Analysis of algorithms for membership problems on trace languages

Massimiliano Goldwurm

Dipartimento di Matematica, Università degli Studi di Milano

L.I.P.N., Université Paris XIII, 26 November 2019

[BMS82] Bertoni, Mauri, Sabadini. Equivalence and membership problems for ... . *Proc. 9th ICALP*, 1982.

. . . . . .

[AG98] Avellone, G., Analysis of algorithms for ... . *RAIRO Theoretical Informatics and Applications* 32: 141–152, 1998.

[GS00] G., Santini. Clique polynomials have a unique root of smallest modulus. *Inform.Proc.Lett.* 75, 2000

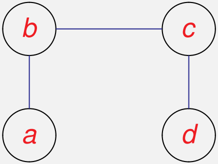
Trace Monoids

Free partially commutative monoids

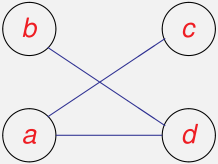
[Cartier-Foata '69, Mazurkiewicz '77, Zielonka '87, ..., Diekert]

concurrent alphabet $(\Sigma, C) \begin{cases} \Sigma & \text{finite alphabet} \\ C \subseteq \Sigma \times \Sigma & \text{irreflexive, symmetric rel.} \end{cases}$

independence graph $(\Sigma, C) =$



dependence graph $(\Sigma, C^c) =$



Definitions

given (Σ, C)

relation $\sim$ — $w = xaby \sim xbay = z \quad \begin{cases} \forall\, x, y \in \Sigma^* \\ \forall\, (a, b) \in C \end{cases}$

relation $\simeq_C$ — reflexive and transitive closure of $\sim$

$\simeq_C$ is a congruence over Σ^*

$$x \simeq_C z,\ y \simeq_C w \Rightarrow xy \simeq_C zw$$

$M(\Sigma, C) = \Sigma^* / \simeq_C$ — trace monoid (f.p.c.m.)

$[x] \cdot [y] = [xy], \quad \forall\, x, y \in \Sigma^*$

$t \in M(\Sigma, C)$ — trace, $t = [x]\ \forall x \in \Sigma^*$

$T \subseteq M(\Sigma, C)$ — trace language

$T = [L] = \{[x] : x \in L\}, \qquad$ where $L \subseteq \Sigma^*$

$Lin(T) = \{x \in \Sigma^* : [x] \in T\}, \ \ T \subseteq M(\Sigma, C)$

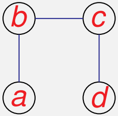
Trace as partially ordered set: $\forall\ t \in M(\Sigma, C)$

$$\mathrm{PO}(t) \;=\; \bigcap_{x \in t} x$$

where $\mathrm{set}(\mathrm{PO}(t)) = \{\sigma_i \mid i\text{-th occurrence of } \sigma \text{ in } t\}$

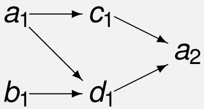
Examples

$(\Sigma, C) =$



$t = [bacda]$ $\qquad t = \{bacda, badca, abdca, abcda, acbda\}$

$t = [bacda]$ $\qquad \mathrm{PO}(t) \;=$



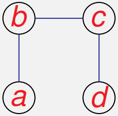
$T = [(ab)^*]$ $\qquad T = \{t \in M(\Sigma, C) \mid t = [a^n b^n], n \in \mathbb{N}\}$

$$Lin(T) = \{x \in \{a, b\}^* \mid |x|_a = |x|_b\}$$

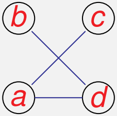
Heaps of pieces

[Viennot '85]

$(\Sigma, C) =$

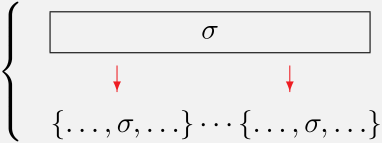


$(\Sigma, C^c) =$

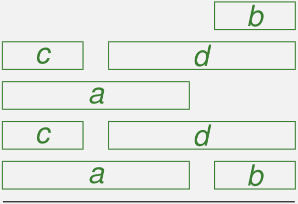


clique cover of $(\Sigma, C^c) = \{\{a, c\}, \{a, d\}, \{b, d\}\}$

$\forall\, \sigma \in \Sigma \longrightarrow$



$t = [bacdadcb] \longrightarrow$



$\{a, c\} \quad \{a, d\} \quad \{b, d\}$

Special cases

1) free monoid $C = \emptyset$ $M(\Sigma, \emptyset) = \Sigma^*$

2) totally commutative monoid

$\Sigma = \{a_1, \ldots, a_m\}$, $C = \{(a_i, a_j) \mid i \neq j\}$, graph (Σ, C) complete

$M(\Sigma, C) \equiv (a_1^* a_2^* \cdots a_m^*)$

$t = [a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m}]$ for $i_1, \ldots, i_m \in \mathbb{N}$

3) direct product of free monoids, C^c transitive

$\Sigma = \bigcup_{i=1}^{k} \Sigma_i$ (disjoint), $C = \bigcup_{i \neq j} (\Sigma_i \times \Sigma_j)$, $\begin{cases} (\Sigma, C) \text{ complete } k\text{-partite} \\ (\Sigma, C^c) \text{ union of } k \text{ cliques} \end{cases}$

$M(\Sigma, C) \equiv \Sigma_1^* \times \Sigma_2^* \times \ldots \times \Sigma_k^*$

$t = [x_1 x_2 \cdots x_k]$, $x_i \in \Sigma_i^*$, $PO(t) =$

$$\begin{array}{l} a_1 \longrightarrow a_2 \longrightarrow \cdots \longrightarrow a_{i_1} \\ b_1 \longrightarrow b_2 \longrightarrow \cdots \longrightarrow b_{i_2} \\ \ldots \qquad \ldots \\ c_1 \longrightarrow c_2 \longrightarrow \cdots \longrightarrow c_{i_k} \end{array}$$

4) free product of commutative monoids, C transitive

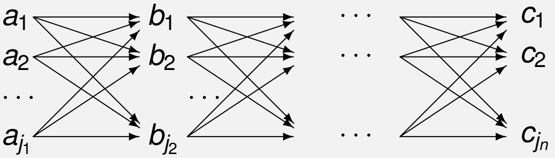
$$\Sigma = \bigcup_{j=1}^{k} \Sigma_j \text{ (disjoint)} \qquad C = \bigcup_{j=1}^{k} (\Sigma_j \times \Sigma_j),$$

$$\begin{cases} (\Sigma, C) \text{ union of } k \text{ cliques} \\ (\Sigma, C^c) \text{ complete } k\text{-partite} \end{cases}$$

$$M(\Sigma, C) \equiv (M(\Sigma_1, C_1), M(\Sigma_2, C_2), \ldots, M(\Sigma_k, C_k))^*$$

$$t = [\gamma_1][\gamma_2]\cdots[\gamma_n], \quad \text{where} \begin{cases} \text{each } \gamma_i \text{ is an antichain, } \gamma_i \subseteq \Sigma_{j_i} \\ \gamma_i \times \gamma_{i+1} \subset C^c \text{ if } j_i \neq j_{i+1} \end{cases}$$

$PO(t) =$



Prefixes of traces

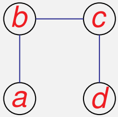
Definition u prefix of $t \in M(\Sigma, C)$ if $\exists v$ s.t.

$$t = u \cdot v$$

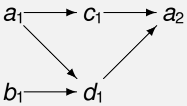
$$t = [xy] \quad \Rightarrow \quad t = [x] \cdot [y] \quad \Rightarrow \quad [x] \text{ prefix of } t$$

Example

$(\Sigma, C) =$



$t = [abcda] \equiv$



$\mathrm{Pre}(t) = \{\varepsilon, [\bar{a}], [\bar{b}], [\overline{ab}], [a\bar{c}], [a\overline{bc}], [ab\bar{d}], [ab\overline{cd}], [abcd\bar{a}]\}$

Prefixes of traces

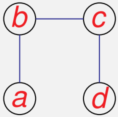
Definition u prefix of $t \in M(\Sigma, C)$ if $\exists v$ s.t.

$$t = u \cdot v$$

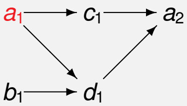
$$t = [xy] \quad \Rightarrow \quad t = [x] \cdot [y] \quad \Rightarrow \quad [x] \text{ prefix of } t$$

Example

$(\Sigma, C) =$



$t = [abcda] \equiv$



$$\mathrm{Pre}(t) = \{\varepsilon, [\overline{a}], [\overline{b}], [\overline{ab}], [a\overline{c}], [a\overline{bc}], [ab\overline{d}], [ab\overline{cd}], [abcd\overline{a}]\}$$

Prefixes of traces

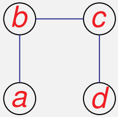
Definition u prefix of $t \in M(\Sigma, C)$ if $\exists v$ s.t.

$$t = u \cdot v$$

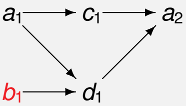
$$t = [xy] \quad \Rightarrow \quad t = [x] \cdot [y] \quad \Rightarrow \quad [x] \text{ prefix of } t$$

Example

$(\Sigma, C) =$



$t = [abcda] \equiv$



$$\text{Pre}(t) = \{\varepsilon, [\overline{a}], [\overline{b}], [\overline{ab}], [a\overline{c}], [a\overline{bc}], [ab\overline{d}], [ab\overline{cd}], [abcd\overline{a}]\}$$

Prefixes of traces

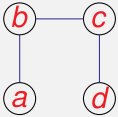
Definition u **prefix** of $t \in M(\Sigma, C)$ if $\exists v$ s.t.

$$t = u \cdot v$$

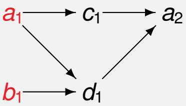
$$t = [xy] \quad \Rightarrow \quad t = [x] \cdot [y] \quad \Rightarrow \quad [x] \text{ prefix of } t$$

Example

$(\Sigma, C) =$



$t = [abcda] \equiv$



$$\mathrm{Pre}(t) = \{\varepsilon, [\overline{a}], [\overline{b}], [\overline{ab}], [a\overline{c}], [a\overline{bc}], [ab\overline{d}], [ab\overline{cd}], [abcd\overline{a}]\}$$

Prefixes of traces

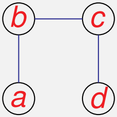
Definition u prefix of $t \in M(\Sigma, C)$ if $\exists v$ s.t.

$$t = u \cdot v$$

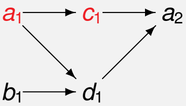
$$t = [xy] \quad \Rightarrow \quad t = [x] \cdot [y] \quad \Rightarrow \quad [x] \text{ prefix of } t$$

Example

$(\Sigma, C) =$



$t = [abcda] \equiv$



$$\text{Pre}(t) = \{\varepsilon, [\overline{a}], [\overline{b}], [\overline{ab}], [a\overline{c}], [a\overline{bc}], [ab\overline{d}], [ab\overline{cd}], [abcd\overline{a}]\}$$

Prefixes of traces

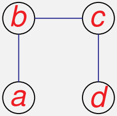
Definition u **prefix** of $t \in M(\Sigma, C)$ if $\exists v$ s.t.

$$t = u \cdot v$$

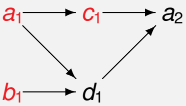
$$t = [xy] \quad \Rightarrow \quad t = [x] \cdot [y] \quad \Rightarrow \quad [x] \text{ prefix of } t$$

Example

$(\Sigma, C) =$



$t = [abcda] \equiv$



$$\mathrm{Pre}(t) = \{\varepsilon, [\overline{a}], [\overline{b}], [\overline{ab}], [a\overline{c}], [a\overline{bc}], [ab\overline{d}], [ab\overline{cd}], [abcd\overline{a}]\}$$

Prefixes of traces

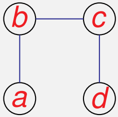
Definition u **prefix** of $t \in M(\Sigma, C)$ if $\exists v$ s.t.

$$t = u \cdot v$$

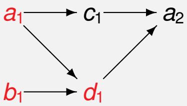
$$t = [xy] \quad \Rightarrow \quad t = [x] \cdot [y] \quad \Rightarrow \quad [x] \text{ prefix of } t$$

Example

$(\Sigma, C) =$



$t = [abcda] \equiv$



$$\text{Pre}(t) = \{\varepsilon, [\overline{a}], [\overline{b}], [\overline{ab}], [a\overline{c}], [a\overline{bc}], [ab\overline{d}], [ab\overline{cd}], [abcd\overline{a}]\}$$

Prefixes of traces

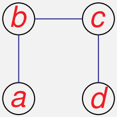
Definition u prefix of $t \in M(\Sigma, C)$ if $\exists v$ s.t.

$$t = u \cdot v$$

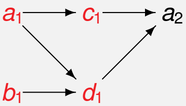
$$t = [xy] \quad \Rightarrow \quad t = [x] \cdot [y] \quad \Rightarrow \quad [x] \text{ prefix of } t$$

Example

$(\Sigma, C) =$



$t = [abcda] \equiv$



$$\mathrm{Pre}(t) = \{\varepsilon, [\overline{a}], [\overline{b}], [\overline{ab}], [a\overline{c}], [a\overline{bc}], [ab\overline{d}], [ab\overline{cd}], [abcd\overline{a}]\}$$

Prefixes of traces

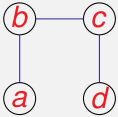
Definition u **prefix** of $t \in M(\Sigma, C)$ if $\exists v$ s.t.

$$t = u \cdot v$$

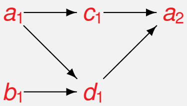
$$t = [xy] \quad \Rightarrow \quad t = [x] \cdot [y] \quad \Rightarrow \quad [x] \text{ prefix of } t$$

Example

$(\Sigma, C) =$



$t = [abcda] \equiv$



$$\mathrm{Pre}(t) = \{\varepsilon, [\overline{a}], [\overline{b}], [\overline{ab}], [a\overline{c}], [a\overline{bc}], [ab\overline{d}], [ab\overline{cd}], [abcd\overline{a}]\}$$

Properties of prefixes

Theorem Set $\alpha = |\text{Max clique}(\Sigma, C)|$. Then, as $n \to +\infty$,

$$\text{Max}\{\#\text{Pre}(t) \mid |t| = n\} = \Theta(n^\alpha)$$

Proof.

1) $u \in \text{Pre}(t) \iff \text{Max}(\text{PO}(u))$ antichain of $\text{PO}(t)$

2) $\text{Pre}(t) \simeq \text{Antichains}(\text{PO}(t))$

3) $\#\text{Pre}(t) = \#\text{Antichains}(\text{PO}(t))$

4) $|A| \leq \alpha$ for every $A \in \text{Antichain}(\text{PO}(t))$

5) for every $t \in M(\Sigma, C)$ s.t. $|t| = n$

$$\#\text{Pre}(t) \leq \sum_{i=0}^{\alpha} \binom{|t|}{i} = O(n^\alpha)$$

$$(\lfloor n/\alpha \rfloor + 1)^\alpha \leq \text{Max}\{\#\text{Pre}(t) \mid |t| = n\}$$

(when $t = [A]^{\lfloor n/\alpha \rfloor}$ with A clique of size α in (Σ, C))

Classes of trace languages

Trace language $T \subseteq M(\Sigma, C)$

$$T = [L] = \{t \in M(\Sigma, C) \mid \exists x \in L : t = [x]\}, \quad L \subseteq \Sigma^*$$

$$Lin(T) = \{x \in \Sigma^* : [x] \in T\}, \;\; T \subseteq M(\Sigma, C)$$

1) T recognizable

$Lin(T)$ regular

$$Lin(T) = \mathcal{L}(\mathcal{A}), \begin{cases} \mathcal{A} \text{ f.s. automaton} \\ \text{partially commutative} \end{cases}$$

2) T rational

finite sets (of traces) $+$ rational operation $(\cup, \cdot, *)$

$T = [L]$, $L \subseteq \Sigma^*$ regular

3) T context-free

algebraic system of equations (over $\mathbb{R}\langle\langle M(\Sigma, C)\rangle\rangle$)

$T = [L]$, $L \subseteq \Sigma^*$ context-free

$\Longrightarrow$ Recognizable $\subsetneq$ Rational $\subsetneq$ Context-free

Kleene's Theorem does not hold:

$$(\Sigma, C) = a \,\text{—}\, b\,, \;\; Lin([(ab)^*]) = \{x \in \Sigma^* \mid \; |x|_a = |x|_b\}$$

Recognition of rational trace languages

Concurrent alphabet (Σ, C)
F.s. det. automaton $\mathcal{A} = \langle Q, q_0, \delta, F \rangle$ over Σ
$L = \mathcal{L}(\mathcal{A})$

Membership problem $(\Sigma, C, \mathcal{A})$
Instance : $x \in \Sigma^*$
Question : $[x] \in [L]$?
(i.e. $\exists\, y \in [x] : \delta(q_0, y) \in F$?)

Idea

Compute $S[x] = \{q \in Q \mid \; q = \delta(q_0, y) \text{ for some } \; y \in [x]\}$

Answer $\begin{cases} \textit{Yes} & \text{if } S[x] \cap F \neq \emptyset \\ \textit{No} & \text{otherwise} \end{cases}$

Observe: Uniform Rational Membership (with Instance $= \Sigma, C, \mathcal{A}, x$) is NP-complete

Recursive procedure

Function $S[x]$

```
begin
    if |x| = 1 then Return {δ(q_0, x)}
       else
         begin
              V := ∅
              for (u, σ) ∈ Pre[x] × Σ  s.t.  [x] = u · [σ]        (|u| = |x| − 1)
                   do { P := S[u]
                        for p ∈ P do   V := V ∪ {δ(p, σ)}
         end
    Return V
end
```

$\exists$ Iterative version of Function $S[x]$ only working on $\text{Pre}[x]$

TIME: $\Theta\left(\#\text{Pre}[x]\right)$

SPACE: $O\left(\text{Max}_{1 \le i \le |x|} \#\{u \in \text{Pre}[x] \mid |u| = i\}\right)$

Analysis of the algorithm

[BMS82, BMS89, BGS86, AG98]

Worst case

TIME: $O\left(n^{\alpha}\right)$
SPACE: $O\left(n^{\alpha-1}\right)$

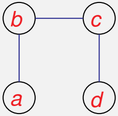
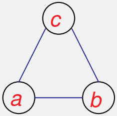
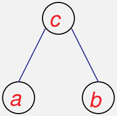
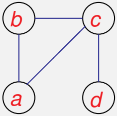
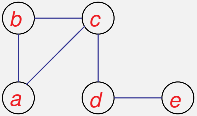
where α is the size of maximum clique in (Σ, C)

Average case

under equiprobable strings of length n

TIME: $\Theta\left(n^{k}\right)$
SPACE: $O\left(n^{\min\{k,\alpha-1\}}\right)$

where k is the number of connected components in (Σ, C^c)
note: $k \leq \alpha$ which often becomes $k << \alpha$

(Σ, C)					
α	2	3	2	3	3
k	1	3	2	2	1

Probabilistic analysis of $\#\mathrm{Pre}[x]$ (equiprobable strings)

[BGS86, G90, G92]

Analysis of the sequence of r.v. $\{\Im_n\}$:
$\Im_n = \#\mathrm{Pre}[z]$ where $z \in \Sigma^n$ under uniform distribution

1) $n+1 \ \leq \ \Im_n \ \leq \ c \cdot n^\alpha, \qquad (c > 0)$

2) $$\mathrm{E}(\Im_n) \ = \ \frac{\sum_{|x|=n} \#\mathrm{Pre}[x]}{\#\Sigma^n} = \eta n^k + \mathrm{O}(n^{k-1}) \qquad (\eta \in \mathbb{Q}_+)$$

3) $$\mathrm{E}(\Im_n^r) \ = \ \eta_r n^{rk} + \mathrm{O}(n^{rk-1}) \qquad (\forall\ r \in \mathbb{N}_+,\ \eta_r \in \mathbb{Q}_+)$$

4) $$\mathrm{var}(\Im_n) \ = \ \mathrm{O}(n^{2k-1})$$

$$\Longrightarrow \ \Pr\left\{\left|\frac{\Im_n}{\eta n^k} - 1\right|\right\} \to 0 \qquad (\forall\ \varepsilon > 0)$$

i.e. $\Im_n \ \sim \ \eta n^k$ with probability $\to 1$

Proof ingredients for $E(\Im_n)$

1) Darboux theorem for rational functions

$$F(z) = \sum_{n=0}^{+\infty} f_n z^n = \frac{a(z)}{b(z)(1 - Hz)^{k+1}}$$

where $a, b \in \mathbb{Z}[z]$, $H \in \mathbb{N}_+$, $k \in \mathbb{N}$, $a(H^{-1}) \neq 0$, roots of $b(z) > H^{-1}$,

$$\Longrightarrow \quad f_n = \eta H^n n^k + O\left(H^n n^{k-1}\right)$$

$$\eta = \frac{a(H^{-1})}{k! b(H^{-1})} \in \mathbb{Q}_+$$

In our cases: $H = \#\Sigma$, $k = \#\text{cc}(\Sigma, C^c) > 0$,
roots of b in $\mathbb{N}_+^{(-1)}$, $b(0) = 1$.

2) Bijective argument

$$\text{let } \Sigma' = \{a' \mid a \in \Sigma\}$$

$$(u,x) \text{ s.t. } u \in \mathrm{Pre}[x] \longleftrightarrow w \in (\Sigma \cup \Sigma')^* \text{ s.t. } \begin{cases} x = w \text{ erasing }' \\ u = [\pi_\Sigma w] \\ a' < b \text{ in } w \Rightarrow (a,b) \in C \end{cases}$$

$$\{(u,x) \mid x \in \Sigma^*, u \in \mathrm{Pre}[x]\} \longleftrightarrow L \subseteq (\Sigma \cup \Sigma')^*$$

L is regular

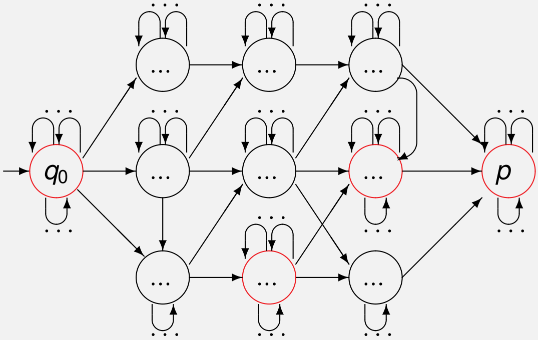
$$\sum_{|x|=n} \#\mathrm{Pre}[x] \equiv f_L(n) = \#(L \cap (\Sigma \cup \Sigma')^n) \qquad (\text{num of } \mathrm{E}(\Im_n))$$

$$F_L(z) = \sum_{n=0}^{+\infty} f_L(n) z^n$$

3) Partially ordered automaton

$L = \mathcal{L}(\mathcal{A})$

- $\mathcal{A} = \langle Q, q_0, \delta, F \rangle$ f.s. automaton over $\Sigma \cup \Sigma'$
- δ defines a partial order over Q, q_0 min, p max
- $\ell(q_0) = \ell(p) = \max\{\ell(q) \mid q \in Q\}$
  where $\ell(q) = \#\{\text{loops in } q\}$
- $Q = F$,



for all $q \in Q$:
$\ell(q) = \#\{\text{loops in } q\}$
$H = \text{Max}\{\ell(q) \mid q \in Q\}$

chain = path in $\mathcal{A}$ from q_0,
$\forall\, \gamma \in \text{Chains}(\mathcal{A}) \;\; m_\gamma = \#\{q \in \gamma \mid \ell(q) = H\}, \quad L_\gamma = \{x \in L \mid \; x \text{ acc. by } \gamma\}$
$m = \text{Max}\{m_\gamma \mid \gamma \in \text{Chains}\}$
$L = \bigcup_\gamma L_\gamma$ disjoint union $\Rightarrow \; F_L(z) = \sum_\gamma F_{L_\gamma}(z)$

$$\Longrightarrow \quad F_L(z) \; = \; \frac{a(z)}{b(z)(1 - Hz)^m} \quad (a, b \text{ as bifore}), \quad H = \#\Sigma, m = k + 1$$

$$f_L(n) = \eta H^n n^k + \mathrm{O}\left(H^n n^{k-1}\right) \qquad (\eta \in \mathbb{Q}_+)$$

Limit distribution of $\{\Im_n\}$ for transitive (Σ, C^c)

5) Assume (Σ, C^c) transitive (and $C \neq \emptyset$) and hence
$M(\Sigma, C) = \Sigma_1^* \times \Sigma_2^* \times \cdots \times \Sigma_k^*$ $(k \geq 2)$
$\Sigma_1, \Sigma_2, \ldots, \Sigma_k$ connected components (cliques) of (Σ, C^c)

1) if $\#\Sigma_i \neq \#\Sigma_j$ for some $i \neq j$ then

$$\frac{\Im_n - n^k \Pi_{i=1}^k p_i}{\sqrt{V} n^{k-1/2}} \longrightarrow \mathcal{N}(0,1) \qquad \text{in distribution}$$

where $p_i = \frac{\#\Sigma_i}{\Sigma}$ and $V > 0$ constant depending on (Σ, C).

2) if $\#\Sigma_i = \#\Sigma/k$ for all $i = 1, 2, \ldots, k$ then

$$2\frac{\Im_n - n^k k^{-k} - n^{k-1} k^{2-k}}{k^{k-3} n^{k-1}} \longrightarrow -\chi^2_{k-1} \qquad \text{in distribution}$$

Proof : based on $\Im_n = \Pi_{i=1}^k (b_i + 1)$ where $(b_1, \ldots b_k) \in \mathcal{M}(n; p_1, \ldots, p_k)$ (multinomial).

Open problem: what about non-transitive C^c ?

Probabilistic analysis of $\#\mathrm{Pre}(t)$ (equiprobable traces)

[..., BGS86, GS00]

Asymptotics for $M_n = \#\{t \in M(\Sigma, C) \mid |t| = n\}$

Based on the clique polynomial :

$$P_{(\Sigma,C)}(z) = \sum_{i=0}^{\alpha} (-1)^i c_i z^i$$

where $c_i = \#\{A \text{ clique of } (\Sigma, C) \mid |A| = i\}$

$$\Longrightarrow \quad \sum_{n=0}^{+\infty} M_n z^n = \frac{1}{P_{(\Sigma,C)}(z)}$$

[Cartier-Foata '69]

$\Longrightarrow$ $P_{(\Sigma,C)}(z)$ has a unique root ρ of smallest modulus

$$\begin{cases} 0 < \rho \leq 1 \\ \rho \text{ with multiplicity } \ell \in \mathbb{N}_+ \end{cases}$$

$$\Longrightarrow \quad M_n = b\rho^{-n} n^{\ell-1} + O\left(\rho^n n^{\ell-2}\right) \qquad (b > 0)$$

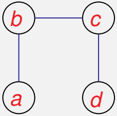
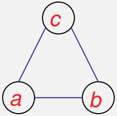
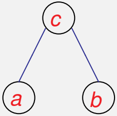
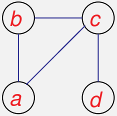
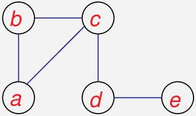
Analysis of the sequence of r.v. $\{\Delta_n\}$:

$\Delta_n = \#\mathrm{Pre}(u)$ where $u \in \{t \in M(\Sigma, C) \mid |t| = n\}$
under uniform distrib.

$$n + 1 \ \leq \ \Delta_n \ \leq \ c \cdot n^{\alpha}, \qquad (c > 0)$$

$$\Longrightarrow \quad \mathrm{E}(\Delta_n) \ = \ \frac{\sum_{|t|=n} \#\mathrm{Pre}(t)}{M_n} = \gamma n^{\ell} + \mathrm{O}(n^{\ell-1}) \qquad (\gamma > 0)$$

where ℓ is the multiplicity of the smallest root of $P_{(\Sigma,C)}(z)$

(Σ, C)					
α	2	3	2	3	3
k	1	3	2	2	1
$P_{(\Sigma,C)}(z)$	$1 - 4z + 3z^2$	$(1 - z)^3$	$1 - 3z + 2z^2$	$1 - 4z + 4z^2 - z^3$	$1 - 5z + 5z^2 - z^3$
ℓ	1	3	1	1	1

Open problems

1) Does $\ell \leq k$ hold for every (Σ, C) ?

2) Limit distributions of $\{\Im_n\}$ for non-transitive (Σ, C^c)

Are they Gaussian ?

3) Asymptotics of $E(\Delta_n^r)$ for $r > 1$

4) Asymptotics of $\mathrm{var}(\Delta_n)$

5) There exist Local limit Laws?

Thank you !

[AG98] A. Avellone, M. Goldwurm. Analysis of algorithms for the recognition of rational and context-free trace languages. *RAIRO Theoretical Informatics and Applications* 32: 141–152, 1998.

[BG89] A. Bertoni and M. Goldwurm. On the prefixes of a random trace and the membership problem for context-free trace languages. In *Proc. AAECC 5*, L. Huguet and A. Poli editors, LNCS n.356, Springer, 35–59, 1989.

[BGS88] A. Bertoni, M. Goldwurm, N. Sabadini, Analysis of a class of algorithms for problems on trace languages, In *Proc. AAECC 4*, Th. Beth and M. Clausen editors, LNCS n. 307, Springer, 202–214, 1988.

[BGMS95] A. Bertoni, M. Goldwurm, G. Mauri, N. Sabadini. Counting techniques for inclusion, equivalence and membership problems. In *The book of traces*, V. Diekert and G. Rozenberg editors, World Scientific, 131–163, 1995.

[BMS82] A. Bertoni, G. Mauri, N. Sabadini. Equivalence and membership problems for regular and context-free trace languages. *Proc. 9th ICALP* LNCS n. 140, Springer, 61–71, 1982.

[BMS89] A. Bertoni, G. Mauri, N. Sabadini. Membership problems for regular and context-free trace languages. *Information and Computation* 82 (2): 135–150, 1989.

[G90] M. Goldwurm. Some limit distributions in analysis of algorithms for problems on trace languages. *International Journal of Foundations of Computer Science*, 1(3):265–276, 1990.

[G92] M. Goldwurm. Probabilistic estimation of the number of prefixes of a trace. *Theoretical Computer Science*, 92:249–268, 1992.

[GS00] M. Goldwurm, M. Santini. Cliques polynomials have a unique root of smallest modulus. *Information Processing Letters*, 75:127–132, 2000.