Knots and their related q-series

(joint with Don Zagier)

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Séminaire CALIN: Combinatorics, Algorithms and Interactions and
GDR EFI: Équations fonctionnelles et interactions
27 April, 2021

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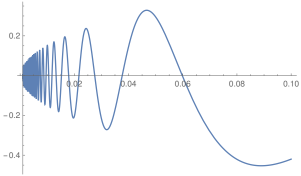
Spin networks are plane trivalent graphs with multiple edges/loops. The evaluation of quantum spin networks produces (multi-parameter) power series in q with integer coefficients. For the simplest spin network, the tetrahedron, the corresponding q-series is

$$\begin{aligned} G_0(q) &= \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)/2}}{(q;q)_n^2} \\ &= 1 - q - 2q^2 - 2q^3 - 2q^4 + q^6 + \dots \end{aligned}$$

where $(q;q)_n = \prod_{j=1}^{n}(1-q^j)$ is the n-th quantum factorial. Integer coefficients with both positive and negative signs.

$G_0(q)$ is an analytic function of $|q| < 1$.
Let $q = e^{2\pi i \tau}$ and $g_0(\tau) = G_0(q)$ for $\mathrm{Im}(\tau) > 0$.
What is the asymptotics as $\tau \to 0$?


0.2
−0.2
−0.4
0.02
0.04
0.06
0.08
0.10

A hard numerical computation shows that the osciallation is about 0.3230659472. I recognized this number to be approximately $V/(2\pi)$ where

$$V = 2\mathrm{Im}(\mathrm{Li}_2(e^{2\pi i/6})), \qquad \mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

An even harder numerical computation (using numerical evaluation of the q-series, extrapolation by Richardson transform and recognition of the numbers) reveals that

$$g_0(\tau) \sim \sqrt{\tau}\left(\widehat{\Phi}(2\pi i \tau) - i\,\widehat{\Phi}(-2\pi i \tau)\right)$$

where

$$\widehat{\Phi}(x) = e^{iV/x}\,\Phi(x)$$

$$\Phi(x) = \sum_{j=0}^{\infty} A_j x^j\,, \qquad A_j = \frac{1}{\sqrt[4]{3}}\left(\frac{1}{72\sqrt{-3}}\right)^j \frac{a_j}{j!}$$

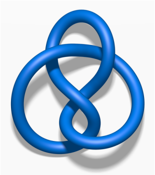
and $a_j \in \mathbb{Q}$ with

j	0	1	2	3	4	5	6	7
a_j	1	11	697	$\frac{724351}{5}$	$\frac{278392949}{5}$	$\frac{244284791741}{7}$	$\frac{1140363907117019}{35}$	$\frac{212114205337147471}{5}$

Don remembered the number 697 in some of the hundreds of joint pari files, grep-ed the answer and found it in relation to the asymptotics of the Kashaev invariant of the 4_1 knot.

We knew 150 terms of the $\widehat{\Phi}$-series of the 4_1 knot and once we matched 11 and 697, we were able to further match 20 more.
But what does $G_0(q)$ have to do with the 4_1 knot?

By hyperbolic metric we mean a complete, finite volume, constant curvature -1 Riemannian metric.



Universal cover $\tilde{M} = \mathbb{H}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$, $ds^2 = \frac{dx^2+dy^2+dz^2}{z^2}$.

$$\mathrm{Isom}^+(\mathbb{H}) = \mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\pm I$$

$$\mathrm{SL}_2(\mathbb{C}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | ad - bc = 1\}$$

$\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ is the discrete faithful representation of a hyperbolic manifold M.
The conjugacy class of $\gamma \in \pi_1(M)$ is represented by a unique geodesic whose complex length is essentially $\mathrm{tr}(\rho(\gamma)) \in \mathbb{C}$.
Trace field

$$F(M) = \mathbb{Q}\langle \mathrm{tr}(\rho(\gamma)) \mid \gamma \in \pi_1(M) \rangle$$

- $\pi_1(M)$ is finitely generated (even finitely presented),
- $\rho : \pi_1(M) \to \mathrm{PSL}_2(\overline{\mathbb{Q}})$

So, $F(M)$ is a number field.

$$\pi_1(4_1) = \langle a, b | bab^{-1}ab = aba^{-1}ba \rangle$$

$$\rho(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \rho(b) = \begin{pmatrix} 1 & 0 \\ -\epsilon^{2\pi i/6} & 1 \end{pmatrix}$$

$$F(4_1) = \mathbb{Q}(\sqrt{-3}).$$

This is the story of finite dimensional representations of $U_q(\mathfrak{sl}_2)$ at roots of unity.

Upside-down cake:

$$J(q) = \sum_{n=0}^{\infty} (-1)^n \frac{(q;q)_n^2}{q^{n(n+1)/2}} = \sum_{n=0}^{\infty} (q;q)_n (q^{-1};q^{-1})_n$$

It can be evaluated when $q = e^{2\pi i/N}$ is a root of unity.

N	1	2	3	4	5	6	$\cdots$	100
$J(q)$	1	5	13	27	$46 + 2\sqrt{5} \approx 50.47$	89	$\cdots$	8.2×10^{16}

Volume Conjecture (Kashaev):

$$\lim_N \frac{1}{N} \log J(e^{2\pi i/N}) = \frac{V}{2\pi}.$$

Asymptotics to all orders in $1/N$:

$$J(e^{2\pi i/N}) \sim N^{3/2}\widehat{\Phi}\left(\frac{2\pi i}{N}\right) .$$

This is the story of Superconformal field theory and a 3d-3d correspondence of a six-dimensional theory X. The 3D-index was introduced by Dimofte-Gaiotto-Gukov.

Building block: the tetrahedron index:

$$I_\Delta(m,e) = \sum_{n=\max\{0,-e\}}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)-(n+\frac{1}{2}e)m}}{(q)_n(q)_{n+e}} .$$

A knot complement is assembled out of ideal tetrahedra, each with its own building block, with contracted indices. For the 4_1 knot:

$$\mathrm{Ind}_{4_1}(q) = \sum_{k_1,k_2\in\mathbb{Z}} I_\Delta(k_1,k_2) I_\Delta(k_2,k_1) = 1-8q-9q^2+18q^3+46q^4+90q^5+\cdots$$

An illegitemate calculation predicts that $\mathrm{Ind}_{4_1}(q) = G_0(q)^2$ but this is not true. So, the search for another series $G_1(q)$ starts.

This is the story of representations of the mapping class group in Hilbert spaces and of quantum hyperbolic geometry, introduced by Andersen-Kashaev. It is also the story of complex Chern–Simons theory introduced by Dimofte and Gukov.

The building block of this theory is Faddeev's quantum dilogarithm.

$$\Phi_b(z) = \exp\left(\frac{1}{4}\int_{\mathbb{R}(+)} \frac{e^{-2ixz}}{\sinh(bx)\sinh(b^{-1}x)}\frac{dx}{x}\right) .$$

$(\tau = b^2)$.

Given an ideal triangulation of a knot complement, we place one quantum dilogarithm at each tetrahedron and contract indices. For the 4_1 knot we have (Andersen-Kashaev):

$$Z_{4_1}(\tau) = \int_{\mathbb{R}+i\varepsilon} \Phi_{\sqrt{\tau}}(x)^2 \, e^{-\pi i x^2} dx \qquad (\tau \in \mathbb{C}' = \mathbb{C} \setminus (-\infty, 0])$$

(G.-Kashaev) When $\mathrm{Im}(\tau) > 0$, we have:

$$2i\,(\tilde{q}/q)^{1/24}\,Z_{4_1}(\tau) = \tau^{1/2}\,G_1(q)\,G_0(\tilde{q}) \,-\, \tau^{-1/2}\,G_0(q)\,G_1(\tilde{q})\,, \qquad (1)$$

where $q = e^{2\pi i\tau}$ and $\tilde{q} = e^{-2\pi i/\tau}$

where

$$G_1(q) = \sum_{m=0}^{\infty} \frac{(-1)^m \, q^{m(m+1)/2}}{(q)_m^2} \left(E_1(q) + 2 \sum_{j=1}^{m} \frac{1+q^j}{1-q^j} \right)$$
$$= 1 - 7q - 14q^2 - 8q^3 - 2q^4 + 30q^5 + 43q^6 + 95q^7 + 109q^8 + \dots$$

where $E_1(q)$ is

$$E_1(q) = 1 \, - \, 4 \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} = 1 \, - \, 4 \sum_{n=1}^{\infty} d(n) \, q^n$$

where $d(n)$ is the number of divisors of n.

The relation of G_0, G_1 and the 3D-index:

$$\mathrm{Ind}_{4_1}(q) = G_0(q) G_1(q)\,.$$

The asymptotics of $g_1(\tau) = G_1(e^{2\pi i \tau})$ at $\tau \to 0$:

$$g_1(\tau) \sim \frac{1}{\sqrt{\tau}} \left(\widehat{\Phi}(2\pi i \tau) + i\, \widehat{\Phi}(-2\pi i \tau) \right).$$

So, the quantum invariants of the 4_1 involve the pair of q-series $(G_0(q), G_1(q))$ and the pair

$$(\widehat{\Phi}^{(\sigma_1)}(h), \widehat{\Phi}^{(\sigma_2)}(h)) := (\widehat{\Phi}(h), i\widehat{\Phi}(-h))$$

of factorially divergent asymptotic power series, labeled by the boundary parabolic $\mathrm{SL}_2(\mathbb{C})$-representations of the fundamental group.

Define:

$$Q(u) = e^{-V/(2\pi)}\Phi(2\pi iu)\Phi\left(-\frac{2\pi iu}{1+u}\right) - e^{V/(2\pi)}\Phi\left(\frac{2\pi iu}{1+u}\right)\Phi(-2\pi iu)$$

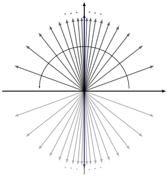
Then, $Q(u)$ is a convergent power series with radius of convergence 1.

k	0	50	100	150
$[h^k]\Phi(h)$	0.75	$6.7 \cdot 10^{71}$	$3.1 \cdot 10^{174}$	$7.4 \cdot 10^{283}$
$[v^k]Q(v)$	-0.379	0.012	-0.007	0.002

In fact, (G.Zagier)

$$Z_{4_1}(u+1) = Q(u)\,.$$

The search for more series is on. $\widehat{\Phi}(h)$ is a resurgent series (G.-Gu-Mariõ) whose Borel transform involves singularities with integer Stokes constants that lead to new q-series that emerge out of the peacock-pattern:



This leads to next-generation descendant q-series $(G_0^{(m)}(q), G_1^{(m)}(q))$ for integers m where

$$G_0^{(m)}(q) = \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)/2+mn}}{(q;q)_n^2}$$

$$G_1^{(m)}(q) = \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)/2+mn}}{(q;q)_n^2} \left(2m + E_1(q) + 2\sum_{j=1}^{n} \frac{1+q^j}{1-q^j}\right) .$$

$G_0^{(m)}(q)$ and $G_1^{(m)}(q)$ are a basis of solutions of the linear q-difference equation (G.-Gu-Mariõ)

$$y_{m+1}(q) - (2 - q^m) y_m(q) + y_{m-1}(q) = 0 \qquad (m \in \mathbb{Z}) .$$

whose Wronskian satisfies the determinant

$$\det W_m(q) = 2$$

and the symmetry

$$W_m(q^{-1}) = W_{-m}(q) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the orthogonality

$$\frac{1}{2} W_m(q) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} W_m(q)^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Likewise, there is a descendant of the pair of the $\widehat{\Phi}$-power series that also satisfies the above linear q-difference equation and relations. So, the quantum invariants of the 4_1 involve a 2×2 matrix of q-series $Q(q)$ and of h-series $\widehat{\Phi}(h)$.

The function

$$W(S,\tau) := Q(e^{-2\pi i/\tau})Q(e^{2\pi i \tau})$$

is holomorphic for $\tau \in \mathbb{C} \setminus (-\infty, 0]$, and together with $W(T,\tau) := 1$ define an $\mathrm{SL}_2(\mathbb{Z})$ cocycle of matrix-valued holomorphic functions on a cut plane that satisfy the equation

$$W(\gamma', x)\, W(\gamma, \gamma' x) = W(\gamma\gamma', x)$$

for all γ and γ' in $\mathrm{SL}_2(\mathbb{Z})$. Recall that $\mathrm{SL}_2(\mathbb{Z})$ is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

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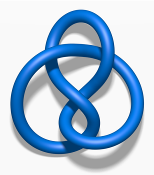
For the next simplest knot, the 5_2 knot



the trace field $F(5_2)$ is cubic $x^3 - x^2 + 1 = 0$, of discriminant -23, giving rise to 3 boundary parabolic connections.

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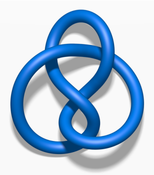

the trace field $F(5_2)$ is cubic $x^3 - x^2 + 1 = 0$, of discriminant -23, giving rise to 3 boundary parabolic connections.
Q^{5_2} is a 3x3 matrix that consists of 9 q-series defined for $|q| < 1$ and 9 more defined for $|q| > 1$ giving a total of 18 q-series.

For the next simplest hyperbolic knot, the $(-2,3,7)$ pretzel knot, $F((-2,3,7)) = F(5_2)$ but there are 6 boundary parabolic connections, 3 defined over the trace field and 3 more defined over $\mathbb{Q}(2cos(2\pi/7))$.

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These cocycles are new, and their entries are *holomorphic quantum modular forms*.

If you want to learn more about these fascinating objects, Don Zagier is giving an online course in MPI-ICTP-SISSA.
https://zoom.us/j/96952516566?pwd=Z3NyZW04M2YxSHo2MWdlOHJ4MlNpUT09
Meeting ID: 969 5251 6566
Passcode: 307018

Merci beaucoup!