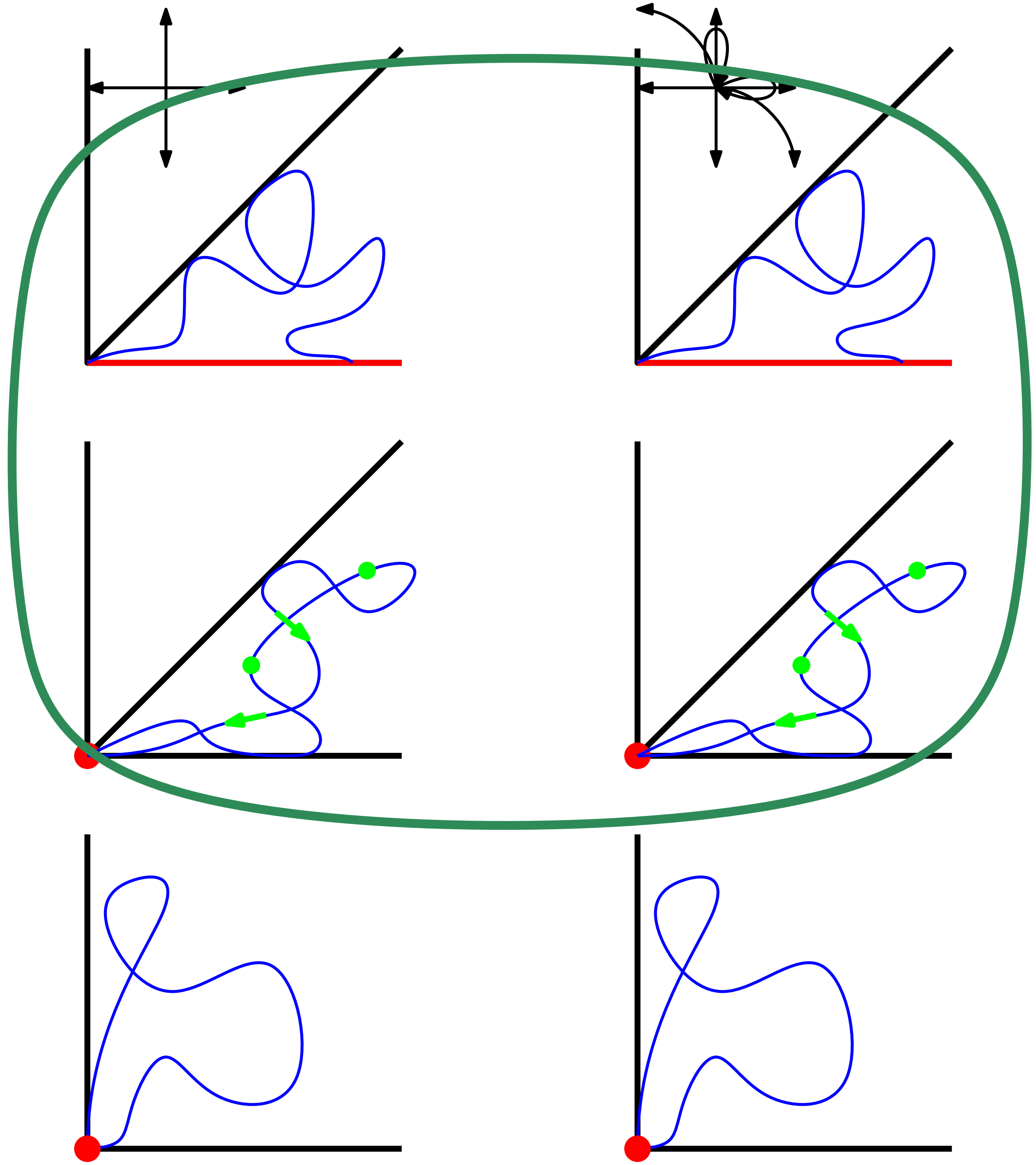
Domain constraint $\leftrightarrow$ ending constraint

The simple case | The Hesitating case

Axis-walk in the octant

Excursion in the octant with marking

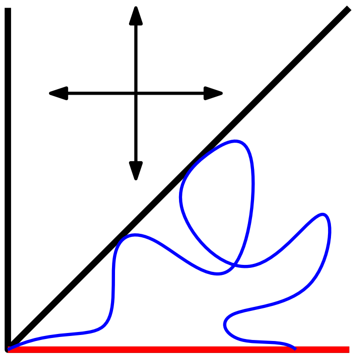
Excursion in the quarter-plane



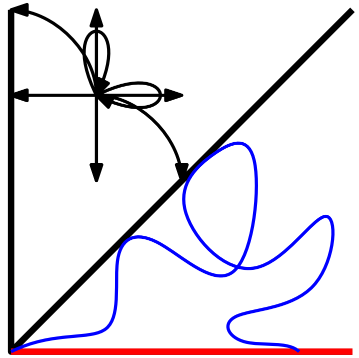
A new approach via open arc diagrams

Remove the open arcs in order to get marked excursions in the octant

Simple axis-walk in the octant



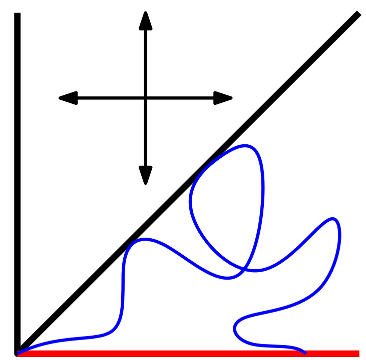
Hesitating axis-walk in the octant



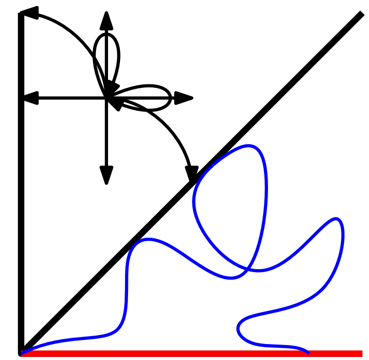
A new approach via open arc diagrams

Remove the open arcs in order to get marked excursions in the octant

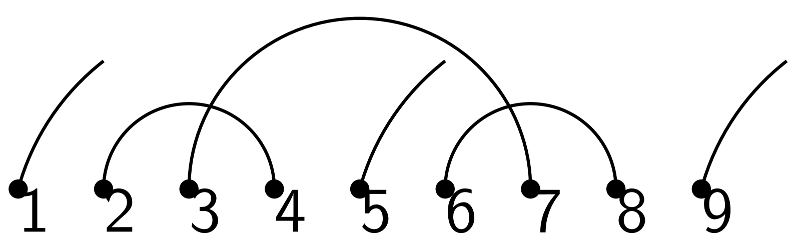
Simple axis-walk in the octant



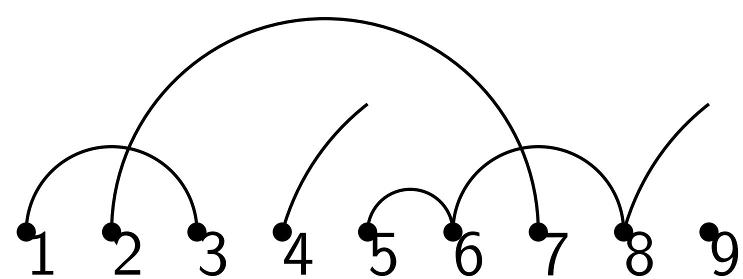
Hesitating axis-walk in the octant



Open matcing diagram without 3-crossing



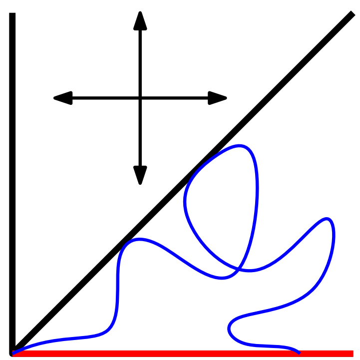
Open partition diagram without enhanced 3-crossing



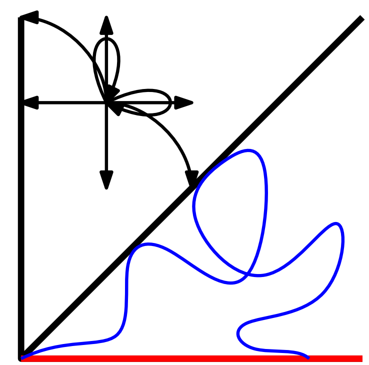
A new approach via open arc diagrams

Remove the open arcs in order to get marked excursions in the octant

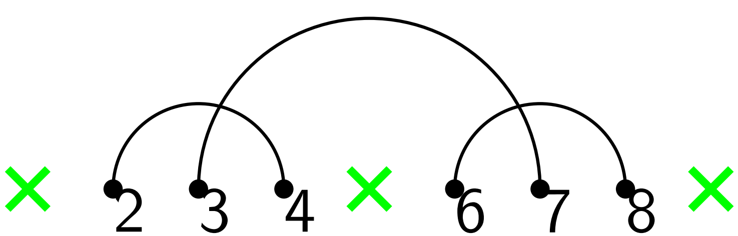
Simple axis-walk in the octant



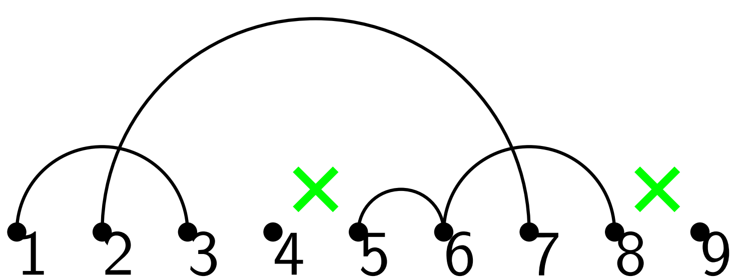
Hesitating axis-walk in the octant



Matching diagram without 3-crossing, with marking



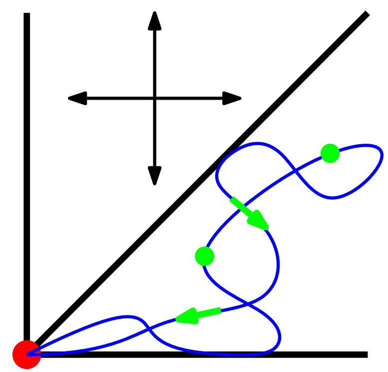
Partition diagram without enhanced 3-crossing, with marking



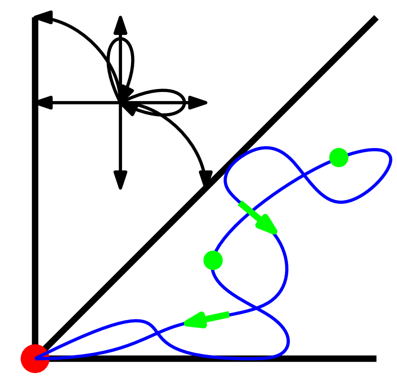
A new approach via open arc diagrams

Remove the open arcs in order to get marked excursions in the octant

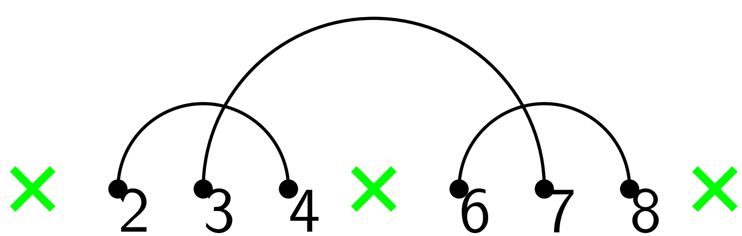
Simple excursion in the octant with marking



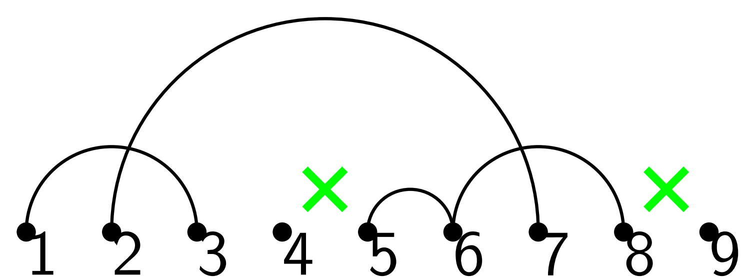
Hesitating excursion in the octant with marking



Matching diagram without 3-crossing, with marking

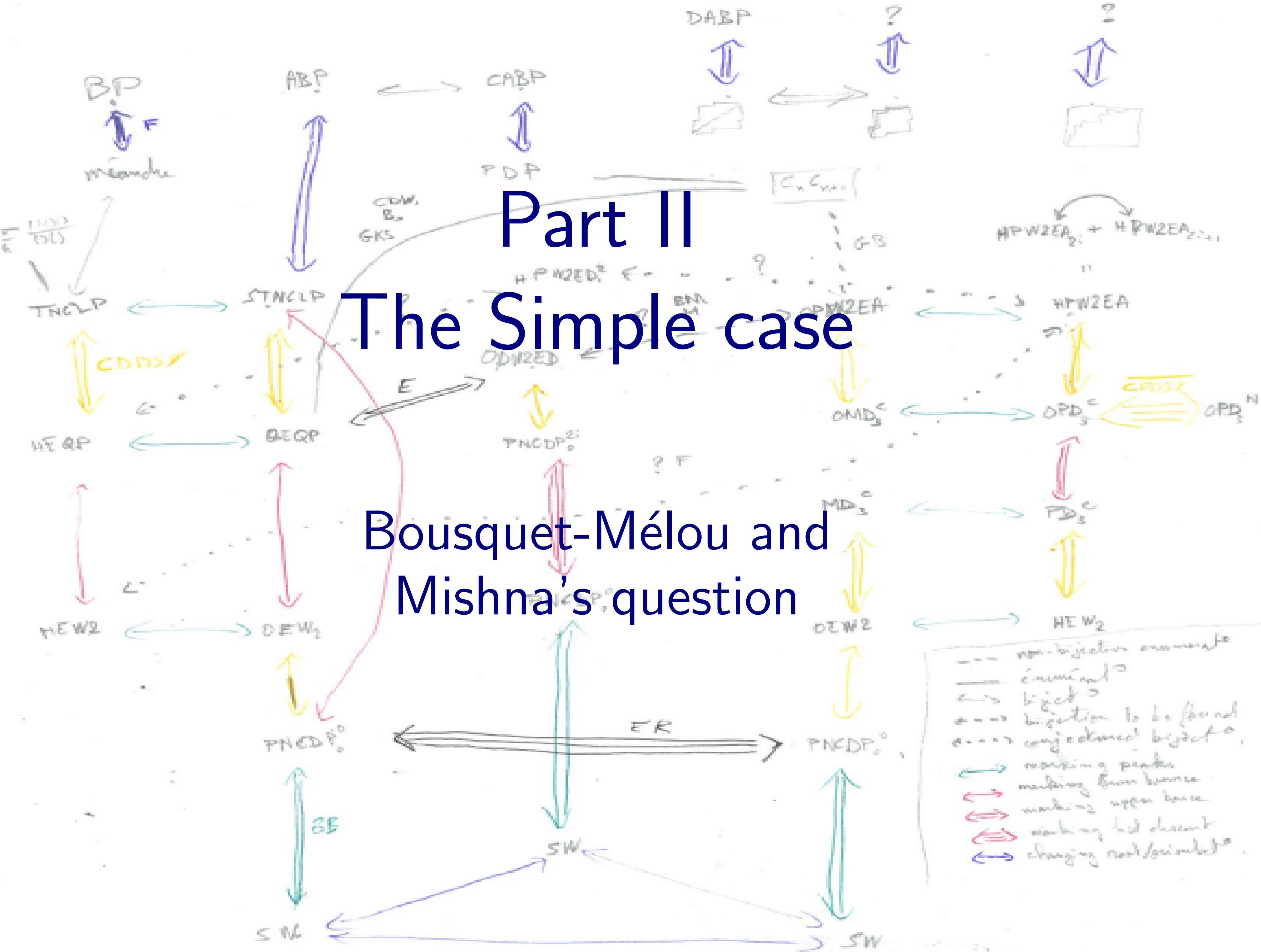


Partition diagram without enhanced 3-crossing, with marking

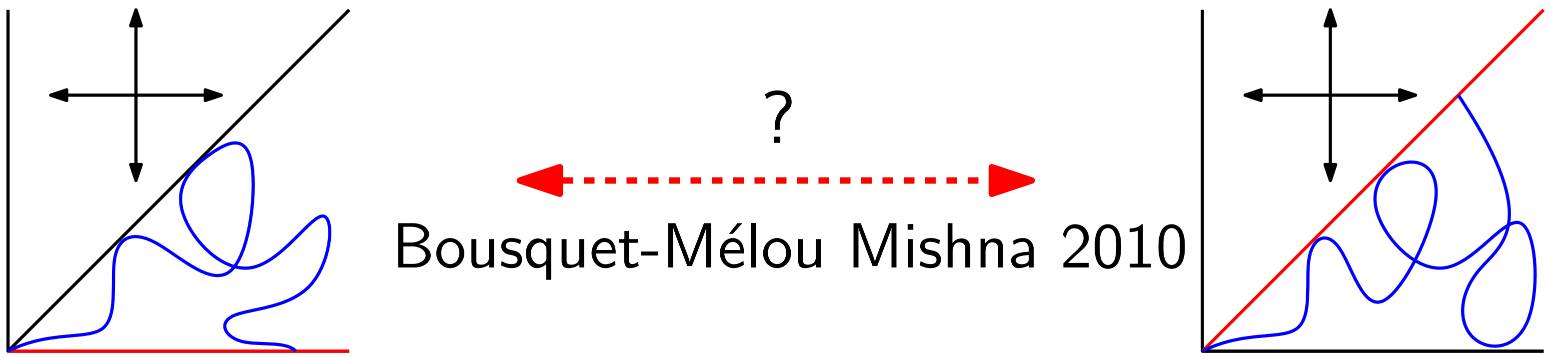


Part II
The Simple case

Bousquet-Mélou and Mishna's question



Bousquet-Mélou and Mishna's question



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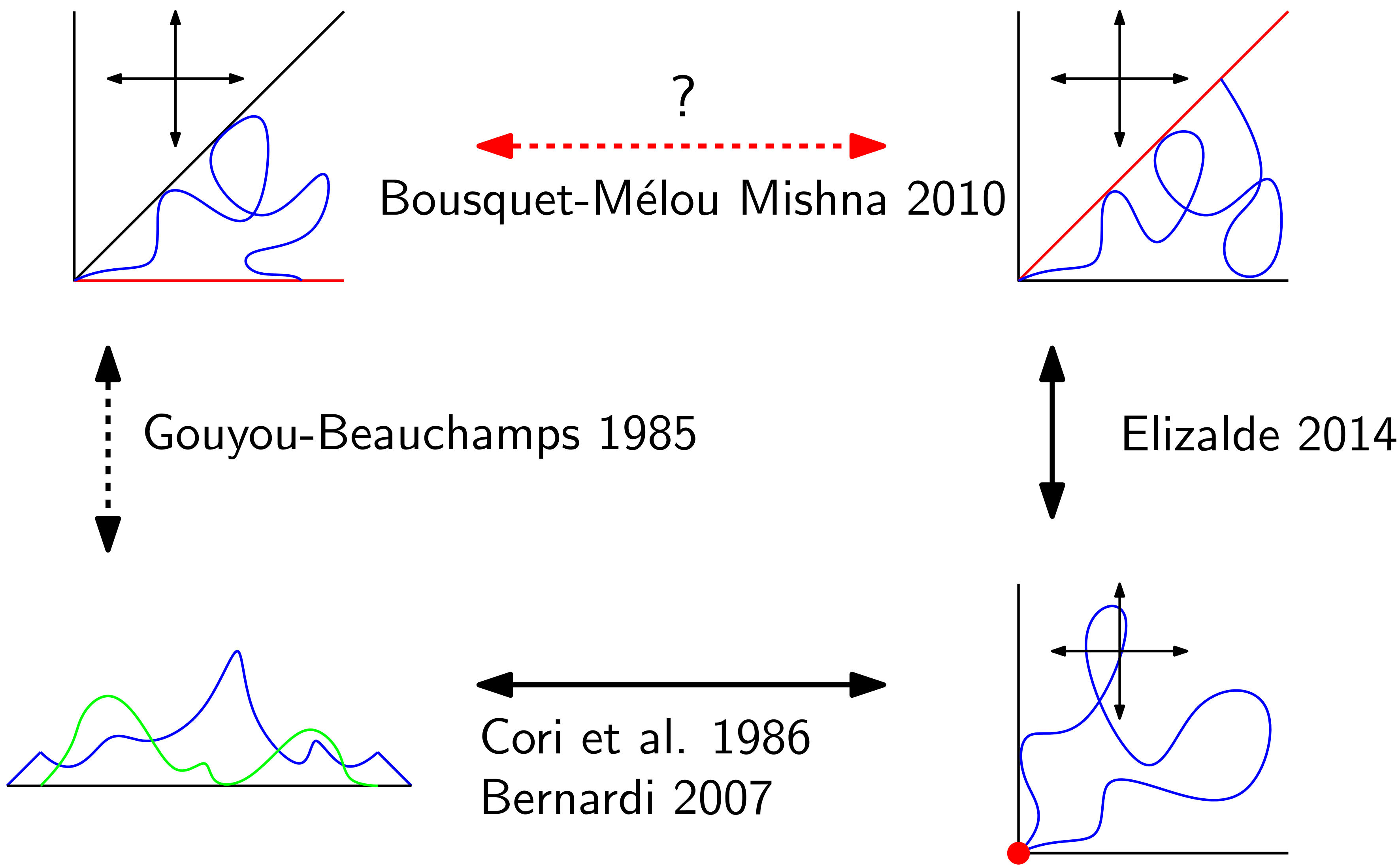
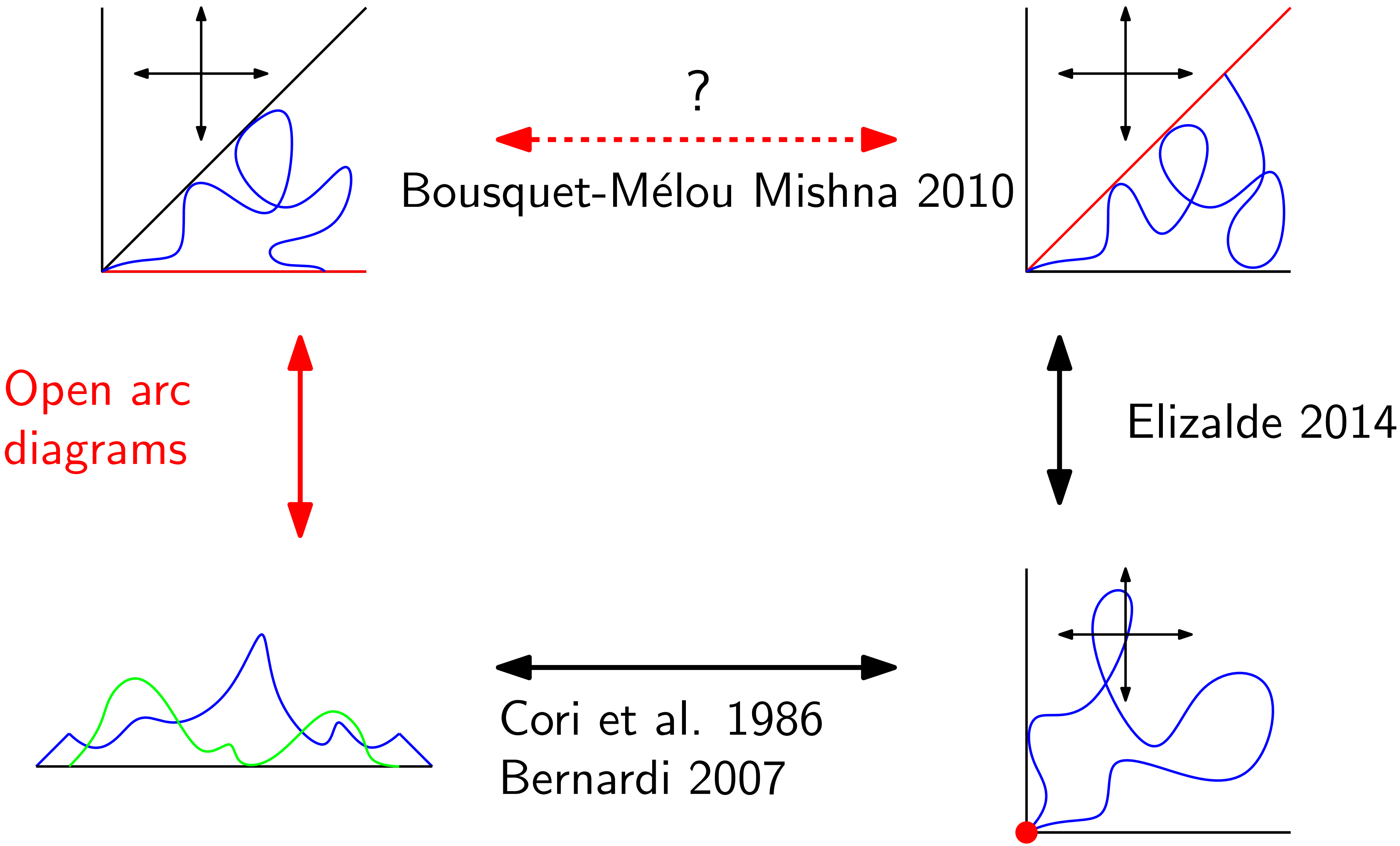


Bousquet-Mélou and Mishna's question



The missing part

Reminder :

Gouyou-Beauchamps 1985 (non-bijective) :

Simple axis-walks in the octant are counted by $C_{\lfloor \frac{n+1}{2} \rfloor} \cdot C_{\lceil \frac{n+1}{2} \rceil}$, where C_n is the n-th Catalan number.

The missing part

Reminder :

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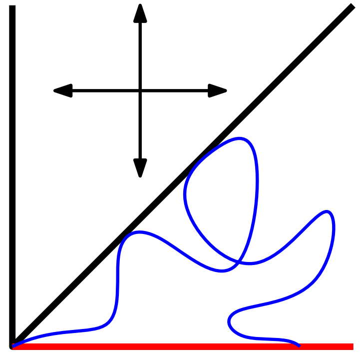
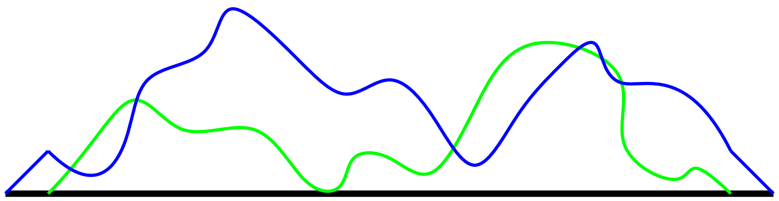
Simple axis-walks in the octant are counted by $C_{\lfloor \frac{n+1}{2} \rfloor} \cdot C_{\lceil \frac{n+1}{2} \rceil}$, where C_n is the n-th Catalan number.

Objective :

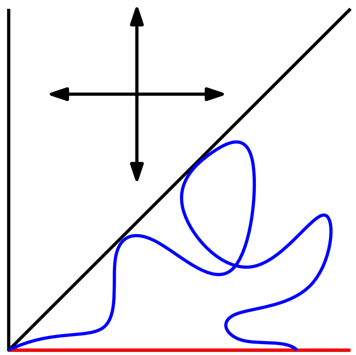
- Build a bijection between Simple axis-walks in the octant of length $2n$ and pairs of Dyck paths of half-lengths n and $n+1$.



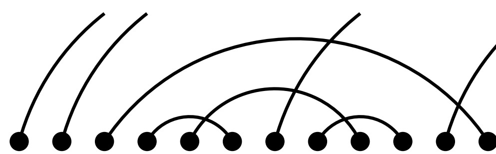


The even case

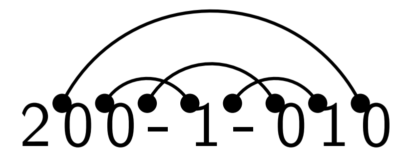
Simple axis-walks in the octant of length $2n$



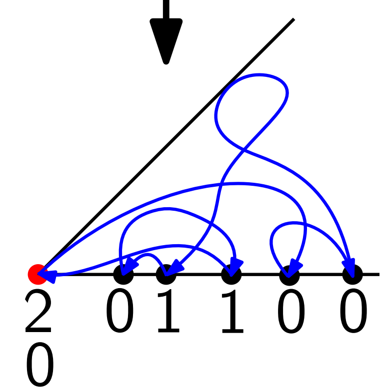
Open matching diagram with no 3-crossing of length $2n$



Matching diagram without 3-crossing with weights on open intervals, of *size* $2n$



size $=$ length $+$ weight



Simple excursion in the octant, with weights on the axis, of size $2n$

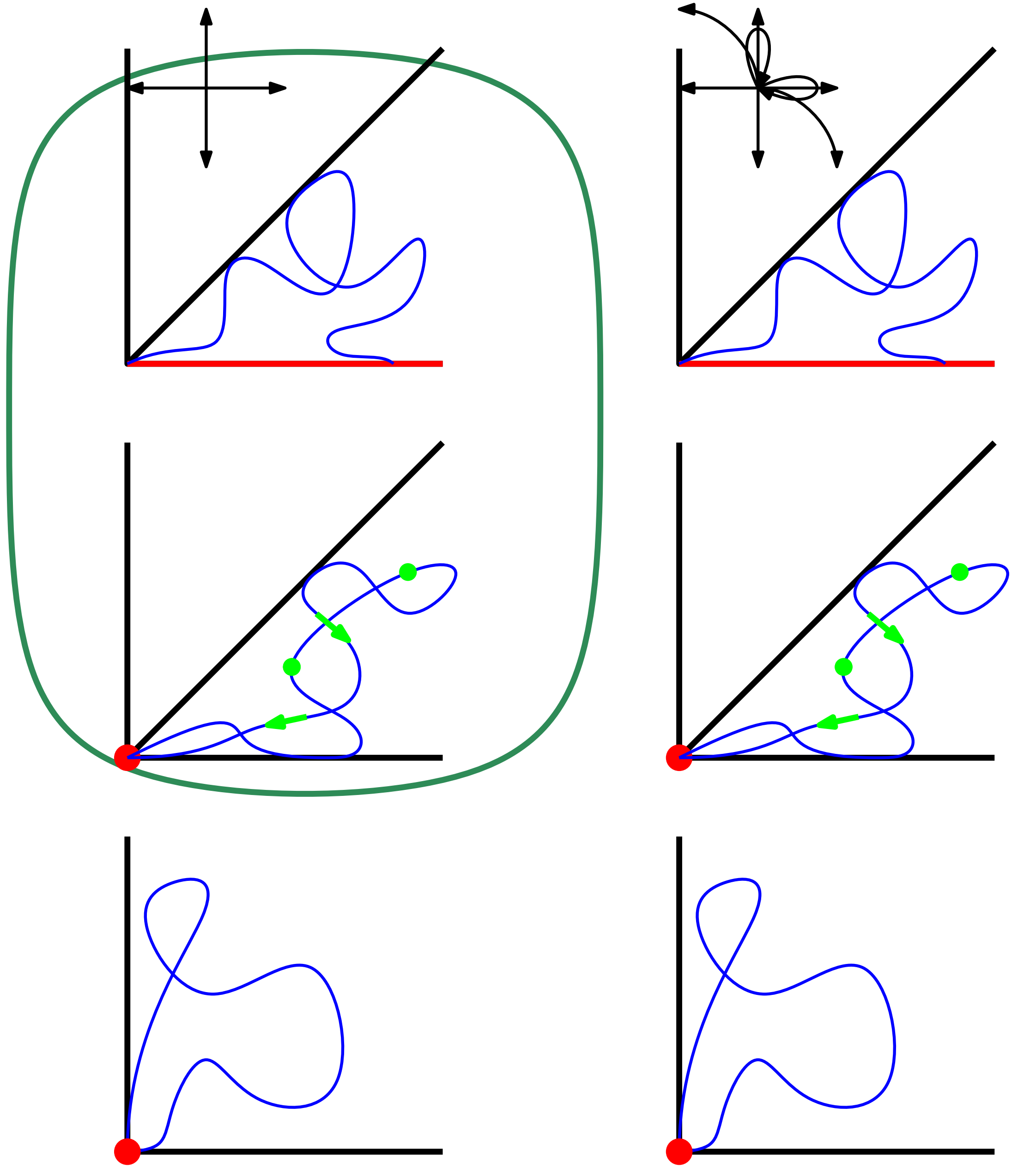
Domain constraint ↔ Ending constraint

Simple case | Hesitating case

Axis-walk in the octant

Excursion in the octant with marking

Excursion in the quarter-plane



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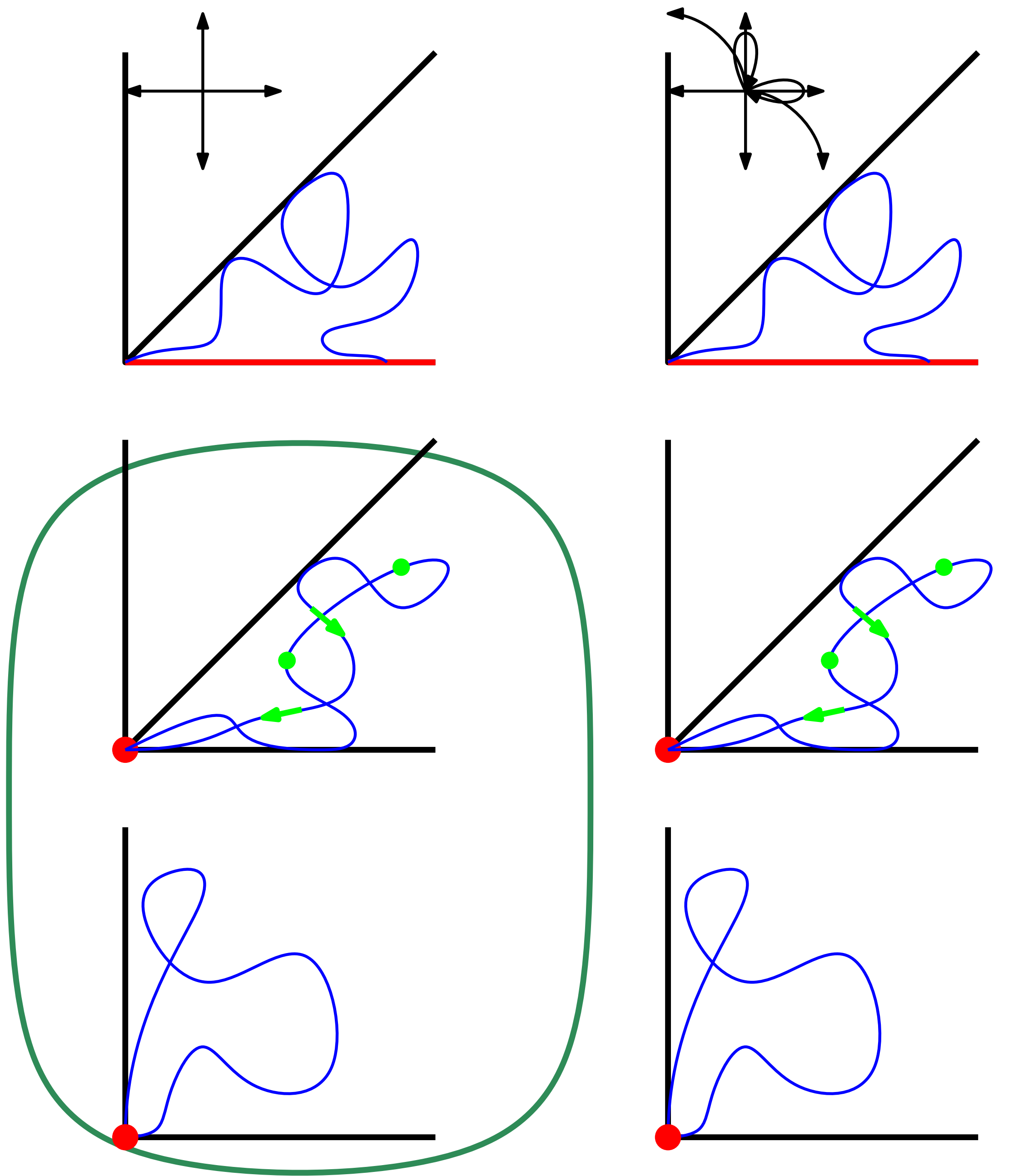
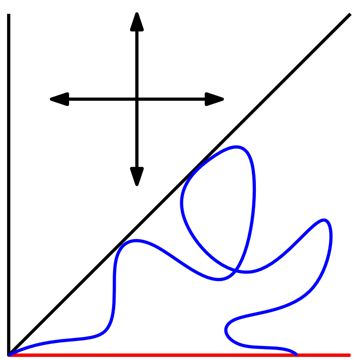
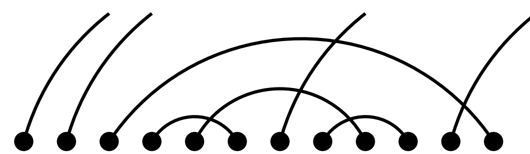


The even case

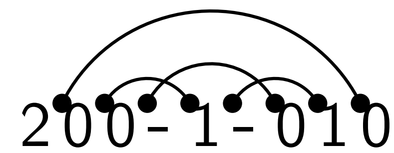
Simple axis-walks in the octant of length $2n$

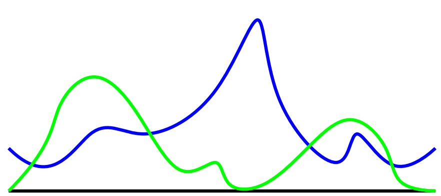


Open matching diagram with no 3-crossing of length $2n$

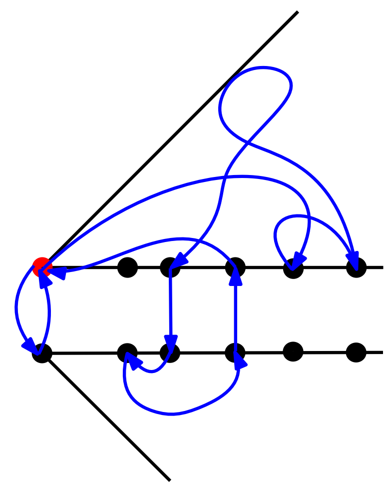


Matching diagram without 3-crossing with weights on open intervals, of *size* $2n$

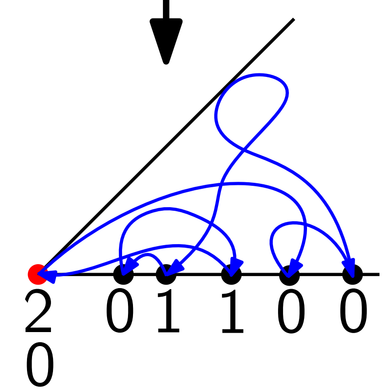




Pair of positive paths of length $2n$ going from $(1,0)$ to $(1,0)$



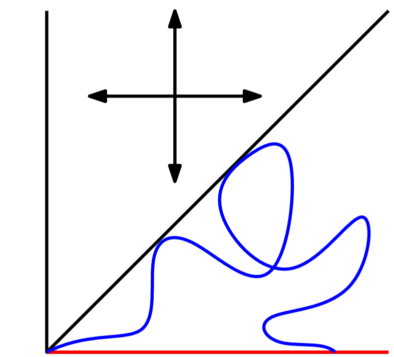
Simple inter-diagonals excursion of length $2n$



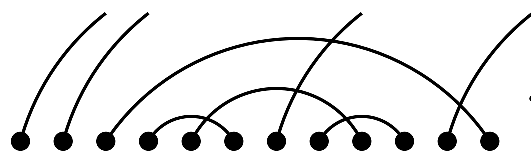
Simple excursion in the octant, with weights on the axis, of size $2n$

The even case

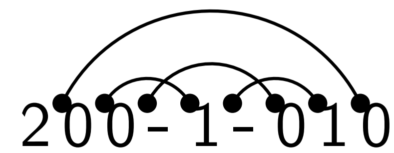
Simple axis-walks in the octant of length $2n$

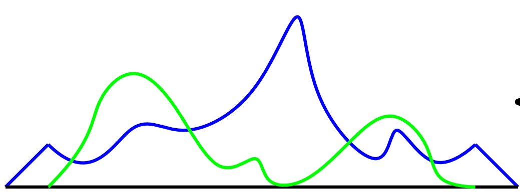


Open matching diagram with no 3-crossing of length $2n$

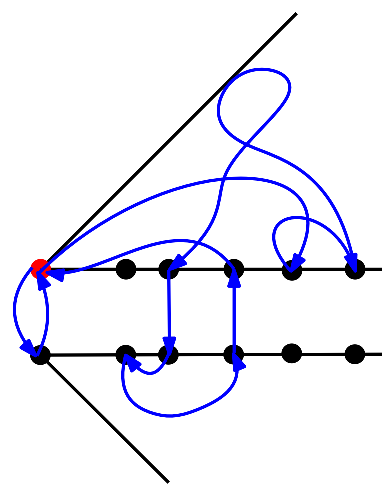


Matching diagram without 3-crossing with weights on open intervals, of *size* $2n$

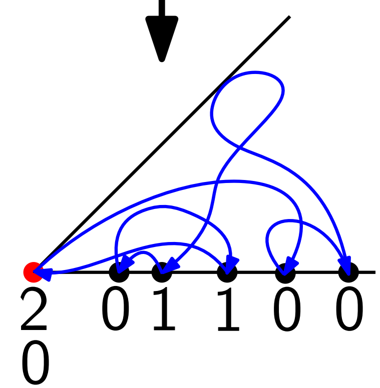




Pair of Dyck paths of half-lengths $(n, n+1)$



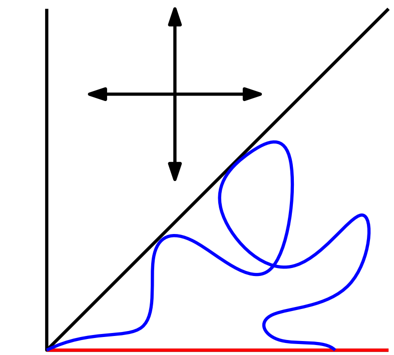
Simple inter-diagonals excursion of length $2n$



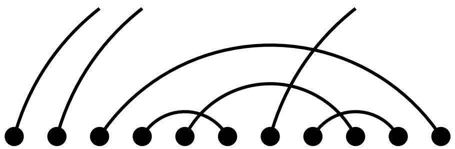
Simple excursion in the octant, with weights on the axis, of size $2n$

The odd case

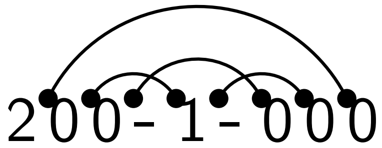
Simple axis-walks in the octant of length $2n+1$

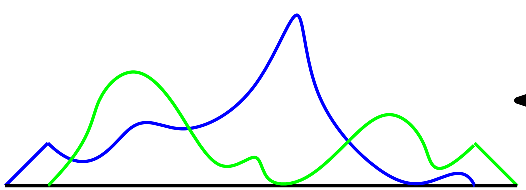


Open matching diagram with no 3-crossing of length $2n+1$

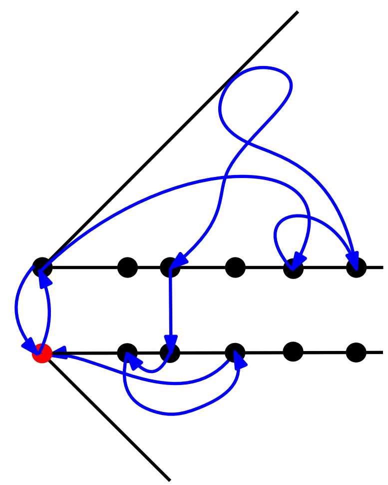


Matching diagram without 3-crossing with weights on open intervals, of *size* $2n+1$

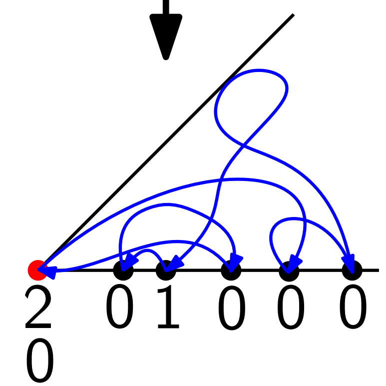




Pair of Dyck paths of half-lengths $n+1$

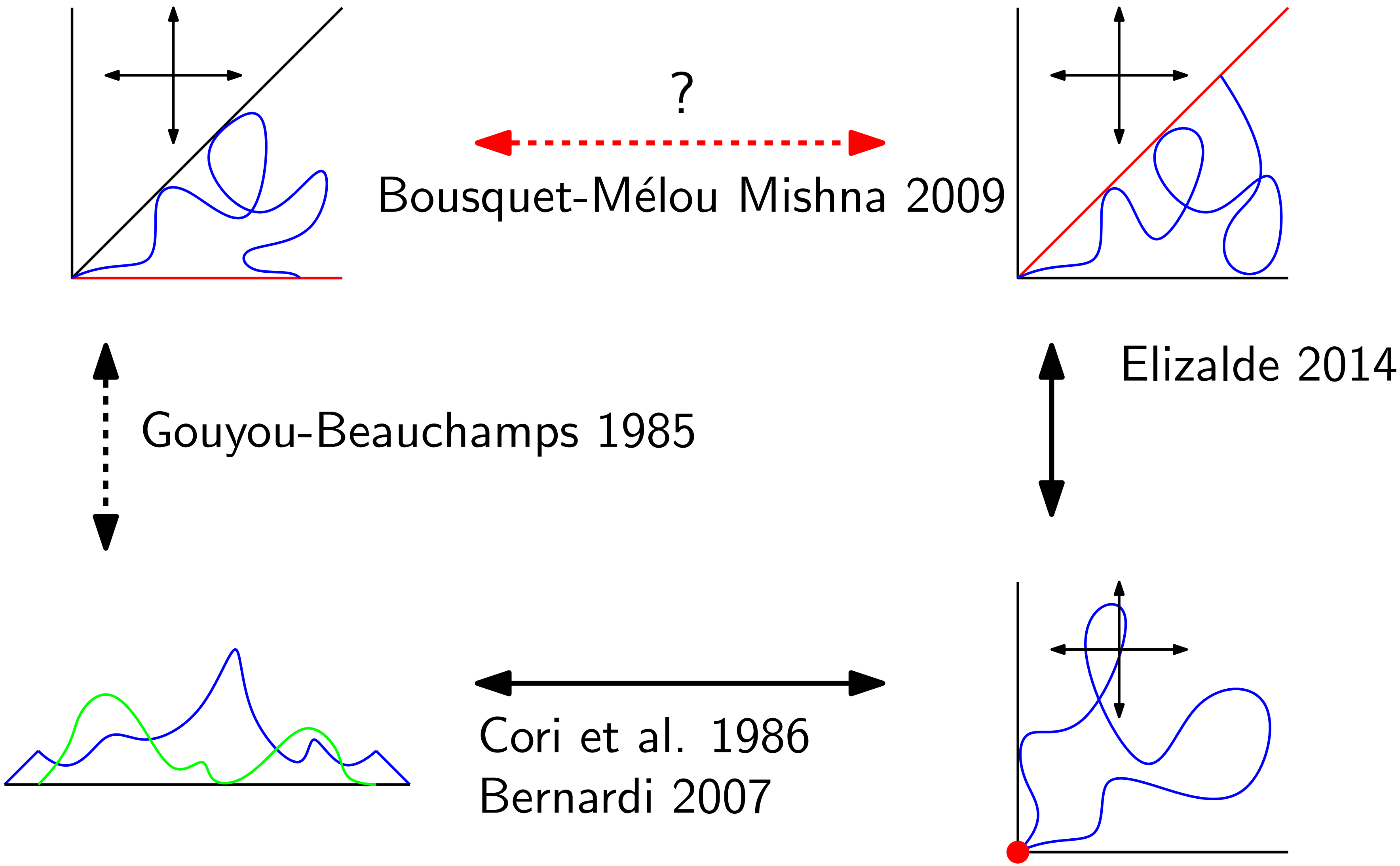


Simple inter-diagonals walk of length $2n+1$

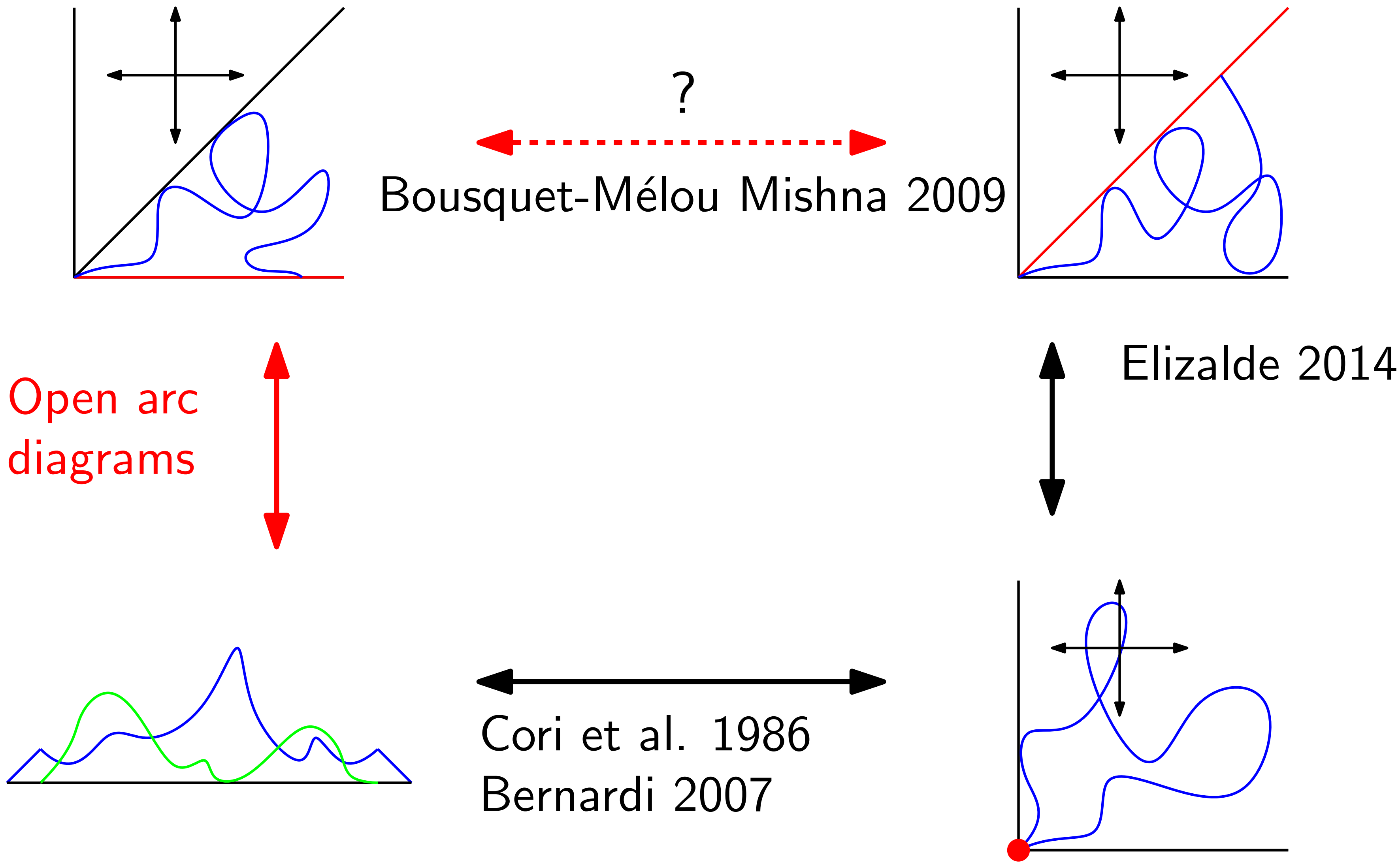


Simple excursion in the octant, with weights on the axis, of size $2n+1$

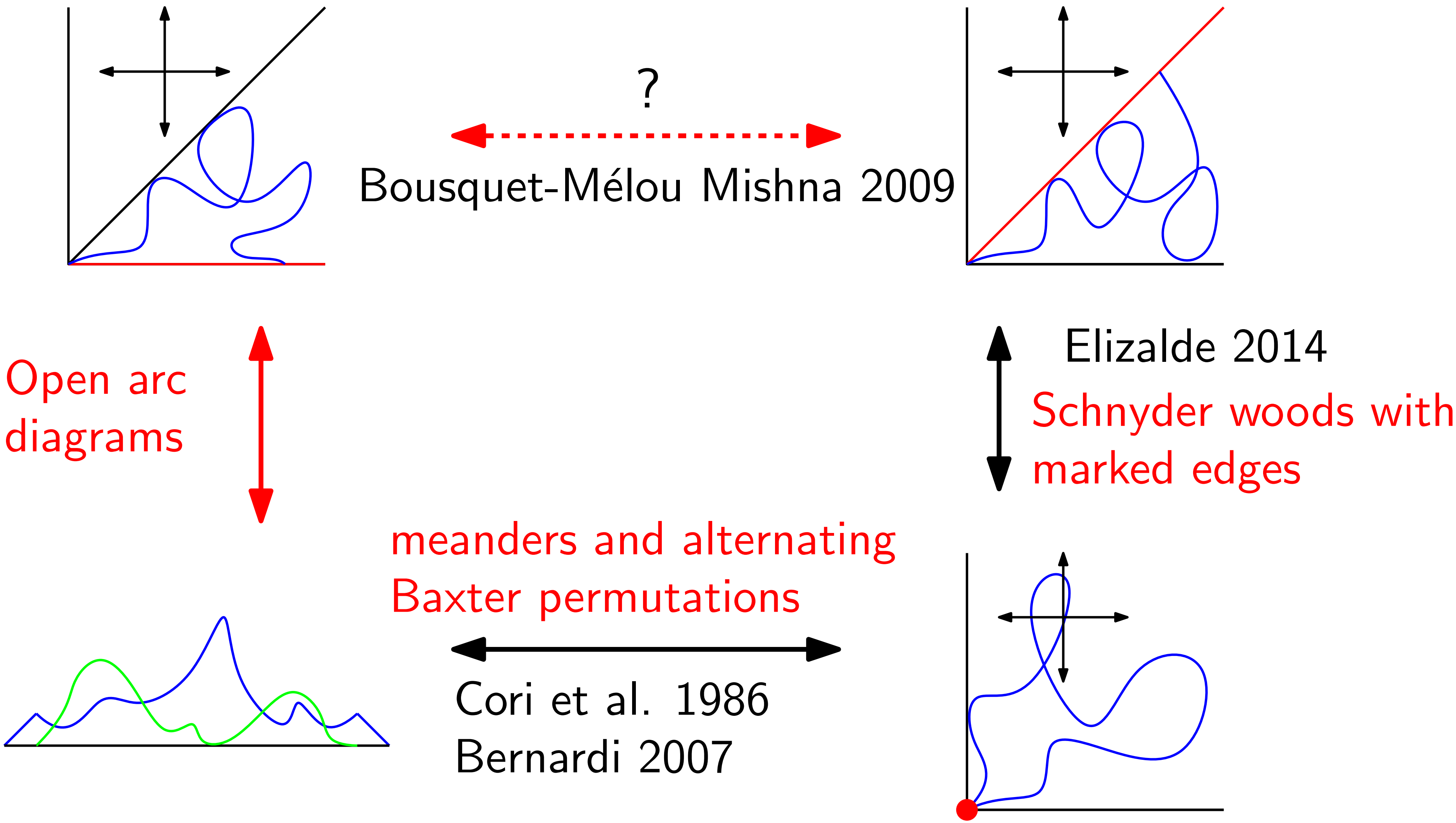
Answering Bousquet-Mélou et Mishna's question : three new bijections

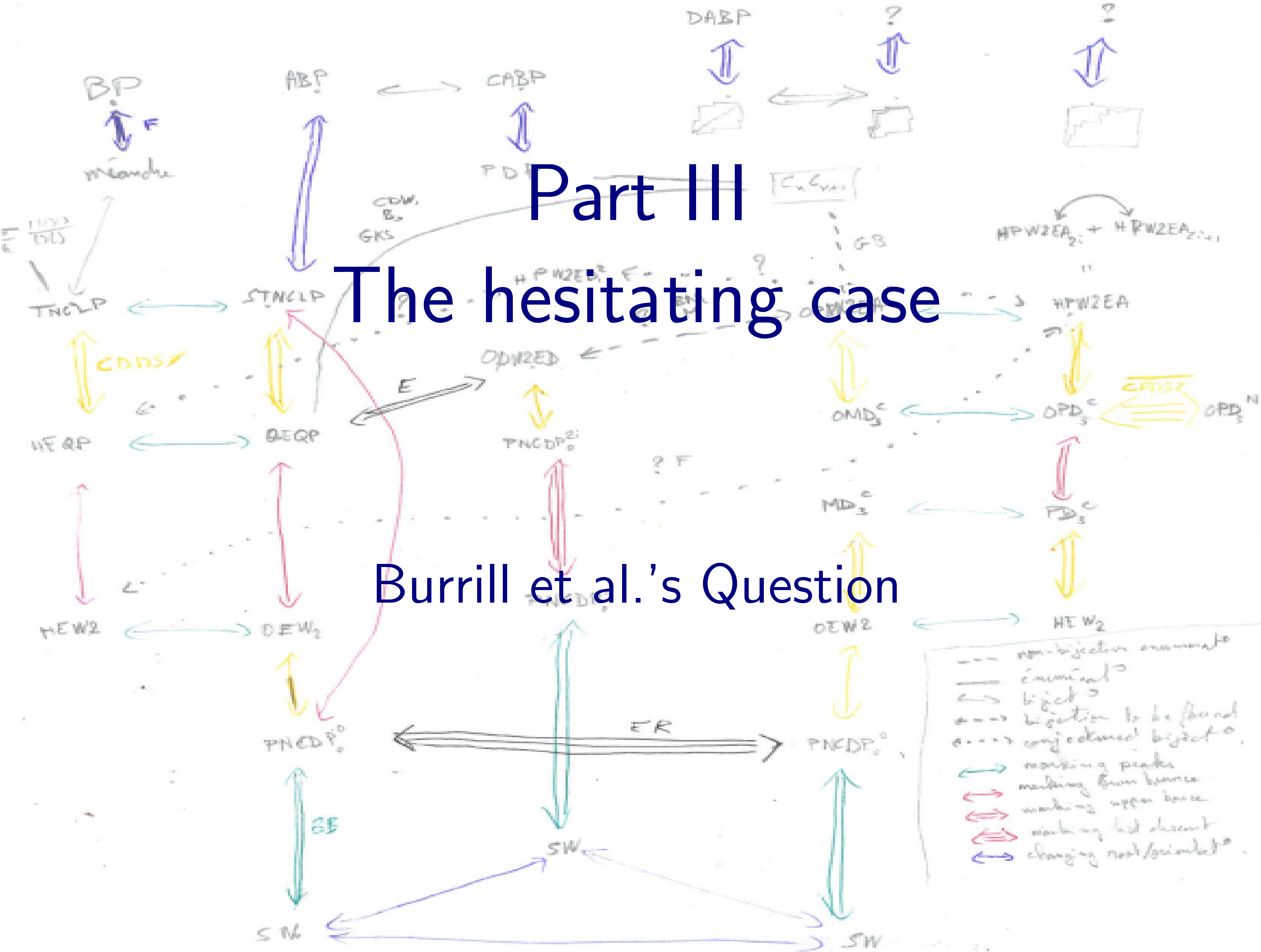


Answering Bousquet-Mélou et Mishna's question : three new bijections



Answering Bousquet-Mélou et Mishna's question : three new bijections

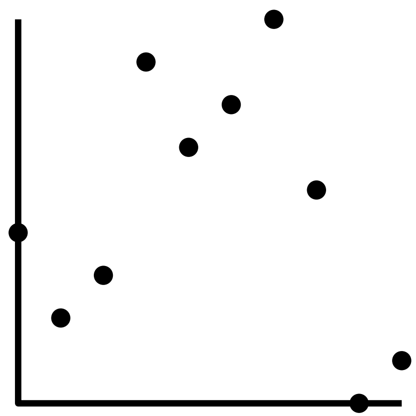




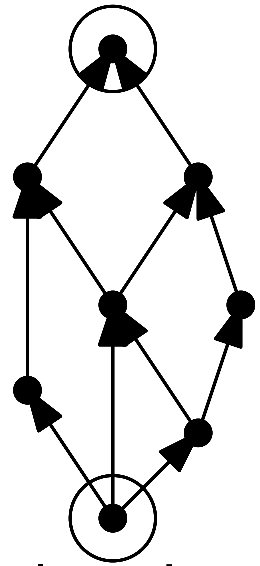
Part III
The hesitating case

Burrill et al.'s Question

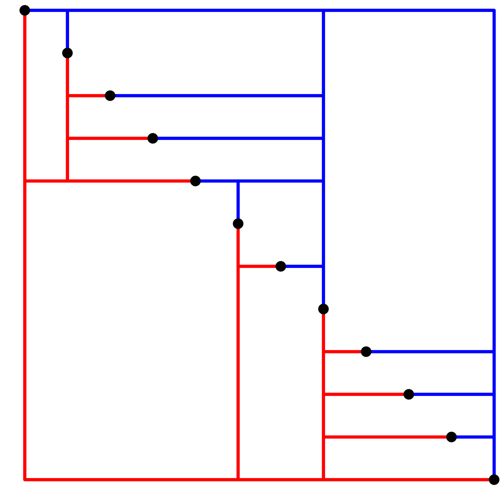
Symmetric Baxter families



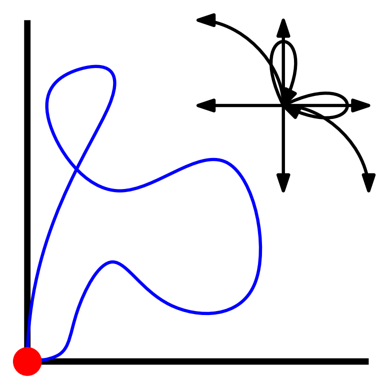
Baxter permutations



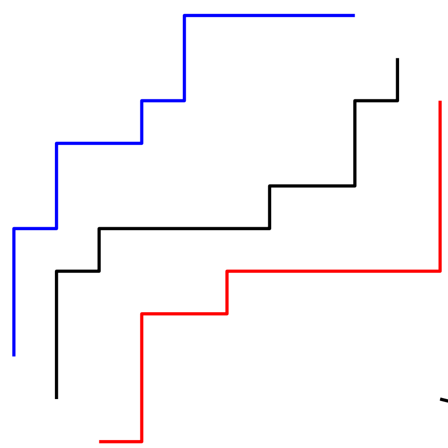
Plane bipolar orientations



Rectangulations of the square

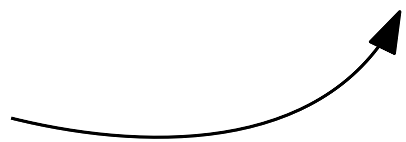


Hesitating excursions
in the quarter-plane



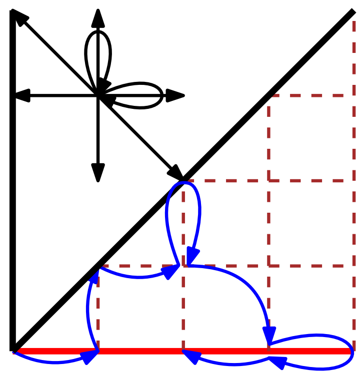
Non-crossing
triples of paths

$$B_n = \sum_{k=1}^{n} \frac{\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}}{\binom{n+1}{1}\binom{n+1}{2}}$$

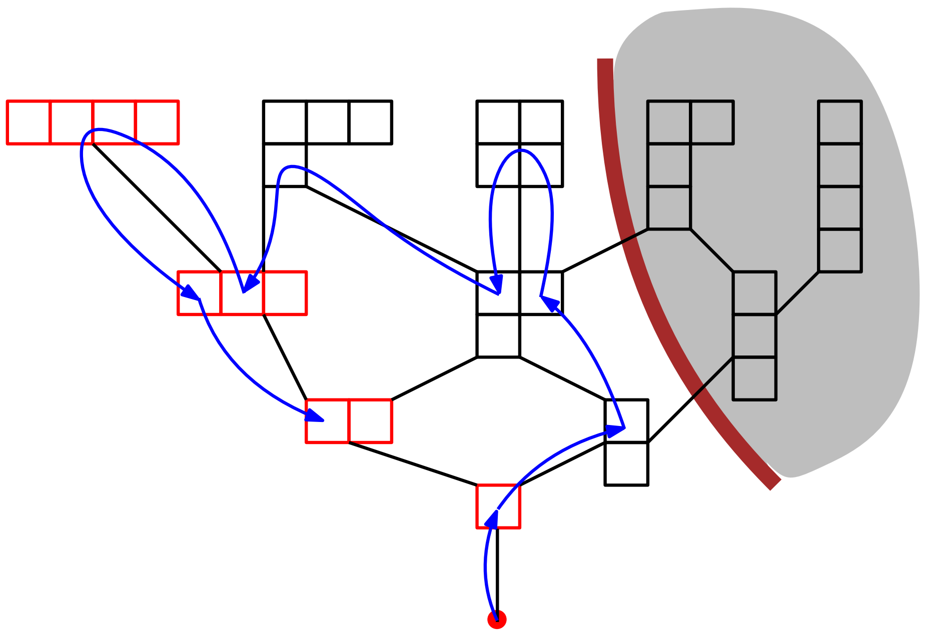


Gessel-Viennot

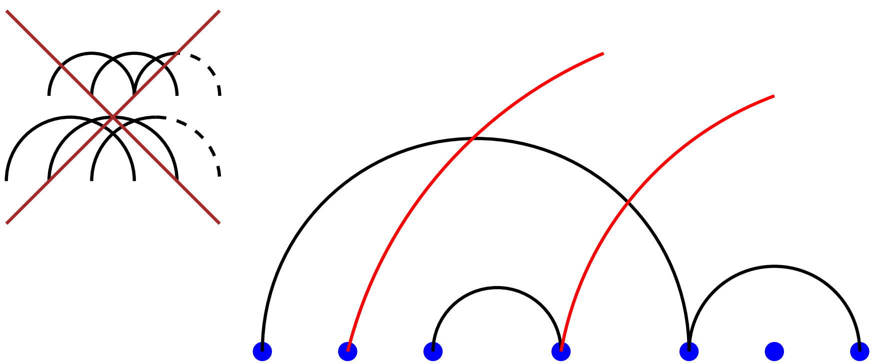
Asymmetric Baxter families



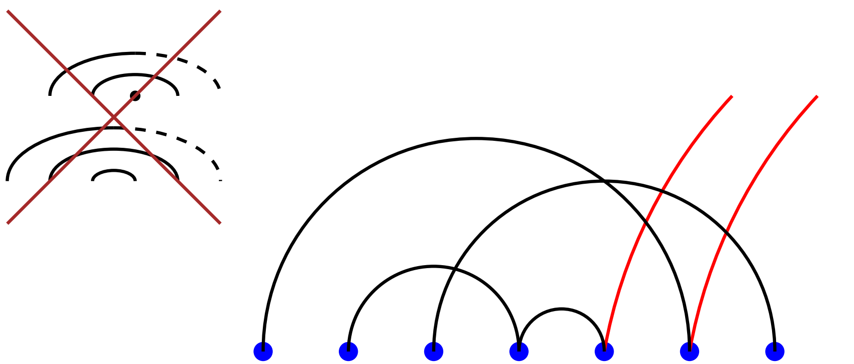
Hesitating axis-walks in the octant



Hesitating tableaux of height at most 2 with a line shape

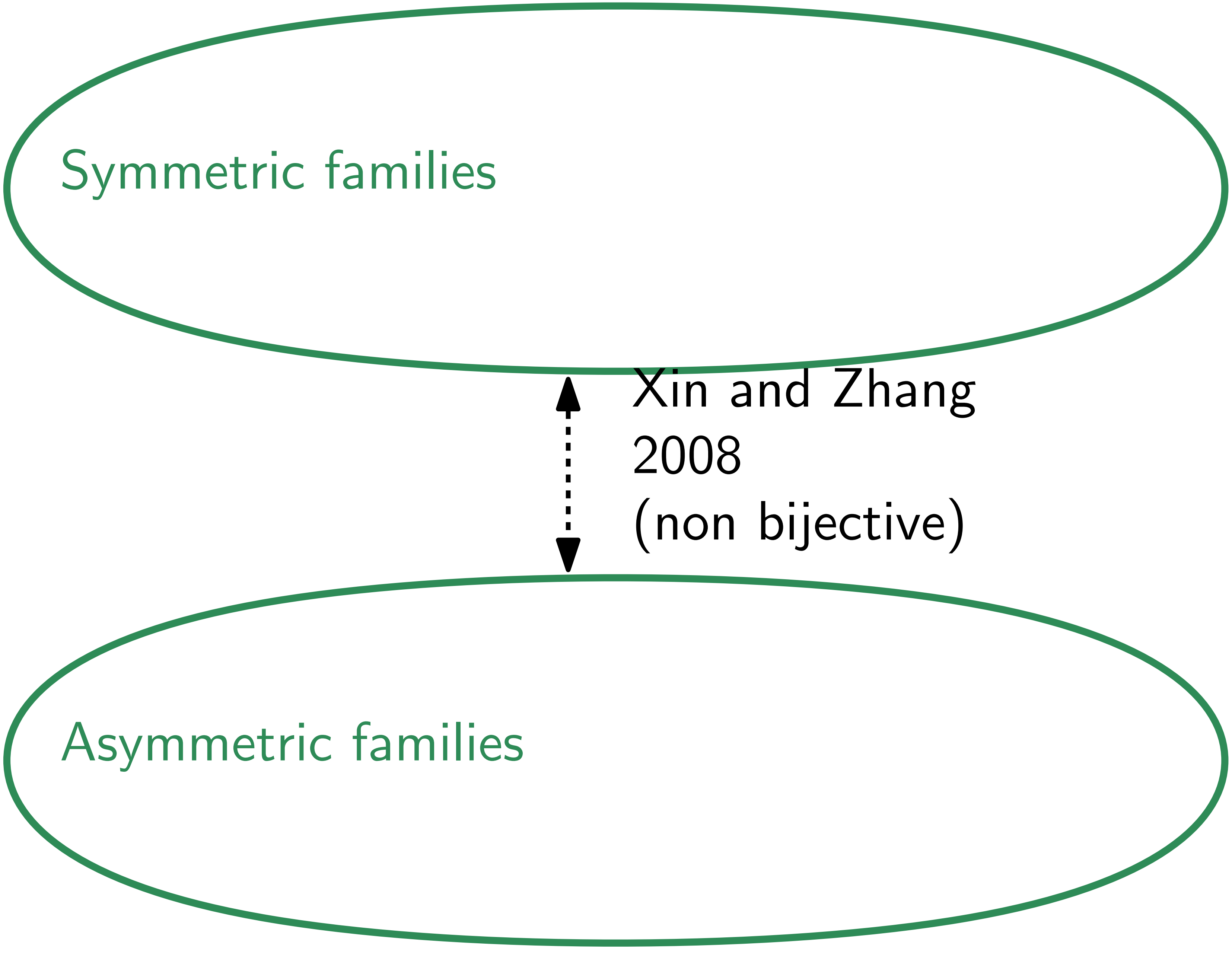


Open partition diagrams with no enhanced 3-crossings

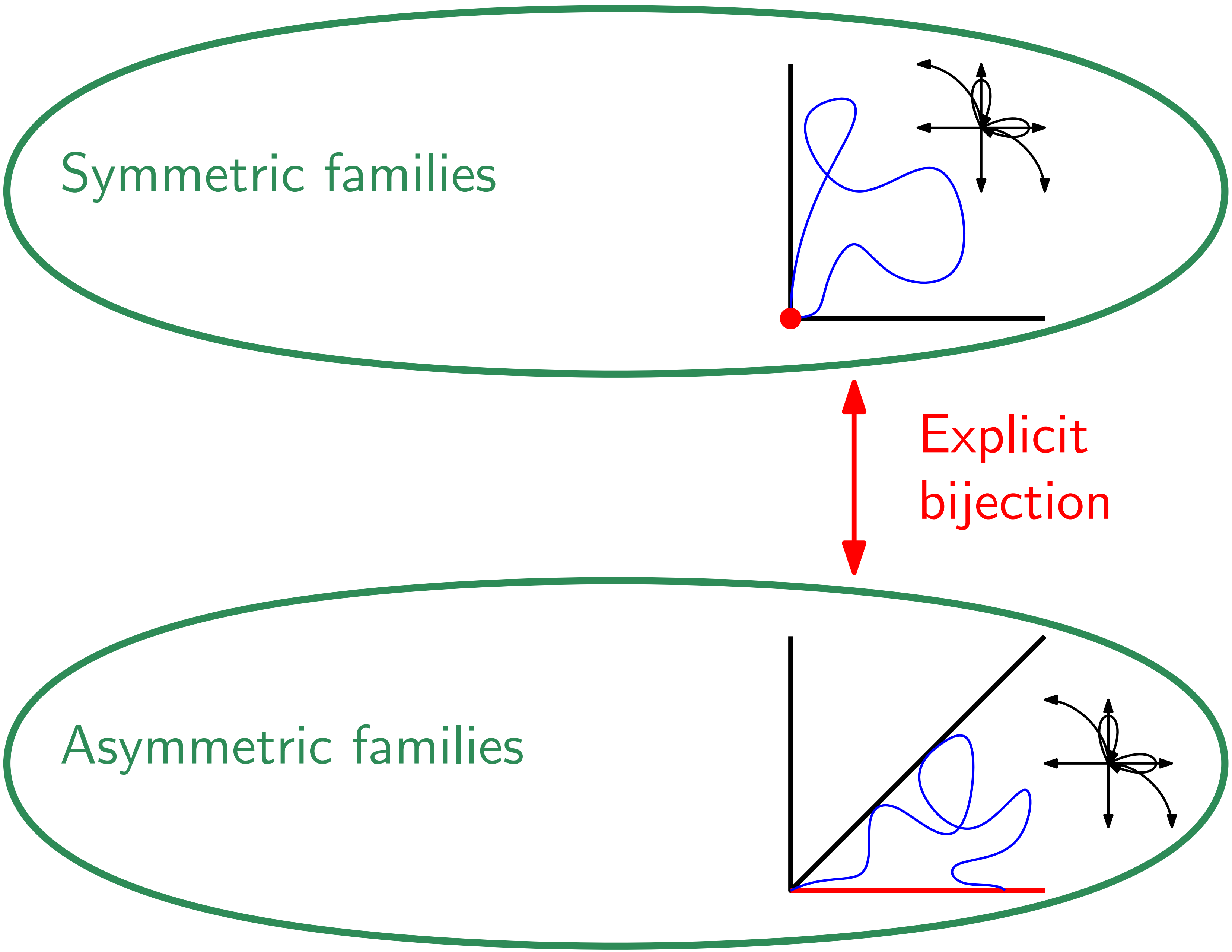


Open partition diagrams with no enhanced 3-nestings

Baxter families



Baxter families



Strategy:

Go through marked excursions in the octant

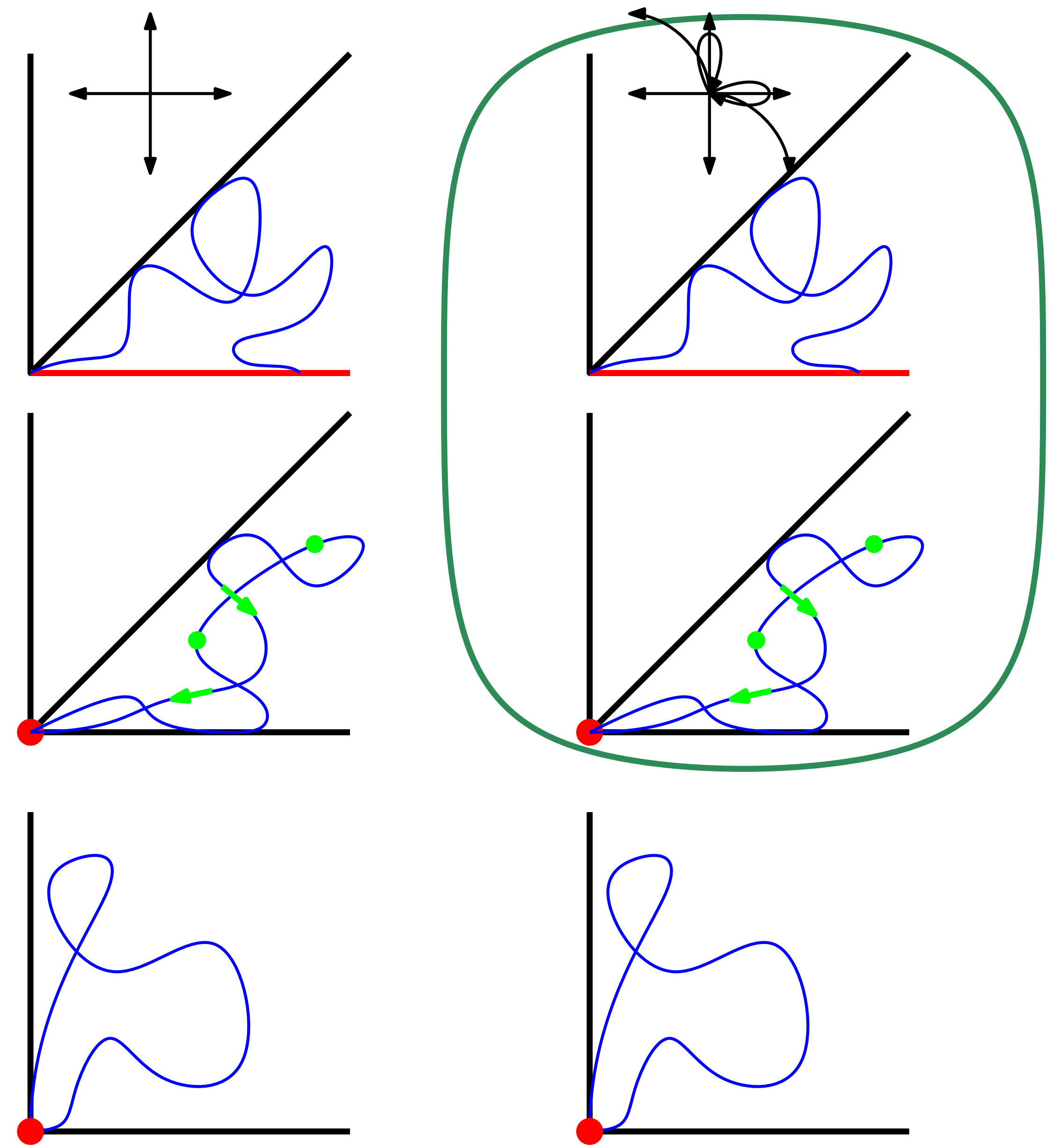
Domain contraint $\leftrightarrow$ Ending constraint

Simple case | Hesitating case

Axis-walk in the octant

Excursion in the octant with marking

Excursion in the quarter-plane



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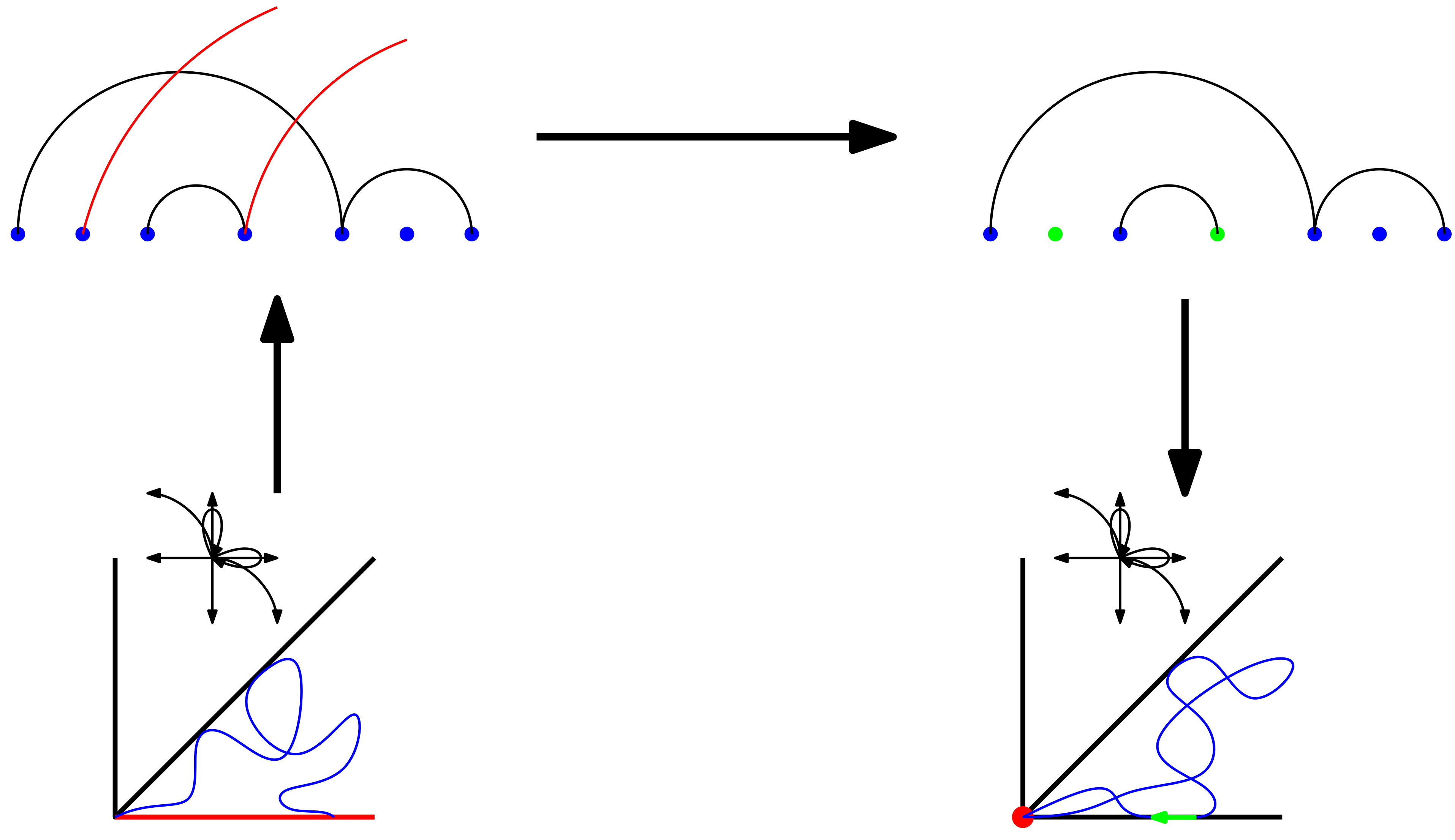

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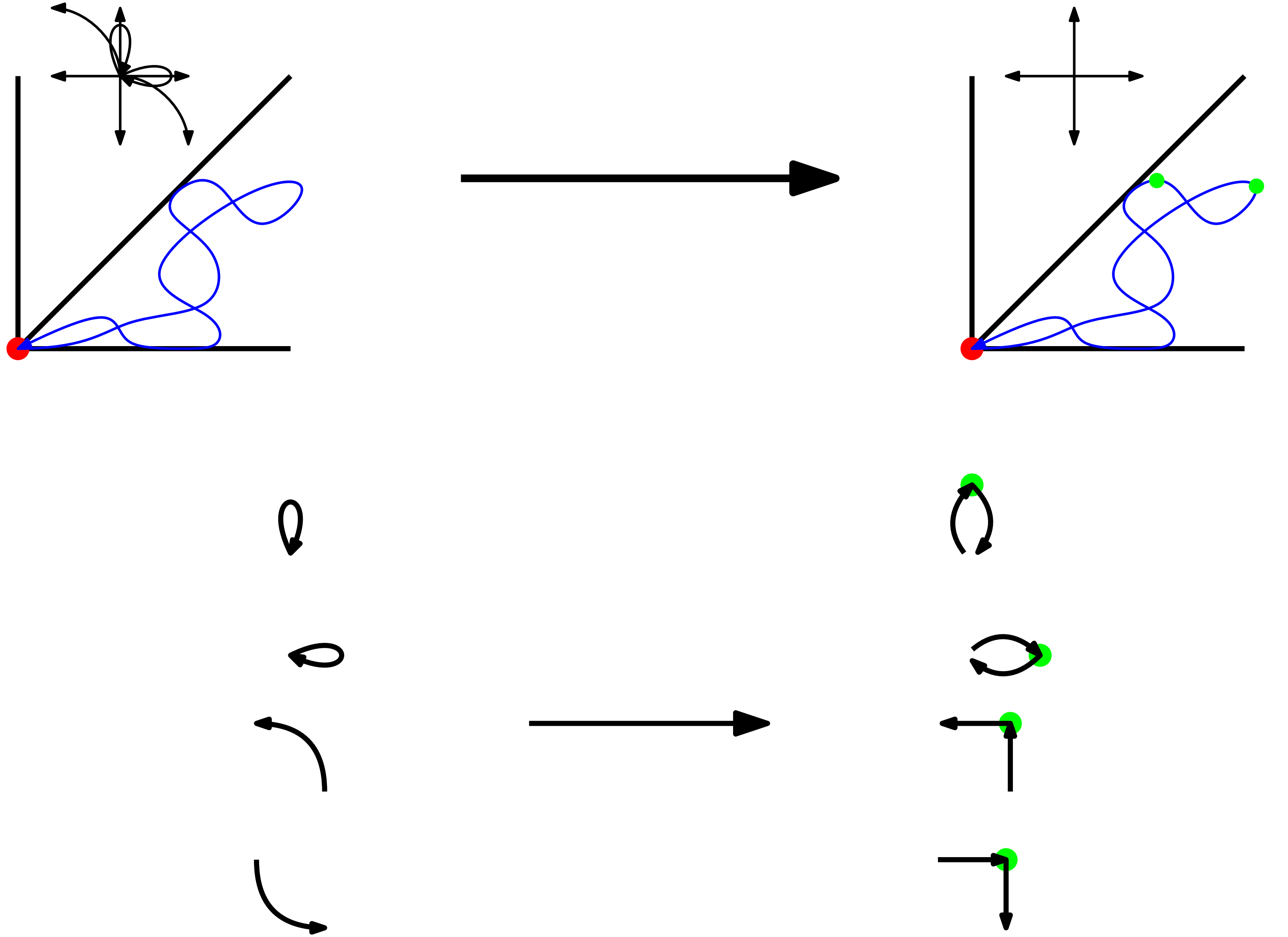
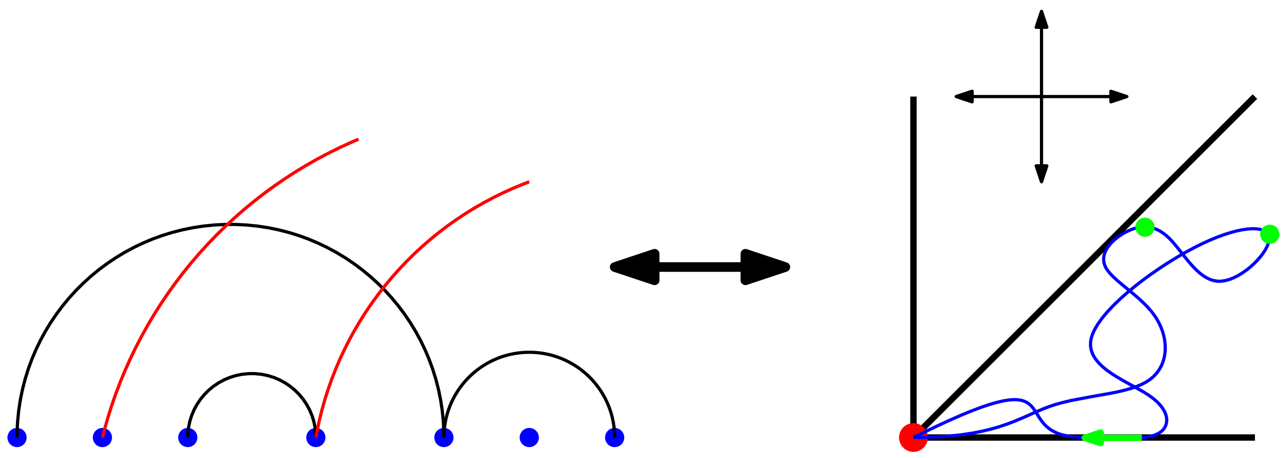


Strategy:
Go through marked excursions in the octant

Open partition diagrams of length n without enhanced 3-crossings

are in bijection with

Simple excursions in the octant of length n with marked peaks and marked *W-steps on the axis*.



Domain contraint $\leftrightarrow$ Ending constraint

Simple case | Hesitating case

Axis-walk in the octant

Excursion in the octant with marking

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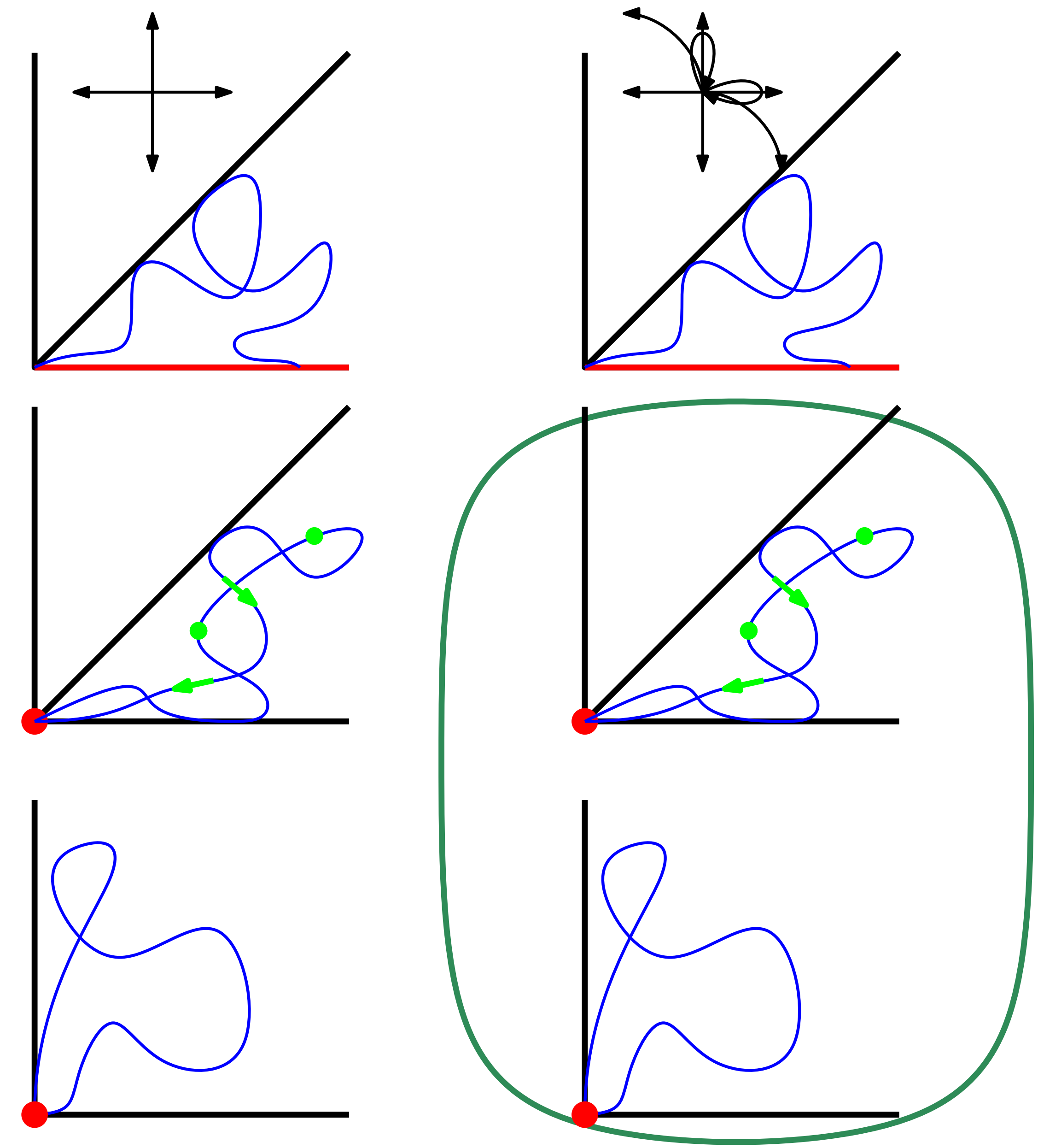
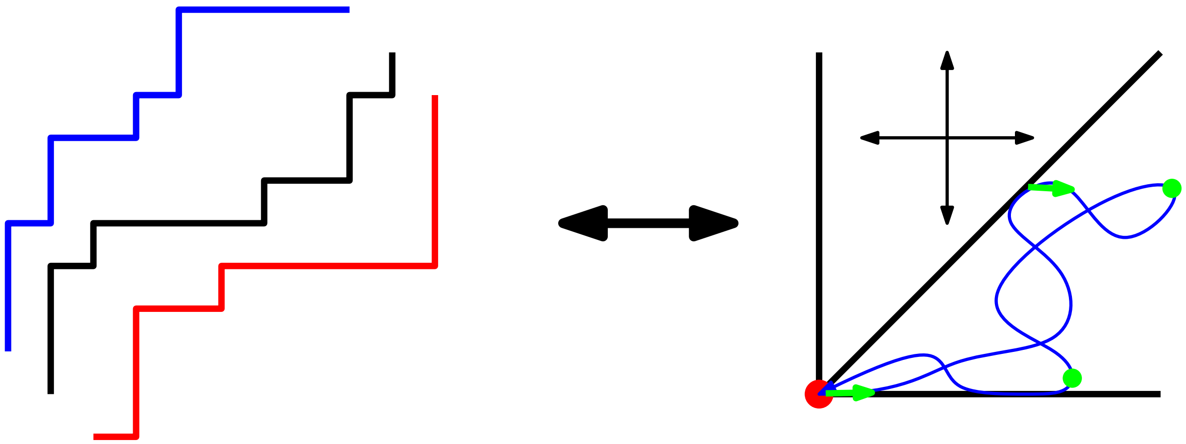


Strategy:
Go through marked excursions in the octant



Triples of non-crossing lattice paths of length n
are in bijection with
Simple excursions in the octant of length n with marked peaks and marked *steps leaving the diagonal*.

Strategy:
Go through marked excursions in the octant

Open partition diagrams of length n without enhanced 3-crossings

are in bijection with

Simple excursions in the octant of length n with marked peaks and marked *W-steps on the axis*.

Triples of non-crossing lattice paths of length n

are in bijection with

Simple excursions in the octant of length n with marked peaks and marked *steps leaving the diagonal*.

Symmetric distribution of the statistics

Theorem [ER,CFLM]:

There is an explicit bijection between:

- Simple excursions of length n in the octant, with p peaks, i *steps leaving the diagonal*, and j *W-steps on the axis*,

  and :
- Simple excursions of length n in the octant, with p peaks, j *steps leaving the diagonal*, and i *W-steps on the axis*

Symmetric distribution of the statistics

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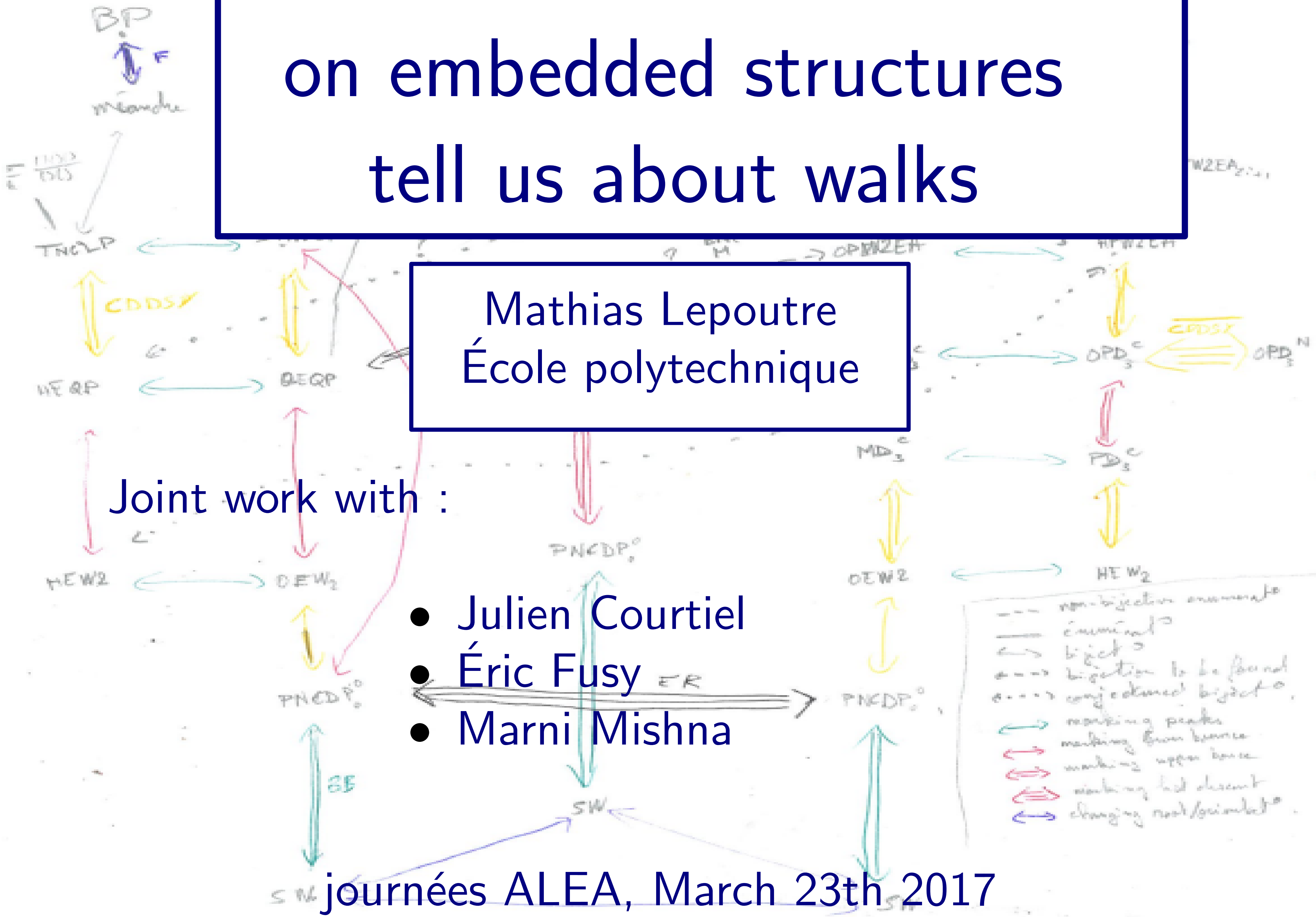
Proofs :

- Local operations on pairs of non-crossing Dyck paths. (Elizalde, Rubey 2012)
- Reflection of a Schnyder wood (Courtiel, Fusy, L., Mishna 2017)

What reflexions on embedded structures tell us about walks

Mathias Lepoutre
École polytechnique



Joint work with :

- Julien Courtiel
- Éric Fusy
- Marni Mishna

journées ALEA, March 23th 2017

Introductory example : Narayana numbers

$N(n,p) = \frac{1}{p}\binom{n}{p-1}\binom{n-1}{p-1}$:

Number of Dyck paths of length $2n$ with p peaks.

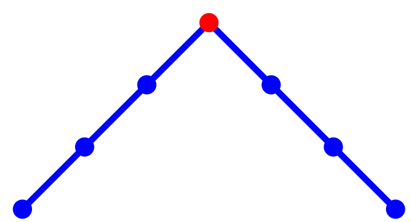
Introductory example : Narayana numbers

$N(n,p) = \frac{1}{p}\binom{n}{p-1}\binom{n-1}{p-1}$:

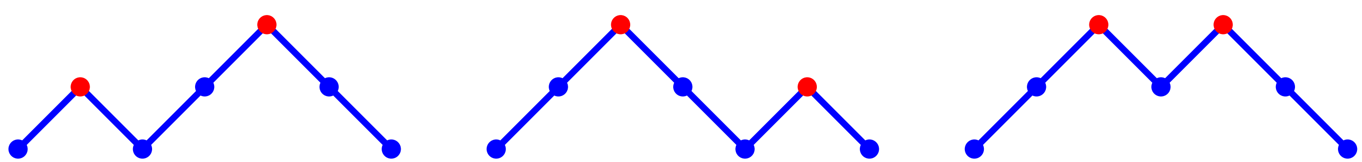
Number of Dyck paths of length $2n$ with p peaks.

Example for $n = 3$

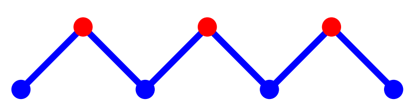
$N(3,1) = 1$



$N(3,2) = 3$



$N(3,3) = 1$



Introductory example : Narayana numbers

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Number of Dyck paths of length $2n$ with p peaks.

Properties :

- $\sum_{p=1}^{n} N(n,p) = C_n$
- $N(n,p) = N(n,n-p+1)$

Introductory example : Narayana numbers

$N(n,p) = \frac{1}{p}\binom{n}{p-1}\binom{n-1}{p-1}$:

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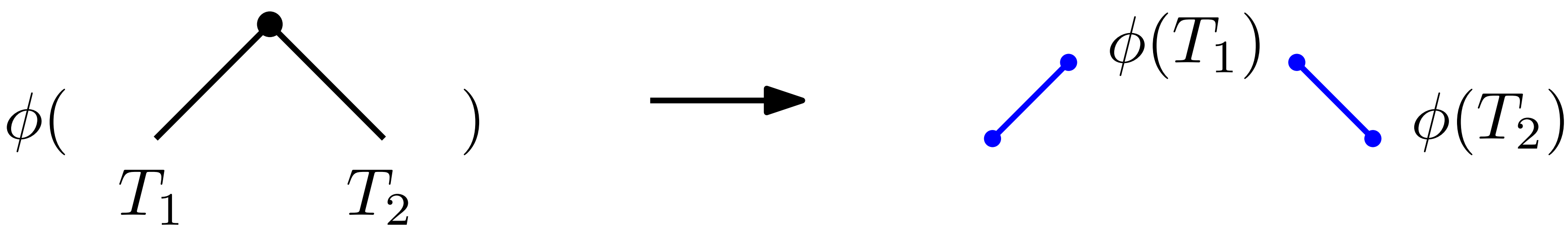
Introductory example : Narayana numbers

A bijection between plane binary trees with n leaves and Dyck paths of length $2n$:



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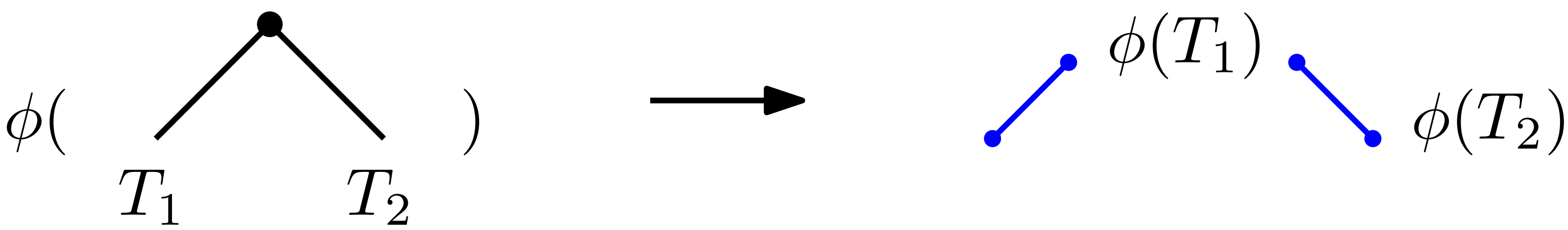
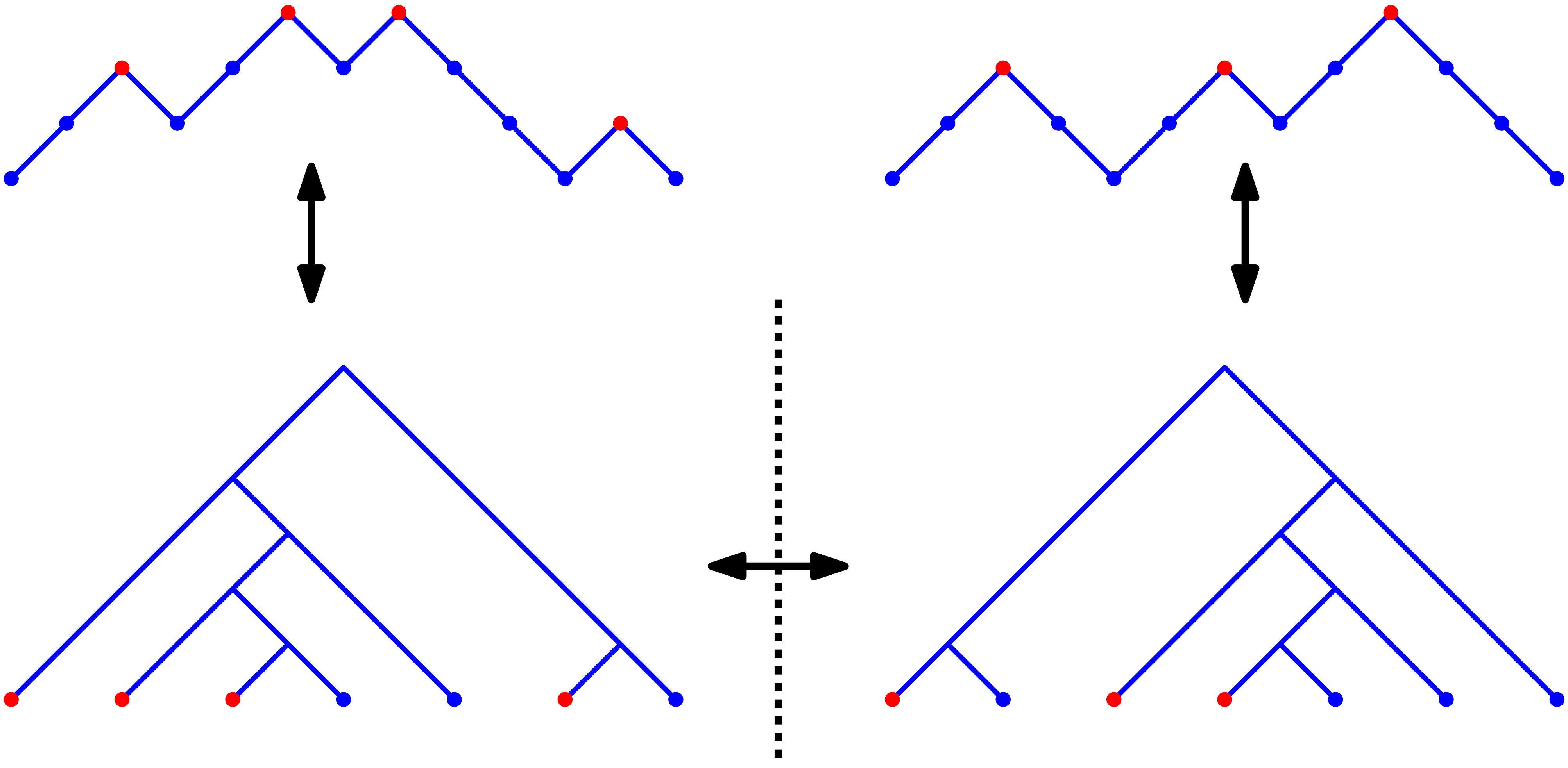


Tracking an interesting parameter :

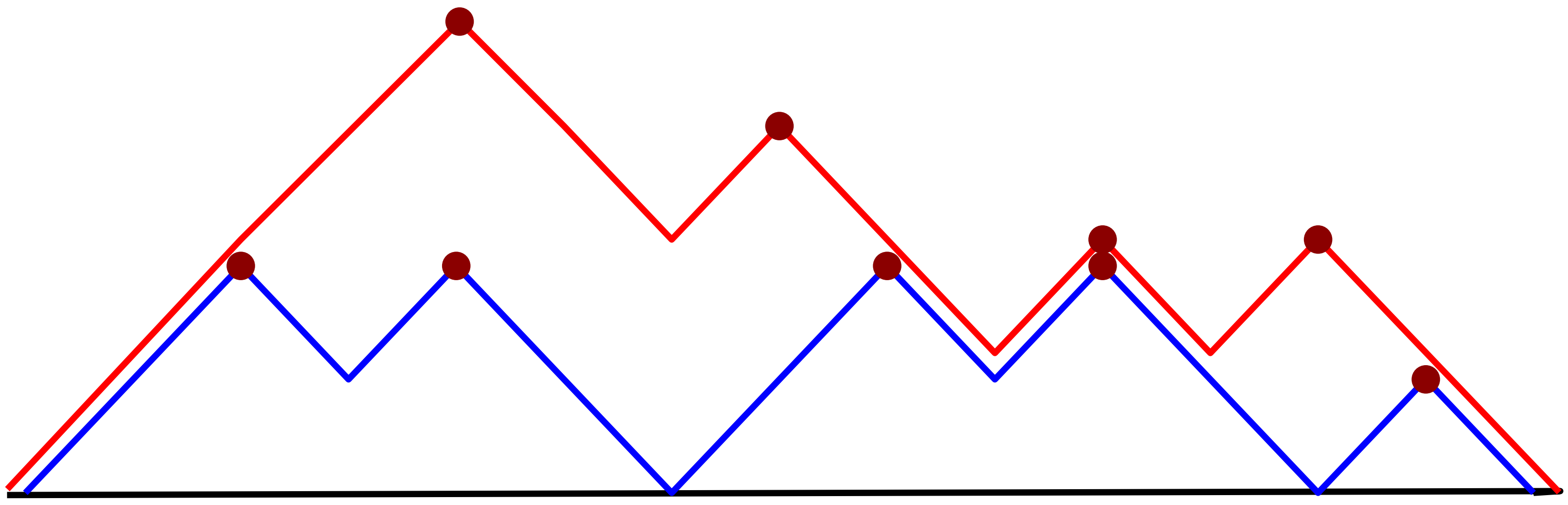
Number of left leaves $\longrightarrow$ Number of peaks

Introductory example : Narayana numbers

A bijective proof of Narayana numbers symmetry:



Generalisation :
Peaks of the pairs of non-crossing Dyck paths



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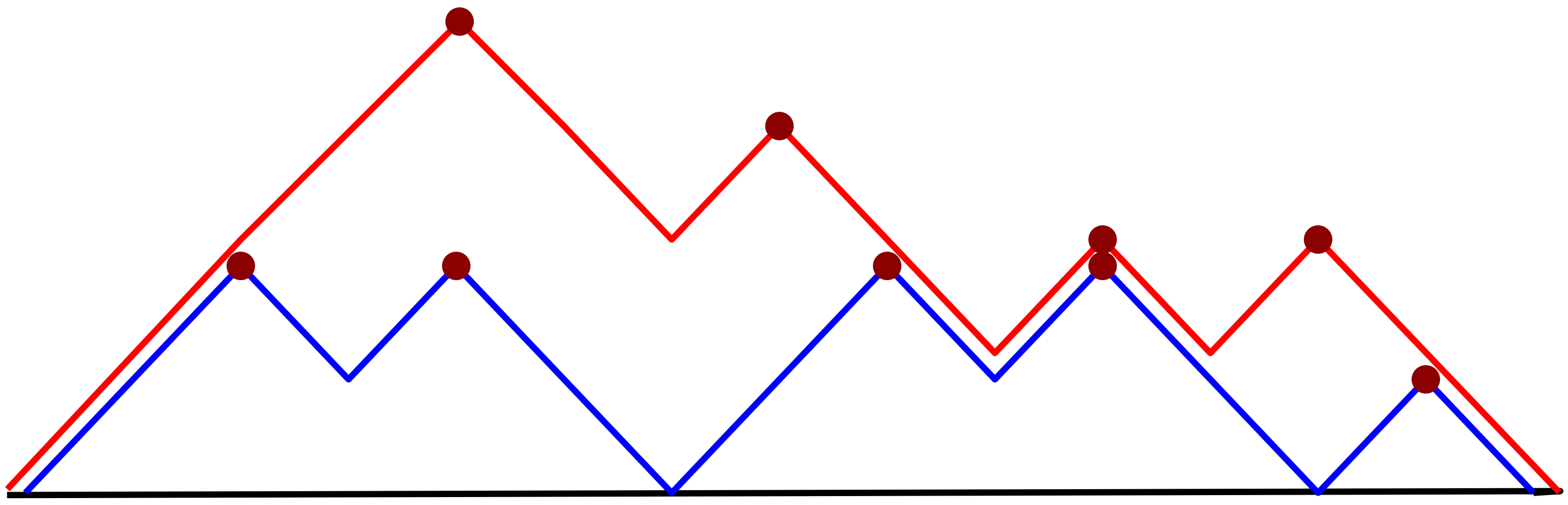
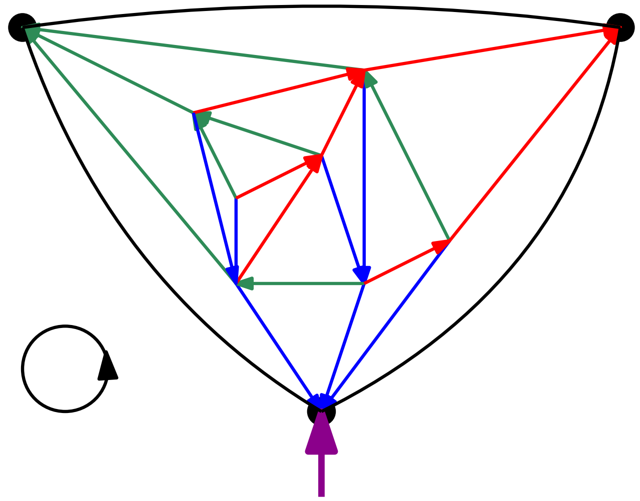
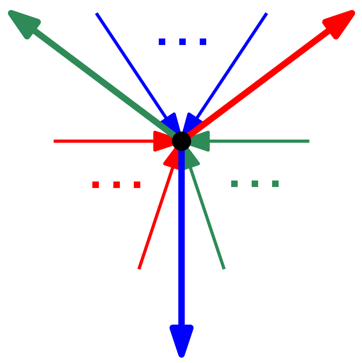


Let $N(n, p, q)$ be the number of pairs of non-crossing Dyck paths of length $2n$ with p upper peaks and q lower peaks.
Then: $N(n, p, q) = N(n, n - q + 1, n - p + 1)$

Generalisation :
Peaks of the pairs of non-crossing Dyck paths

Schnyder woods of triangulations





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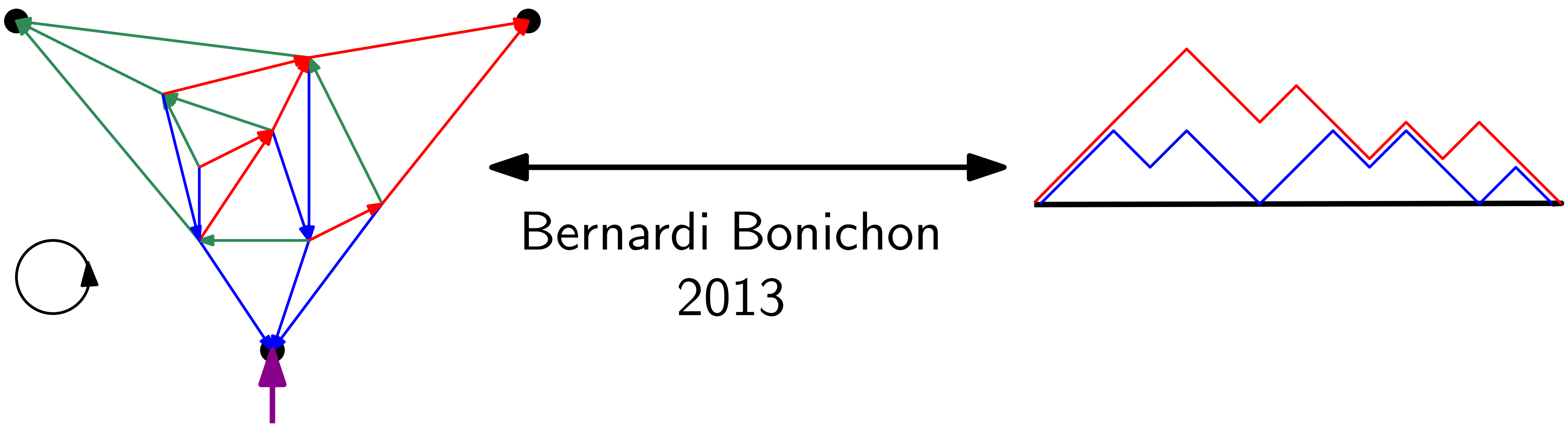
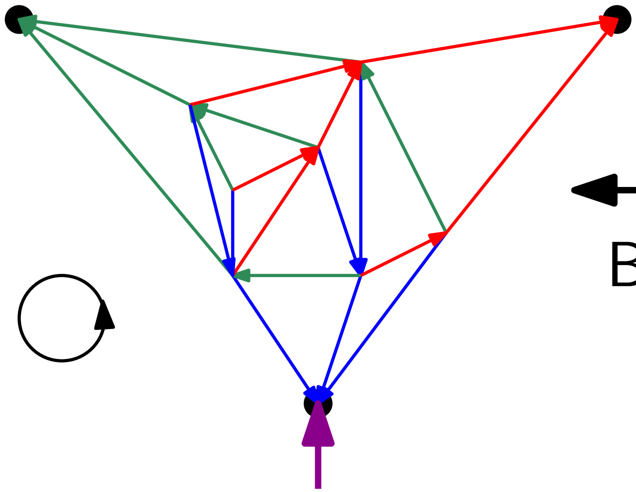


Generalisation : Peaks of the pairs of non-crossing Dyck paths



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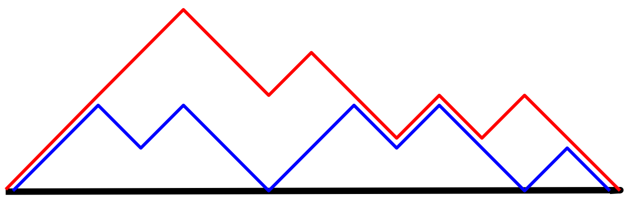
2013

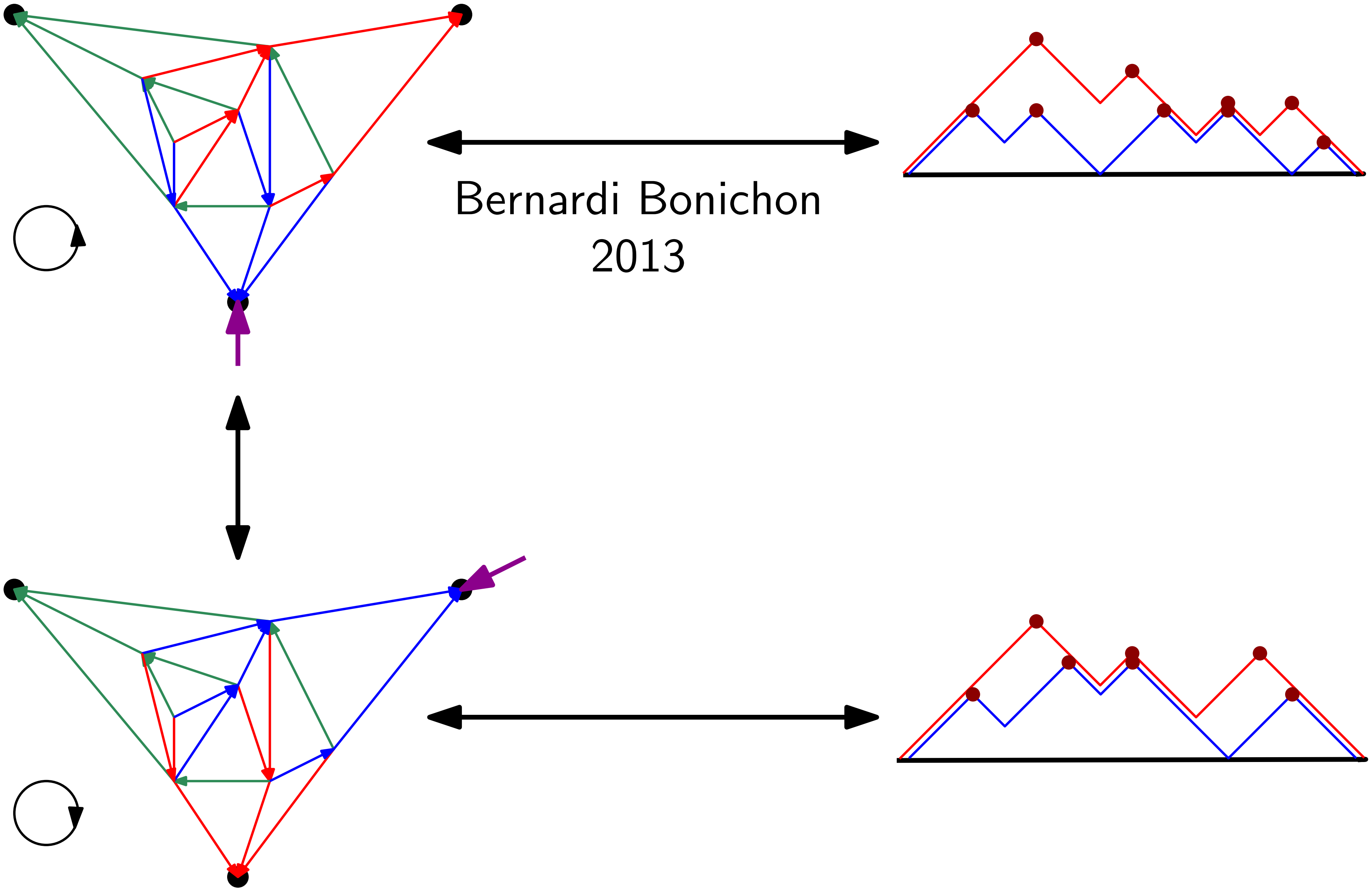


- Number of blue leaves
- Number of red internal vertices

↔

- Number of blue peaks
- Number of red peaks

Generalisation : Peaks of the pairs of non-crossing Dyck paths



Can this be further generalized?

Let $N(n, p_1 ... p_k)$ be the number of k-tuples of non-crossing Dyck paths of length $2n$ with p_i peaks on the i-th paths from the top.

Do we have: $N(n, p_1 ... p_k) = N(n, n - p_k + 1 ... n - p_1 + 1)$?

Can this be further generalized?

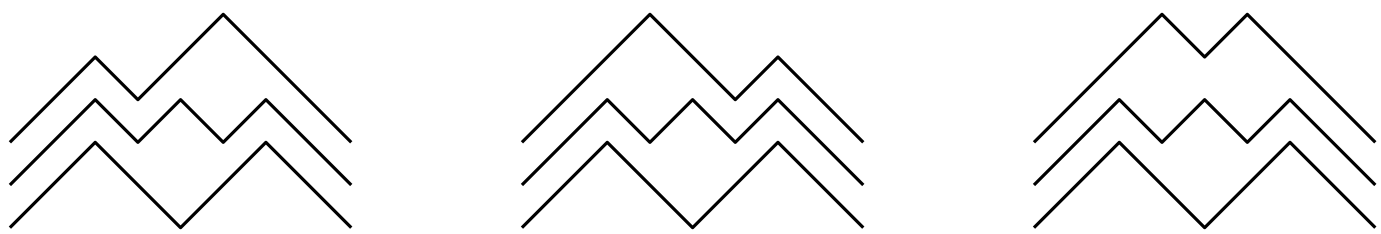
Let $N(n, p_1 ... p_k)$ be the number of k-tuples of non-crossing Dyck paths of length $2n$ with p_i peaks on the i-th paths from the top.

Do we have: $N(n, p_1 ... p_k) = N(n, n - p_k + 1 ... n - p_1 + 1)$?

No!

$N(4, 3, 2, 3) = 2$

$N(4, 2, 3, 2) = 3$



A result on walks in the plane

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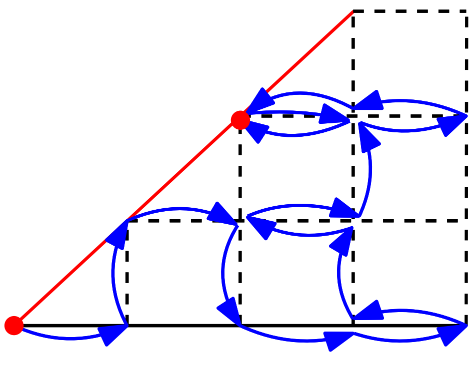
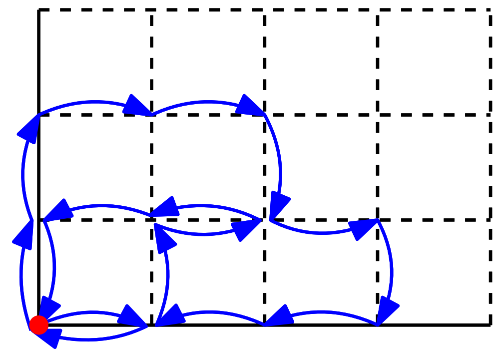
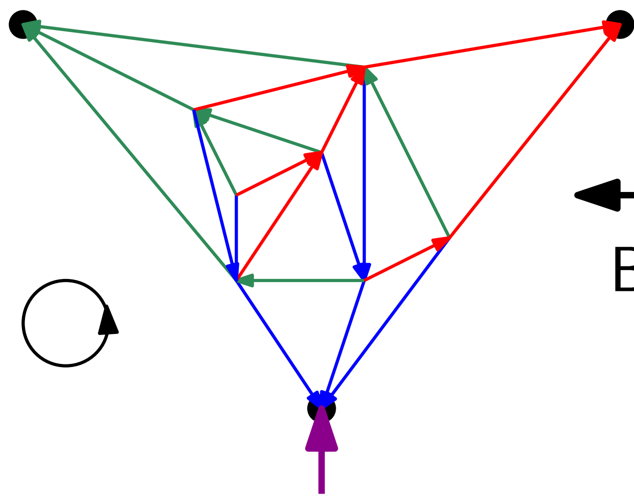


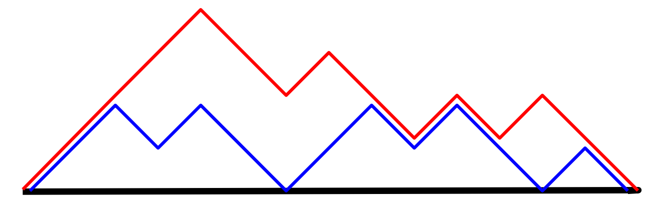


At given size, there are as many walks in the first octant that end on the x-axis than excursions in the quarter plane.

A result on walks in the plane



Bernardi Bonichon
2013



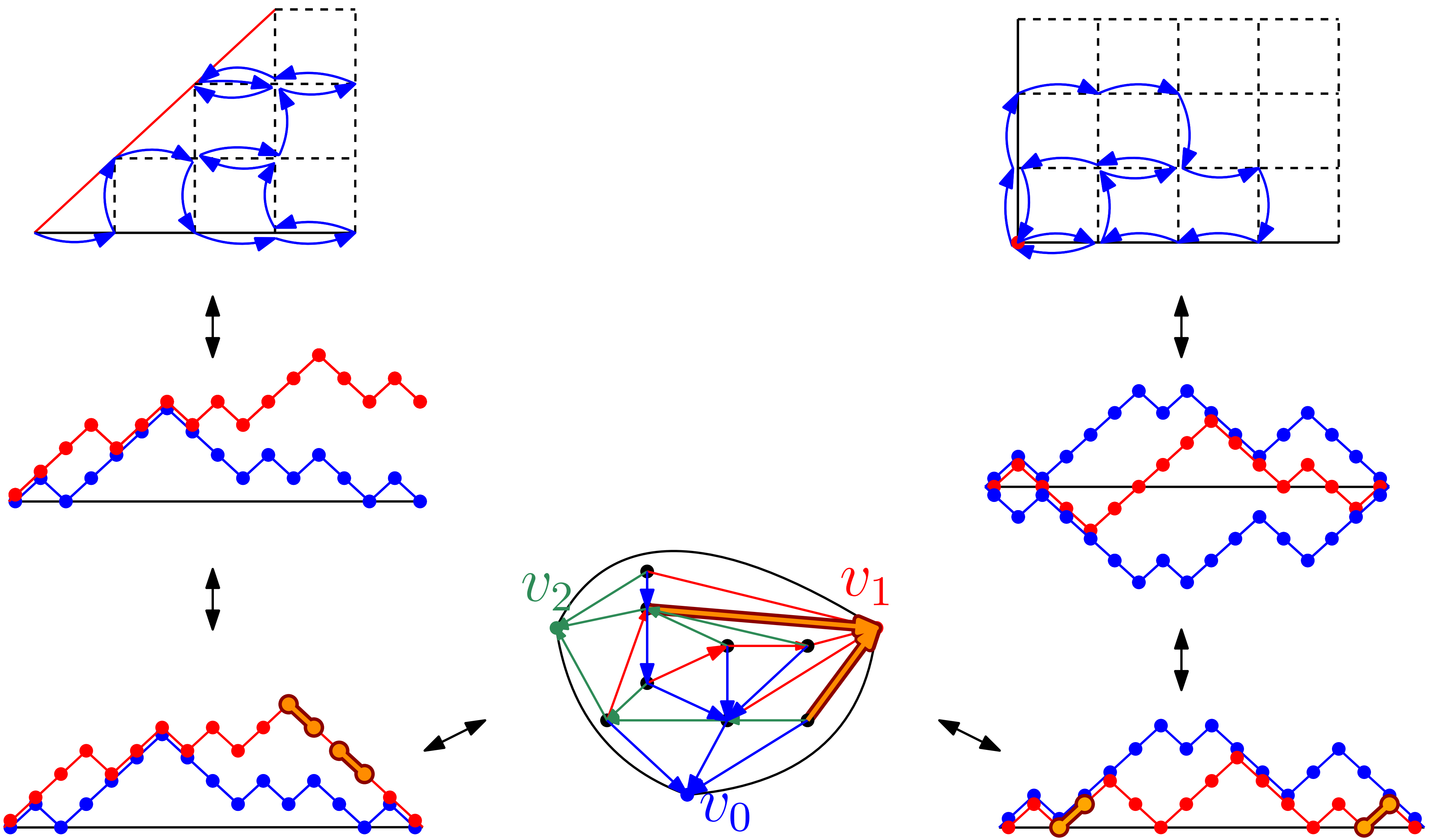
- Number of blue leaves
- Number of red internal vertices

- Number of blue peaks
- Number of red peaks

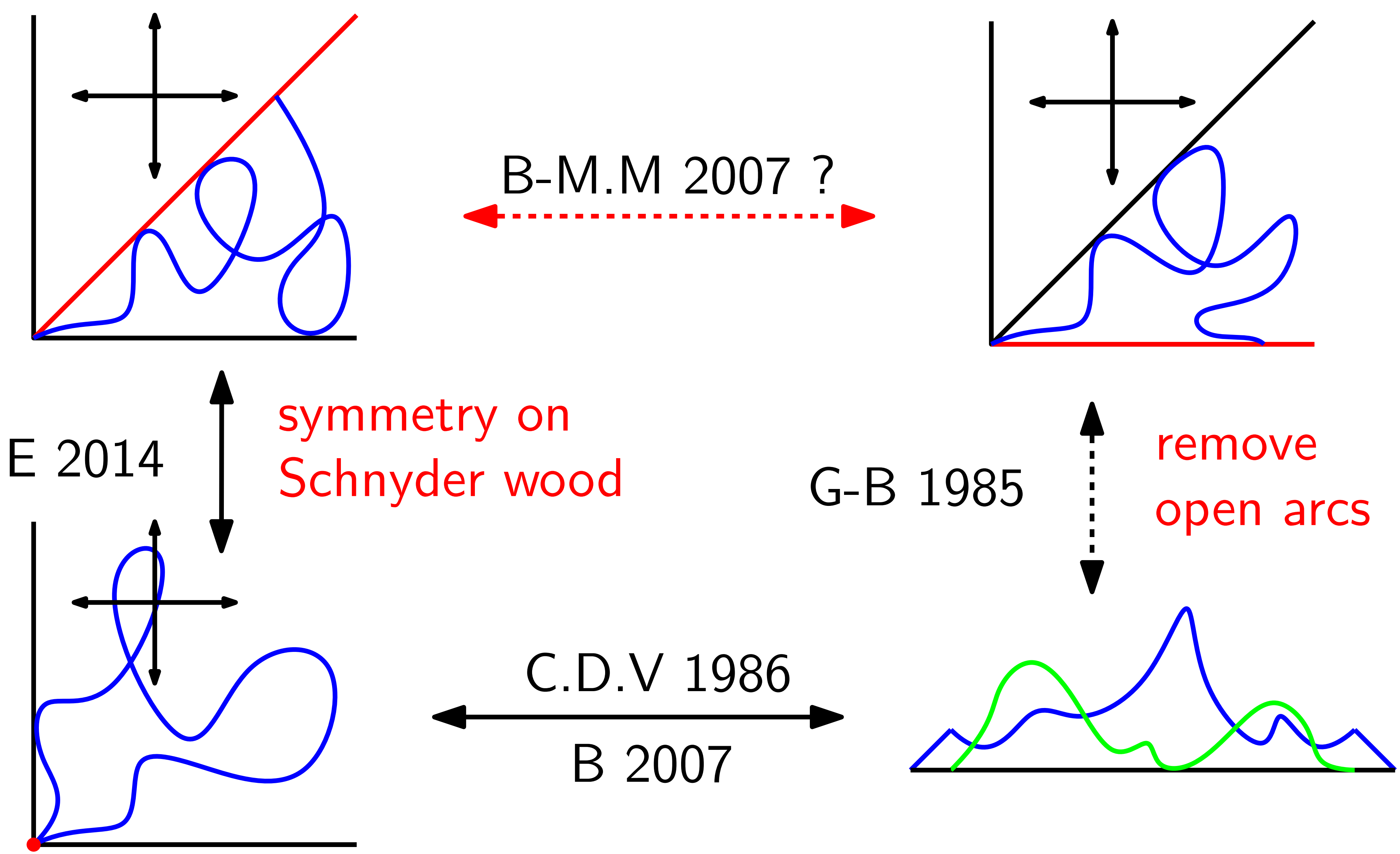
- Blue root-degree
- Red root-degree

- Number of blue steps leaving the axis
- Length of the red last descent

A result on walks in the plane



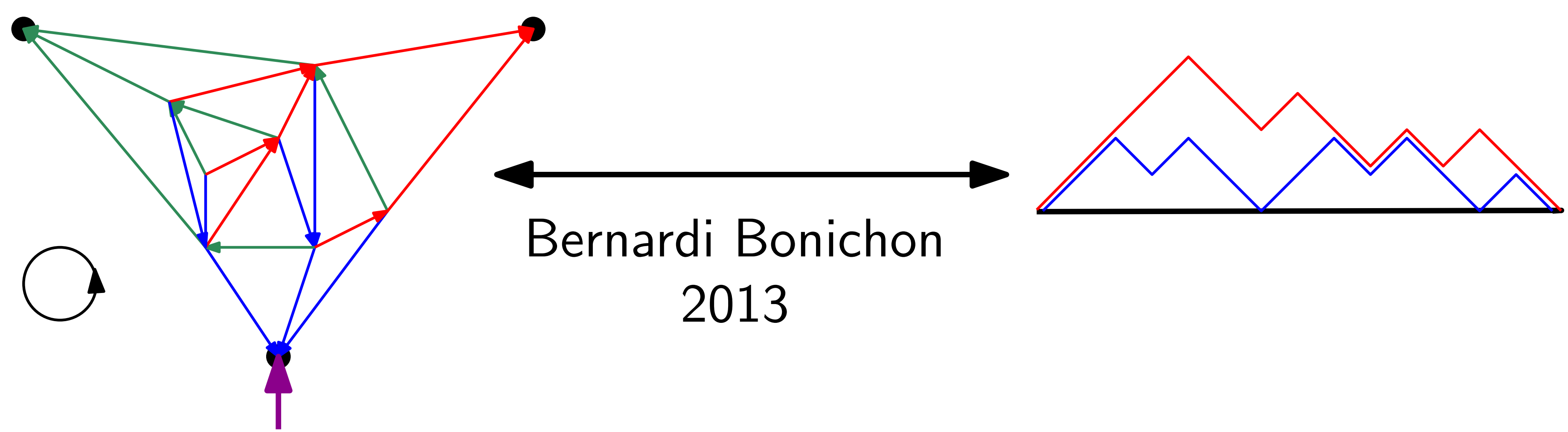
A look at another problem



Another result to prove a conjecture

There exists an explicit involution on pairs of non-crossing Dyck paths that preserves the size and the number of upper peaks, while exchanging the number of lower steps leaving the axis and the number of common up-steps.

Another result to prove a conjecture



- Number of blue leaves
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- Number of blue peaks
- Number of red peaks

- Blue root-degree
- Red root-degree

↔

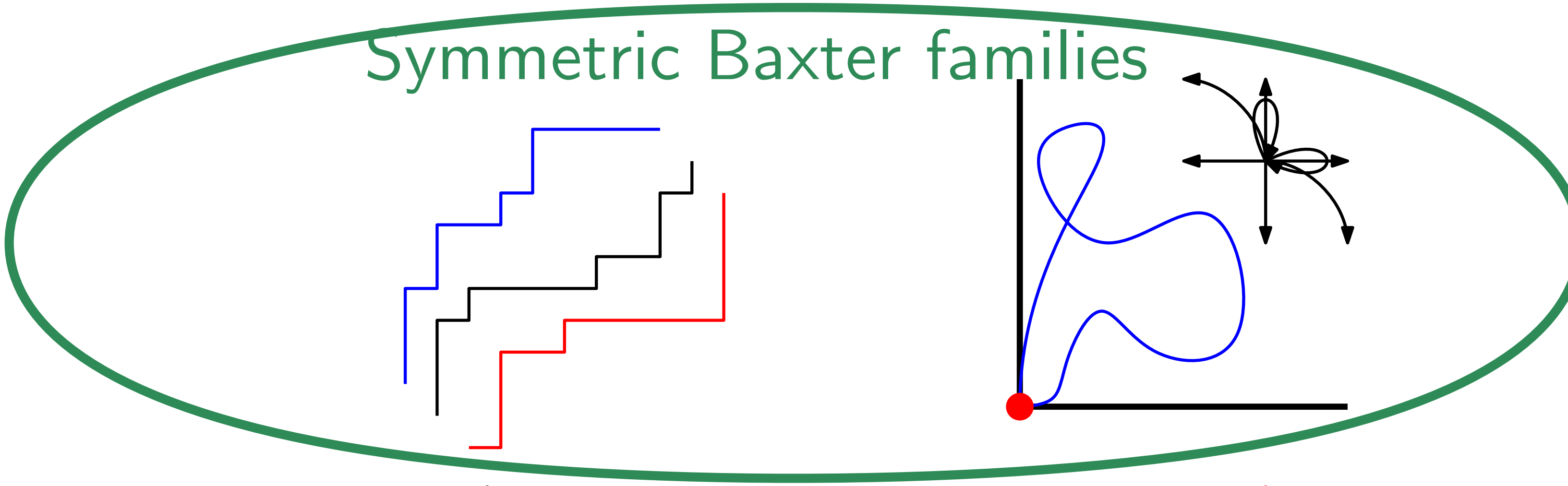
- Number of blue steps leaving the axis
- Length of the red last descent

- Green root-degree

↔

- Number of common up-steps

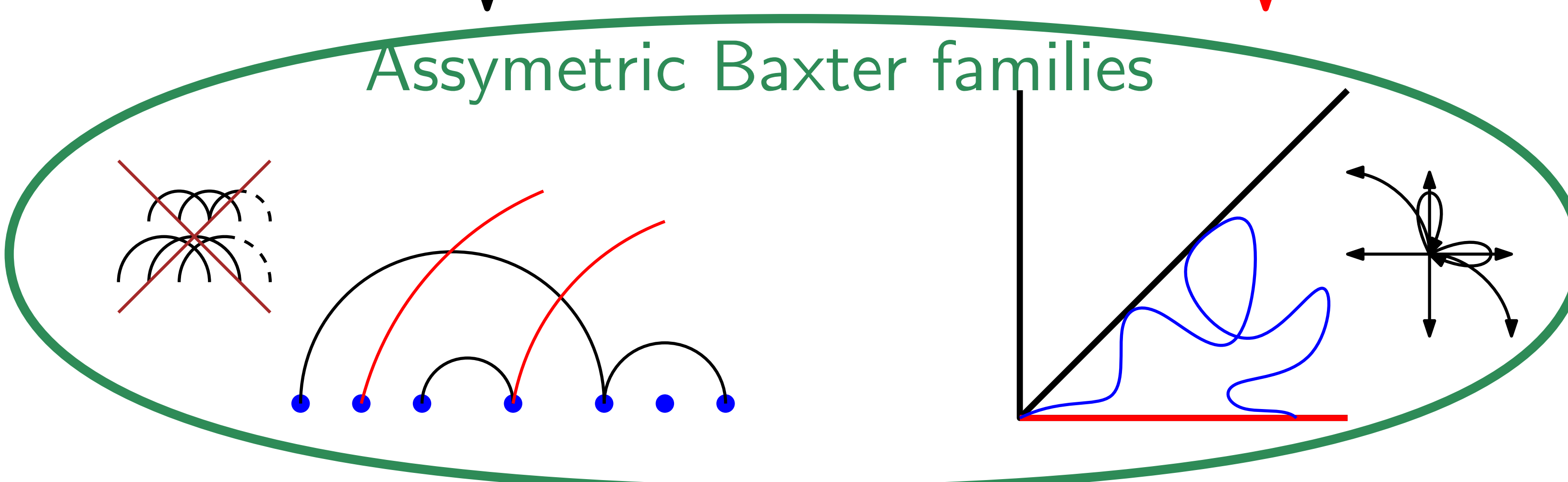
Another result to prove a conjecture



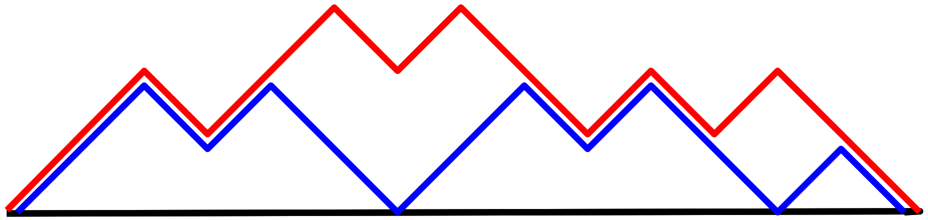
Xin et Zhang 2009
(non-bijective)

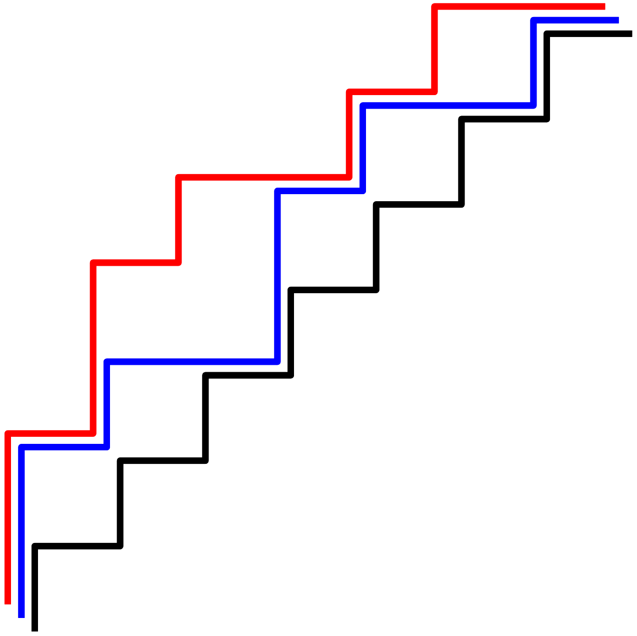
Burrill & al 2015
(non-bijective)

bijective
proof?



Extending this last result to triples of paths making use of plane bipolare orientations





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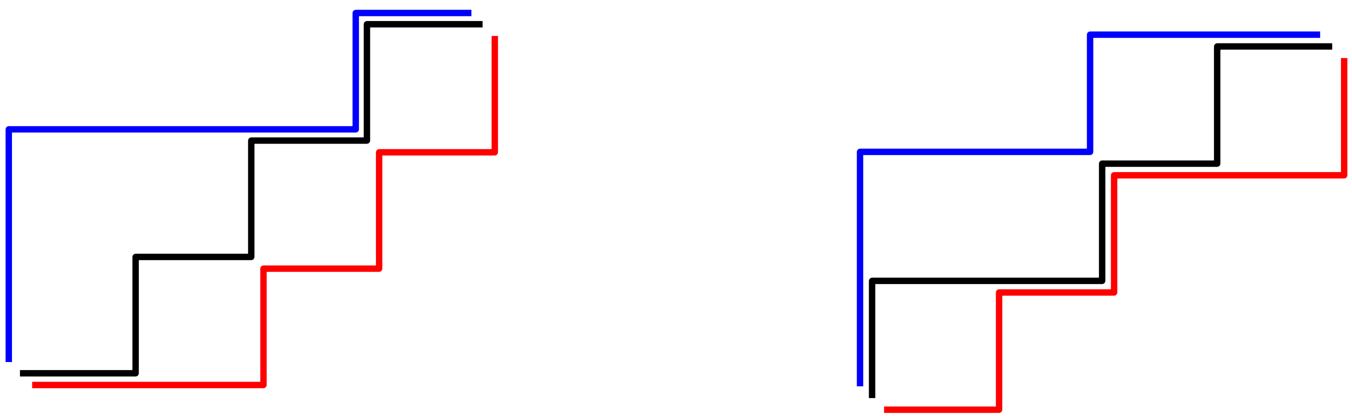
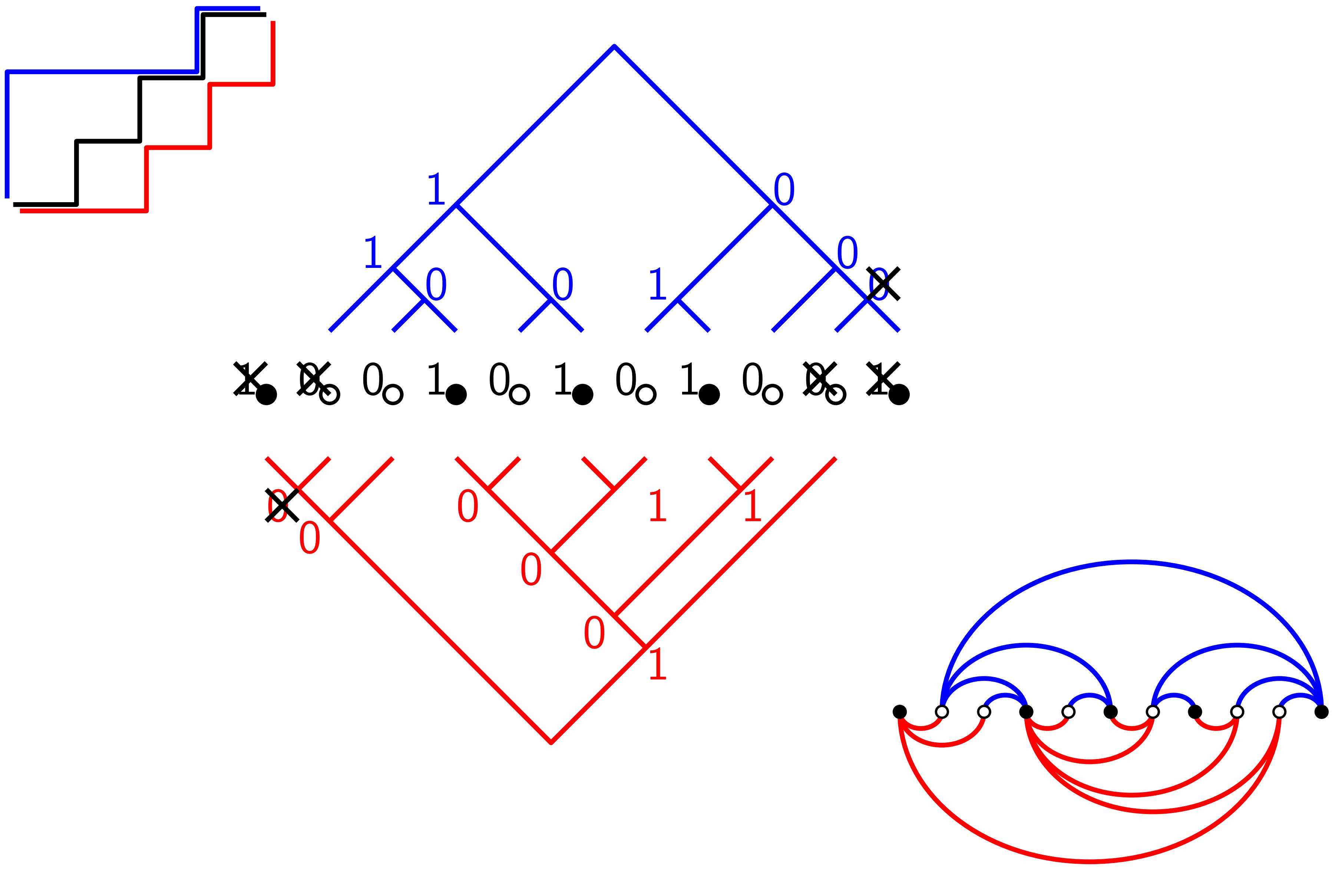


There exists an explicit involution on triples of non-crossing lattice paths that preserves the size, the number of upper peaks, and the number of lower valleys, while exchanging the number higher horizontal contacts and the number of lower horizontal contacts.

Plane bipolar orientations



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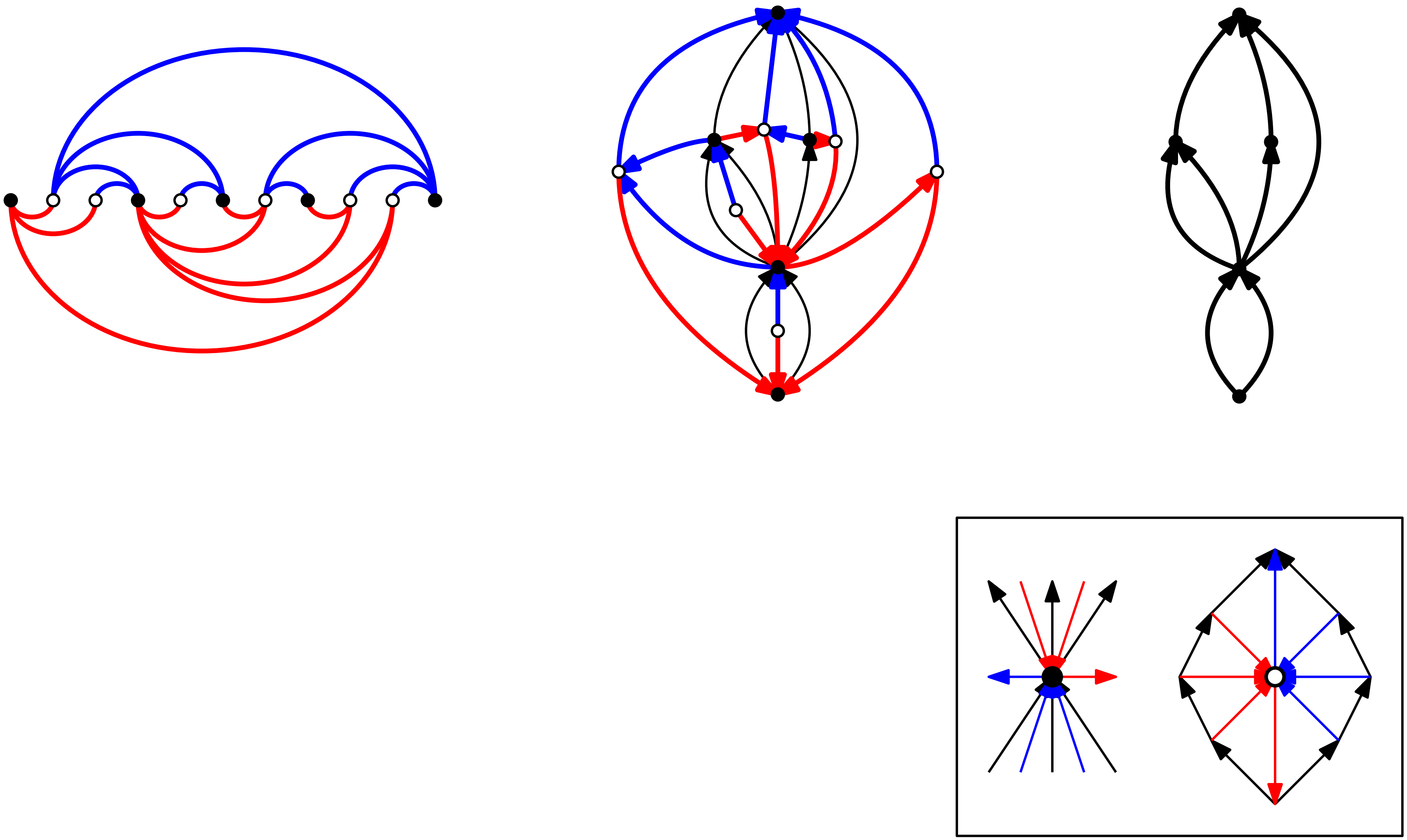

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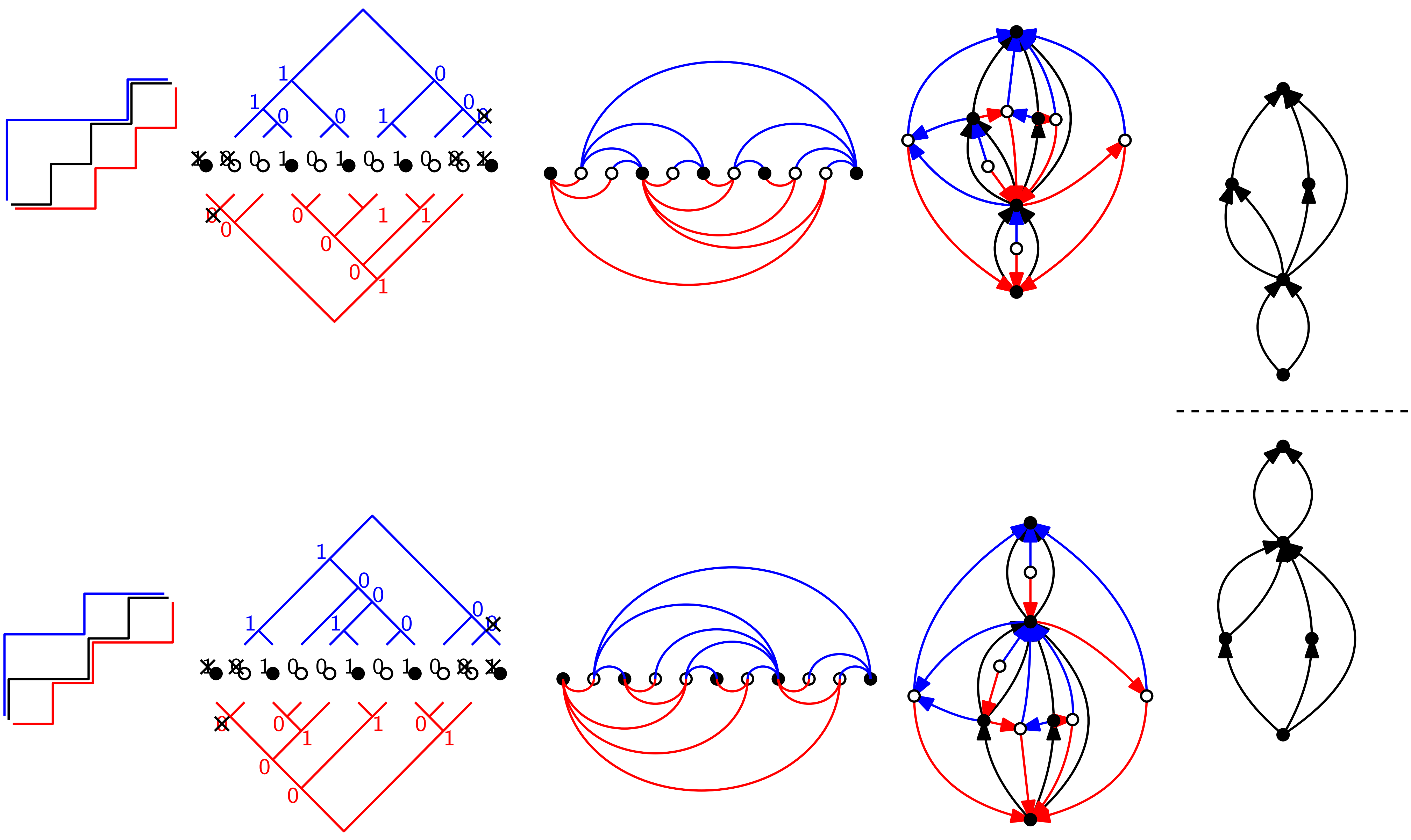

Thanks for your attention