Universal Asymptotics for Positive Catalytic Equations

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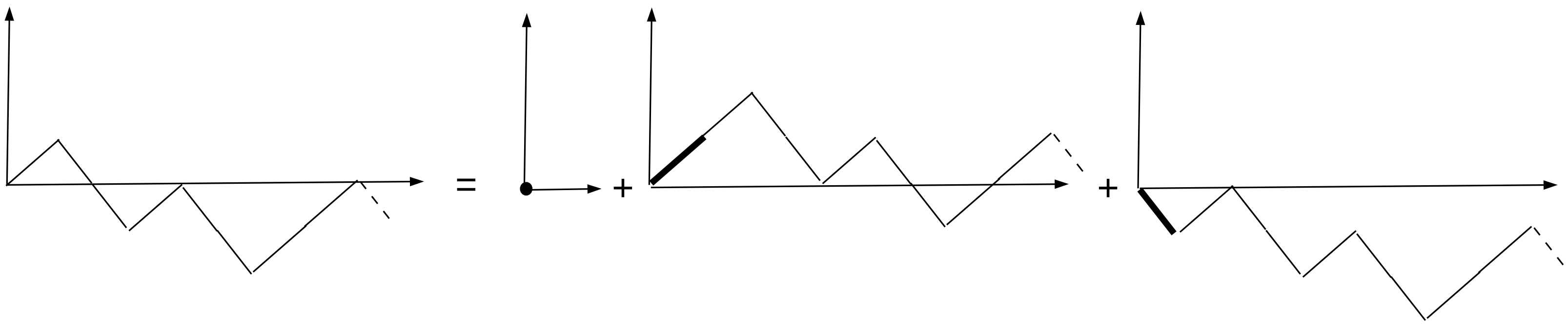
One Functional Equation

Unrestricted paths



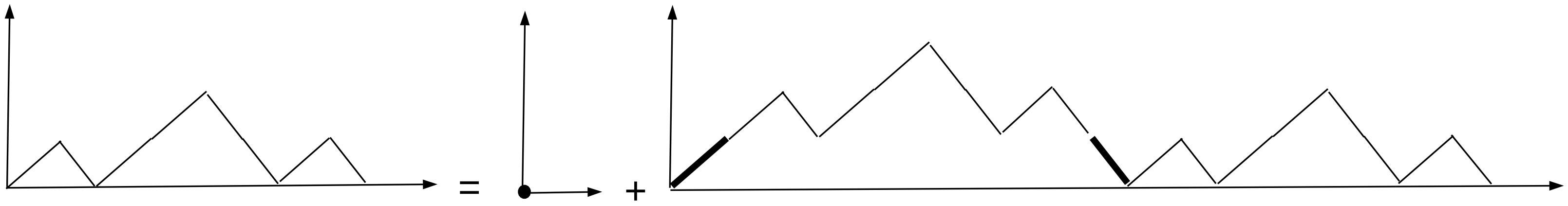
$$\boxed{B(z) = 1 + 2zB(z)}$$

$$B(z) = \frac{1}{1-2z}$$ **(polar singularity)**

$$\boxed{b_n = [z^n]B(z) = 2^n}$$

One Functional Equation

Dyck paths



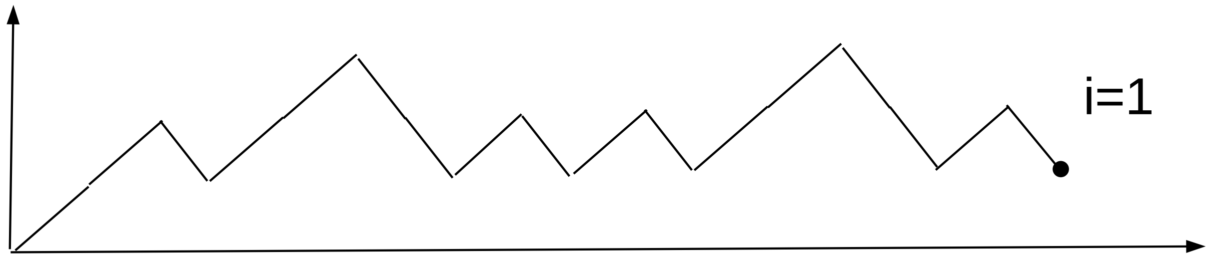
$$\boxed{B(z) = 1 + z^2\, B(z)^2}$$

$$B(z) = \frac{1 - \sqrt{1 - 4z^2}}{2z^2}$$ **(squareroot singularity)**

$$\boxed{b_{2n} = [z^{2n}]B(z) = \frac{1}{n}\binom{2n}{n} \sim \sqrt{\frac{8}{\pi}} n^{-3/2} 2^n}$$

One Functional Equation

Non-negative lattice paths



$f_{n,i}$ ... number of non-negative paths from $(0,0) \to (n,i)$

$$f_i(z) = \sum_{n\ge 0} f_{n,i} z^i \qquad F(z,u) = \sum_{i\ge 0} f_i(z) u^i = \sum_{n,i\ge 0} f_{n,i} z^n u^i$$

$$\begin{aligned} f_0(z) &= 1 + z f_1(z), \\ f_i(z) &= z f_{i-1}(z) + z f_{i+1}(z) \quad (i \ge 1) \end{aligned}$$

$$\boxed{F(z,u) = 1 + zuF(z,u) + z\frac{F(z,u) - F(z,0)}{u}}$$

u ... **"catalytic variable"**

One Functional Equation

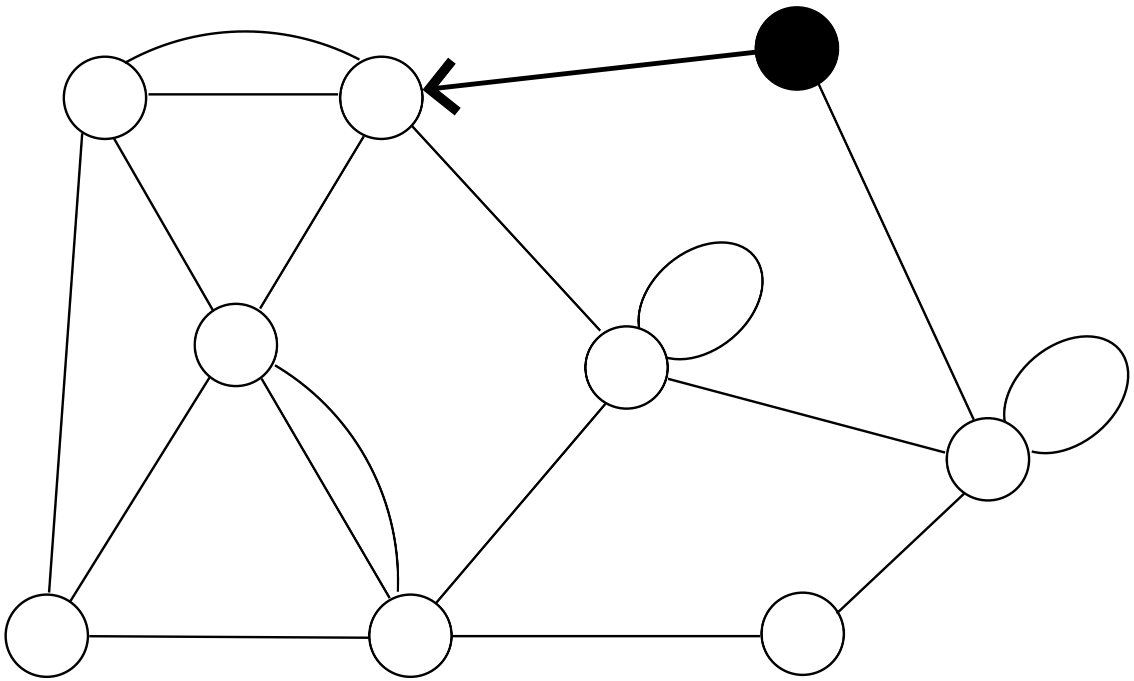
Non-negative lattice paths

$$F(z,0) = \frac{1-\sqrt{1-4z^2}}{2z^2} \quad \textbf{(squareroot singularity)}$$

$$\boxed{f_{2n,0} = [z^{2n}]F(z,0) = \frac{1}{n}\binom{2n}{n} \sim \sqrt{\frac{8}{\pi}} n^{-3/2} 2^n}$$

One Functional Equation

Planar Maps



$M_{n,k}$ ... number of planar maps with n edges and outer face valency k

$$M(z,u) = \sum_{n,k} M_{n,k} z^n u^k$$

One Functional Equation

Planar Maps

$$\boxed{M(z,u) = 1 + zu^2 M(z,u)^2 + uz\frac{uM(z,u) - M(z,1)}{u-1}.}$$

u … **"catalytic variable"**

$$M(z,1) = -\frac{1}{54z^2}\left(1 - 18z - \boxed{(1-12z)^{3/2}}\right) \qquad \text{(3/2-singularity)}$$

$$\boxed{M_n = [z^n]M(z,1) = \frac{2(2n)!}{(n+2)!n!}3^n \sim \frac{2}{\sqrt{\pi}} \cdot n^{-5/2} 12^n}$$

One Functional Equation

One positive linear equation

Theorem 1. Polar singularity:

$Q_0(z)$, $Q_1(z)$ ... polynomials with **non-negative coefficients**.

$$\boxed{B(z) = Q_0(z) + zQ_1(z)B(z)}$$

$$\Longrightarrow \quad \boxed{b_n = [z^n]B(z) \sim c_j \cdot z_0^{-n}}, \quad n \equiv j \bmod m$$

for $j \in \{0, 1, \ldots, m-1\}$ and some $m \geq 1$.
$z_0 > 0$ is given by $z_0 Q_1(z_0) = 1$.

Remark. Proof is simple analysis of $B(z) = Q_0(z)/(1 - zQ_1(z))$.

One Functional Equation

One positive non-linear equation

Theorem 2. [Bender, Canfield, Meir+Moon, ...] **Squareroot sing.**:

$Q(z,y)$ ... polynomial with **non-negative coefficients** and $Q(0,0) = 0$ and $Q_{yy} \neq 0$.

$$\boxed{B(z) = Q(z, B(z))}$$

$$\Longrightarrow \quad \boxed{b_n = [z^n]B(z) \sim c \cdot n^{-3/2} z_0^{-n}.}, \qquad n \equiv j_0 \bmod m,$$

and $\boxed{b_n = 0}$ for $n \not\equiv j_0 \bmod m$, where $m \geq 1$.
$z_0 > 0$ satisfies $b_0 = Q(z_0, b_0)$ and $1 = Q_y(z_0, b_0)$ for some $b_0 > 0$.

Remark. Proof is based on the analysis of the singular point (z_0, b_0) of the curve $b = Q(z, b)$ that leads to the squareroot singularty $B(z) = g(z) - h(z)\sqrt{1 - z/z_0}$.

One Functional Equation

One positive linear catalytic equation

Theorem 3. [D.+Noy+Yu] **Squareroot singularity**:

$Q_0(z,u), Q_1(z,u), Q_2(z,u)$ ... polynomials with **non-negative coefficients** such that $Q_{1,u} \neq 0$ and $u \not| Q_2$.

$$\boxed{M(z,u) = Q_0(z,u) + zM(z,u)Q_1(z,u) + z\frac{M(z,u)-M(z,0)}{u}Q_2(z,u)}$$

$$\Longrightarrow \quad \boxed{M_n = [z^n]M(z,0) \sim c \cdot n^{-3/2} z_0^{-n}}, \quad n \equiv j_0 \bmod m,$$

(for some constants $c, z_0 > 0$) and $\boxed{M_n = 0}$ for $n \not\equiv j_0 \bmod m$, where $m \geq 1$.

One Functional Equation

One positive non-linear catalytic equation

Theorem 4. [D.+Noy+Yu] 3/2-**Singularity**:

$Q(y_0, y_1, z, u)$ ... polynomial with **non-negative coefficients** that is **non-linear** in y_0, y_1 (and depends on y_0, y_1) and $Q_0(u)$ a non-negative polynomial in u.

$$\boxed{M(z,u) = Q_0(u) + zQ\left(M(z,u), \frac{M(z,u) - M(z,0)}{u}, z, u\right)}$$

$$\Longrightarrow \quad \boxed{M_n = [z^n]\, M(z,0) \sim c \cdot n^{-5/2} z_0^{-n}.}, \qquad n \equiv j_0 \bmod m,$$

(for some constants $c, z_0 > 0$) and $\boxed{M_n = 0}$ for $n \not\equiv j_0 \bmod m$, where $m \geq 1$.

System of Functional Equations

$Q_1, \dots Q_d$ ... polynomials with **non-negative** coefficients.
$y_1 = y_1(z), \dots, y_d = y_d(z)$ ... solution of the system:

$$\begin{aligned} y_1 &= Q_1(z, y_1, \dots, y_d), \\ &\vdots \\ y_d &= Q_d(z, y_1, \dots, y_d). \end{aligned}$$

Recall that if $\boxed{d = 1}$ then the single equation $\boxed{y = Q(z, y)}$ has either a **polar singularity** (if it is linear) or a **squareroot singularity** (if it is non-linear).

Question. *What happends for* $\boxed{d > 1}$ *??*

Systems of functional equations

Strongly connected dependency graph

Theorem 5 [D., Lalley, Woods]

$\boxed{\mathbf{y} = \mathbf{Q}(z, \mathbf{y})}$ ... **non-negative** (and well defined) polynomial system of $d \ge 1$ equations such that the dependency graph is **strongly connected**.

Then the situation is the **same as for a single equation**.

It the system is **linear** then we have a common **polar singularity** and

$$\boxed{[z^n]y_1(z) \sim c_j \cdot z_0^{-n}}, \qquad n \equiv j \bmod m$$

whereas if it is **non-linear** then we have a **squareroot singularity** and

$$\boxed{[z^n]y_1(z) \sim c \cdot n^{-3/2} z_0^{-n}.}, \qquad n \equiv j_0 \bmod m.$$

Systems of functional equations

General dependency graph

Theorem 6 [Banderier+D.]

$\boxed{\mathbf{y} = \mathbf{Q}(z, \mathbf{y})}$ ... **non-negative** (and well defined) polynomial system of equations.

$$\Longrightarrow \quad \boxed{[z^n]\, y_1(z) \sim c_j\, n^{\alpha_j}\, \rho_j^{-n}} \qquad (n \equiv j \bmod m),$$

for $j \in \{0, 1, \ldots, m-1\}$ for some $m \geq 1$, where

$$\boxed{\alpha_j \in \{-2^{-k} - 1 : k \geq 1\} \cup \{m2^{-k} - 1 : m \geq 1, k \geq 0\}}.$$

Theorem 3: Kernel Method

$$M(z,u) = Q_0(z,u) + zM(z,u)Q_1(z,u) + z\frac{M(z,u)-M(z,0)}{u}Q_2(z,u)$$

rewrites to

$$M(z,u)\boxed{\left(1 - zQ_1(z,u) - \frac{z}{u}Q_2(z,u)\right)} = Q_0(z,u) - \frac{z}{u}M(z,0)Q_2(z,u).$$

If $u = u(z)$ satisfies the **kernel equation**

$$\boxed{1 - zQ_1(z,u(z)) - \frac{z}{u(z)}Q_2(z,u(z)) = 0}$$

Then the right hand side is also zero and we obtain

$$M(z,0) = \frac{Q_0(z,u(z))}{1 - zQ_1(z,u(z))}$$

Theorem 3: Kernel Method

The **kernel equation**

$$1 - zQ_1(z, u(z)) - \frac{z}{u(z)} Q_2(z, u(z)) = 0$$

rewrites to

$$\boxed{u(z) = zQ_2(z, u(z)) + zu(z)Q_1(z, u(z))}$$

By **Theorem 2** we, thus, obtain a **squareroot singularity** for $u(z)$ which implies a **squareroot singularity** for

$$M(z, 0) = \frac{Q_0(z, u(z))}{1 - zQ_1(z, u(z))}.$$

Theorem 4: Bousquet-Melou–Jehanne Method

Let $P(x_0, x_1, z, u)$ be an analytic function such that $(y(z) = M(z, 0))$

$$\boxed{P(M(z,u), y(z), z, u) = 0.}$$

By taking the derivative with respect to u we get

$$\boxed{P_{x_0}(M(z,u), y(z), z, u)} M_u(z,u) + P_u(M(z,u), y(z), z, u) = 0.$$

Key observation:

$$\boxed{\exists\, u(z) : P_{x_0}(M(z,u(z)), y(z), z, u(z)) = 0 \Longrightarrow P_u(M(z,u(z)), y(z), z, u(z)) = 0}$$

Thus, with $f(z) = M(z, u(z))$ we get the system for $f(z), y(z), u(z)$

$$P(f(z), y(z), z, u(z)) = 0$$
$$P_{x_0}(f(z), y(z), z, u(z)) = 0$$
$$P_u(f(z), y(z), z, u(z)) = 0.$$

Theorem 4: Bousquet-Melou–Jehanne Method

Set (as given in our case)

$$\boxed{P(x_0, x_1, z, u) = Q_0(u) + zQ(x_0, (x_0 - x_1)/u, z, u) - x_0.}$$

Then the system $P = 0$, $P_{x_0} = 0$, $P_u = 0$ rewrites to

$$\begin{aligned}
f(z) &= Q_0(u(z)) + zQ(f(z), w(z), z, u(z)),\\
u(z) &= zu(z)Q_{y_0}(f(z), w(z), z, u(z)) + zQ_{y_1}(f(z), w(z), z, u(z)),\\
w(z) &= Q_{0,u}(u(z)) + zQ_v(f(z), w(z), z, u(z)) + zw(z)Q_{y_0}(f(z), w(z), z, u(z)),
\end{aligned}$$

where

$$w(z) = \frac{f(z) - y(z)}{u(z)}.$$

This is a **positive strongly connected polynomial system**.

Theorem 4: Bousquet-Melou–Jehanne Method

Thus, by **Theorem 5** the solution functions $f(z), u(z), w(z)$ have a **squareroot singularity** at some common singularity z_0:

$$f(z) = g_1(z) - h_1(z)\sqrt{1 - \frac{z}{z_0}},$$

$$u(z) = g_2(z) - h_2(z)\sqrt{1 - \frac{z}{z_0}},$$

$$w(z) = g_3(z) - h_3(z)\sqrt{1 - \frac{z}{z_0}}.$$

$\Longrightarrow$ $y(z) = f(z) - u(z)w(z)$ has also a squareroot singularity at z_0

$$y(z) = g_4(z) - h_4(z)\sqrt{1 - \frac{z}{z_0}} = a_0 + a_1\sqrt{1 - \frac{z}{z_0}} + a_2\left(1 - \frac{z}{z_0}\right) + a_3\left(1 - \frac{z}{z_0}\right)^{3/2} + \cdots$$

but maybe there are **cancellations of coefficients** a_j (**and actually this happens!!!**): we have $\boxed{a_1 = 0}$ and $\boxed{a_3 > 0}$.

Bousquet-Melou–Jehanne Method – General Case

1st difference

$$M(z,u) = Q_0(u) + zQ\left(M(z,u), \frac{M(z,u) - M(z,0)}{u}, z, u\right)$$

Higher differences

$$\boxed{M(z,u) = Q_0(u) + zQ\left(M(z,u), \Delta^{(1)}(z,u), \ldots, \Delta^{(d)}(z,u), z, u\right)}$$

where

$$\Delta^{(j)}(z,u) = \frac{M(z,u) - M(z,0) - M_u(z,0)u - \cdots - M_{u^{j-1}}(z,0)u^{j-1}}{u^j}$$

Theorem (Bousquet-Melou–Jehanne). Such an equation has always an **algebraic solution**.

Kernel Method for the Linear Case ($d = 2$)

$$M(z,u) = Q_0(z,u) + zM(z,u)Q_1(z,u) + z\frac{M(z,u)-M(z,0)}{u}Q_2(z,u) + z\frac{M(z,u)-M(z,0)-M_u(z,0)u}{u^2}Q_3(z,u)$$

rewrites to

$$M(z,u)\boxed{\left(1 - zQ_1(z,u) - \frac{z}{u}Q_2(z,u) - \frac{z}{u^2}Q_3(z,u)\right)} = Q_0(z,u) - M(z,0)\left(\frac{z}{u}Q_2(z,u) + \frac{z}{u^2}Q_3(z,u)\right) - M_u(z,0)\frac{z}{u}Q_3(z,u)$$

Here **two functions** $u = u_1(z)$ and $u = u_2(z)$ satisfy the **kernel equation**

$$1 - zQ_1(z,u(z)) - \frac{z}{u(z)}Q_2(z,u(z)) - \frac{z}{u(z)^2}Q_3(z,u(z)) = 0$$

The right hand side is then zero for $u = u_1(z)$ and $u = u_2(z)$ which is a **linear system** for $M(z,0)$ and $M_u(z,0)$

Kernel Method for the Linear Case ($d = 2$)

The **kernel equation** for $u = u_1(z)$ and $u = u_2(z)$

$$1 - zQ_1(z, u(z)) - \frac{z}{u(z)} Q_2(z, u_{1,2}(z)) - \frac{z}{u(z)^2} Q_3(z, u(z)) = 0$$

rewrites to

$$u(z)^2 = u(z)^2 z Q_1(z, u(z)) + z u(z) Q_2(z, u_{1,2}(z)) + z Q_3(z, u(z))$$

or to

$$\boxed{u_1(z) = \sqrt{z u_1(z)^2 Q_1(z, u_1(z)) + z u_1(z) Q_2(z, u_1(z)) + Q_3(z, u_1(z))}}$$

$$\boxed{u_2(z) = -\sqrt{z u_2(z)^2 Q_1(z, u_2(z)) + z u_2(z) Q_2(z, u_2(z)) + Q_3(z, u_2(z))}}$$

We **lose the property** that $u_1(z)$ and $u_2(z)$ have just **non-negative coefficients** and **it is not clear** that there is a **squareroot singularity**.

Bousquet-Melou–Jehanne Method for the Non-linear Case

Let $P(x_0, x_1, x_2, z, u)$ be an analytic function such that

$$\boxed{P(M(z,u), y_0(z), y_1(z), z, u) = 0.}$$

By taking the derivative with respect to u we get

$$\boxed{P_{x_0}(M(z,u), y_0(z), y_1(z), z, u)} \cdot M_u(z,u) + P_u(M(z,u), y_0(z), y_1(z), z, u) = 0.$$

Key obervation:

$$\boxed{P_{x_0}(M(z,u(z)), y_0(z), y_1(z), z, u(z)) = 0 \Longrightarrow P_u(M(z,u(z)), y_0(z), y_1(z), z, u(z)}$$

We need **two functions** $u_1(z)$ and $u_2(z)$. Setting $f_j(z) = M(z, u_j(z))$ we get the system for $\boxed{f_1(z), f_2(z), y_0(z), y_1(z), u_1(z), u_2(z)}$

$$P(f_1(z), y_0(z), y_1(z), z, u_1(z)) = 0, \quad P(f_2(z), y_0(z), y_1(z), z, u_2(z)) = 0$$
$$P_{x_0}(f_1(z), y_0(z), y_1(z), z, u_1(z)) = 0, \quad P_{x_0}(f_2(z), y_0(z), y_1(z), z, u_2(z)) = 0$$
$$P_u(f_1(z), y_0(z), y_1(z), z, u_1(z)) = 0, \quad P_u(f_2(z), y_0(z), y_1(z), z, u_2(z)) = 0$$

Bousquet-Melou–Jehanne Method for the Non-linear Case

Set (as given in our case)

$$P(x_0,x_1,x_2,z,u) = Q_0(u) + zQ(x_0,(x_0-x_1)/u,(x_0-x_1-ux_2)/u^2,z,u) - x_0.$$

Then the above system rewrites to

$$\begin{aligned}
f_{1,2}(z) &= Q_0(u_{1,2}(z)) + \\
&+zQ\Big(f_{1,2}(z),\frac{f_{1,2}(z)-M(z,0)}{u_{1,2}(z)},\frac{f_{1,2}(z)-M(z,0)-u_{1,2}(z)M_u(z,0)}{u_{1,2}(z)^2},z,u_{1,2}(z)\Big),\\
u_{1,2}(z)^2 &= zu_{1,2}(z)^2Q_{y_0}(\cdots)+zu_{1,2}(z)Q_{y_1}(\cdots)+zQ_{y_2}(\cdots),\\
Q_{0,u}(u_{1,2}(z)) &= \frac{f_{1,2}(z)-M(z,0)}{u_{1,2}(z)}(1-zQ_{y_0}(\cdots)\\
&\quad + z\frac{f_{1,2}(z)-M(z,0)-u_{1,2}(z)M_u(z,0)}{u_{1,2}(z)^3}Q_{y_2}(\cdots)
\end{aligned}$$

This **cannot** be rewritten into a **positive strongly connected polynomial system**.

Second Differences: The Linear Case

Theorem 3'. [D.+Hainzl] **Squareroot singularity**:

$Q_0(z,u), Q_1(z,u), Q_2(z,u), Q_3(z,u)$ ... polynomials with non-negative coefficients (+ some technical conditions).

$$M(z,u) = Q_0(z,u) + zM(z,u)Q_1(z,u) + z\frac{M(z,u) - M(z,0)}{u}Q_2(z,u) + z\frac{M(z,u) - M(z,0) - M_u(z,u)u}{u^2}Q_3(z,u)$$

$$\Longrightarrow \quad \boxed{M_n = [z^n]M(z,0) \sim c \cdot n^{-3/2} z_0^{-n}.}, \qquad n \equiv j_0 \bmod m,$$

(for some constants $c, z_0 > 0$) and $\boxed{M_n = 0}$ for $n \not\equiv j_0 \bmod m$, where $m \geq 1$.

Second Differences: The Non-linear Case

Theorem 4'. [D.+Hainzl] 3/2-**Singularity**:

$Q(y_0, y_1, y_2, z, u)$ ... polynomial with non-negative coefficients that is **non-linear** in y_0, y_1, y_2 (+ some technical conditions).

$$M(z,u) = Q_0(u) + zQ\left(M(z,u), \frac{M(z,u)-M(z,0)}{u}, \frac{M(z,u)-M(z,0)-M_u(z,0)u}{u^2}, z, u\right)$$

$$\Longrightarrow \quad \boxed{M_n = [z^n]\, M(z,0) \sim c \cdot n^{-5/2} z_0^{-n}.}, \qquad n \equiv j_0 \bmod m,$$

(for some constants $c, z_0 > 0$) and $\boxed{M_n = 0}$ for $n \not\equiv j_0 \bmod m$, where $m \geq 1$.

Applications

One-dimensional non-negative lattice path with steps ± 1 and ± 2

$$E_0(z) = 1 + z(E_1(z) + E_2(z)),$$
$$E_1(z) = z(E_0(z) + E_1(z) + E_2(z)),$$
$$E_k(z) = z(E_{k-2}(z) + E_{k-1}(z) + E_{k+1}(z) + E_{k+2}(z)) \qquad (k \geq 2),$$

which gives for $E(z,u) = \sum_{k\geq 0} E_k(z) u^k$

$$E(z,u) = 1 + z(u+u^2)E(z,u) + z\frac{E(z,u)-E(z,0)}{u} + z\frac{E(z,u)-E(z,0)-uE_v(u,0)}{u^2}.$$

Applications

3-**Constellations in Eulerian Maps**

$$M(z,u) = 1 + zuM(z,u)^3 + zu(2M(z,u) + M(z,1))\frac{M(z,u) - M(z,1)}{u-1} + zu\frac{M(z,u) - M(z,1) - M_u(z,1)(u-1)}{(u-1)^2}$$

Remark. There are many equations of this type in the context of map enumeration (even more generally with higher differences)

Higher Differences

Conjecture

Consider a **catalytic equation with higher differences**:

$$\boxed{M(z,u) = Q_0(u) + zQ\left(M(z,u), \Delta^{(1)}(z,u), \ldots, \Delta^{(d)}(z,u), z, u\right)}$$

where Q_0 and Q have **non-negative coefficients** (+ some technical conditions)

- If Q is **linear** in $y_0, y_1, \ldots, y_d$ then $M(z,0)$ has a **squareroot singularity**
- If Q is **non-linear** in $y_0, y_1, \ldots, y_d$ then $M(z,0)$ has a 3/2-**singularity**

Theorem 3': Proof Ideas for the Linear Case

Set

$$R(z,u) = zu^2 Q_1(z,u) + zuQ_2(z,u) + Q_3(z,u)$$

Then the **kernel equation** for $u = u_{1,2}(z)$ reads as

$$\boxed{u^2 = R(z,u)}$$

Ansatz

$$\boxed{u_1(z) = g(z) + \sqrt{h(z)} \qquad u_2(z) = g(z) - \sqrt{h(z)}}$$

Proof Ideas for the Linear Case

$$u^2 = (g \pm \sqrt{h})^2 = \boxed{g^2 + h \pm \sqrt{h}\, 2g}$$

$$\begin{aligned} R(z, g \pm \sqrt{h}) &= \sum_k R_k(z)(g \pm \sqrt{h})^k \\ &= \sum_k R_k(z) \sum_{j=0}^{k} \binom{k}{j} g^{k-j} (\pm 1)^j h^{j/2} \\ &= \sum_{k,\ell} R_k(z) \binom{k}{2\ell} g^{k-2\ell} h^{\ell} \pm \sqrt{h} \sum_{k,\ell} R_k(z) \binom{k}{2\ell+1} g^{k-2\ell-1} h^{\ell} \\ &= \boxed{R^+(z, g, h) \pm \sqrt{h} \cdot R^-(z, g, h)} \end{aligned}$$

$$u^2 = R(z, u) \quad \Longrightarrow \quad g^2 + h = R^+(z, g, h), \quad 2g = R^-(z, g, h)$$

Proof Ideas for the Linear Case

The **kernel equation**

$$\boxed{u^2 = R(z,u)}$$

rewrites to

$$\boxed{g^2 + h = R^+(z,g,h), \qquad 2g = R^-(z,g,h)}$$

or to

$$\boxed{h = R^+(z,g,h) - g^2, \qquad g = \frac{1}{2}R^-(z,g,h)}$$

This is **not a positive system**!

Proof Ideas for the Linear Case

Lemma

The functions $\boxed{g(z)}$, $\boxed{h(z)}$ have the following properties:

- they have **non-negative coefficients**
- they have a common **squareroot singularity** z_0
- the function $u_2(z) = g(z) - \sqrt{h(z)}$ is regular at z_0

Corollary. The functions $\boxed{M(z,0)}$, $\boxed{M_u(z,0)}$ have a **squareroot singularity** at z_0, too.

Additional Parameters

Number of vertices in planar maps

$M(z,x,u)$ ... generating function of rooted planar maps, where the variable z corresponds to the number of edges, x to the number of vertices and u to the root face valency.

$$M(z,x,v) = x + zu^2 M(z,x,u)^2 + zu\frac{M(z,x,1) - uM(z,x,u)}{1-u}$$

X_n ... number of vertices in a random planar map with n edges

Central Limit Theorem

X_n satisfies a central limit theorem with $\mathbb{E}[X_n] = \frac{1}{2}n + O(1)$ and $\mathbb{V}\mathrm{ar}[X_n] = \frac{5}{32}n + O(1)$.

Additional Parameters

Theorem 7

Suppose that $M(z,x,u)$ and $M_1(z,x)$ are the solutions of the catalytic equation

$$\boxed{P(M(z,x,u), M_1(z,x), z, x, u) = 0},$$

where the function $P(x_0, x_1, z, x, u)$ is analytic and $M_1(z,1)$ has a singularity at $z = z_0$ of the form

$$M_1(z,1) = y_0 + y_2\left(1 - \frac{z}{z_0}\right) + y_3\left(1 - \frac{z}{z_0}\right)^{3/2} + \cdots,$$

with $y_3 \neq 0$ (+ some technical conditions)

Then $M_1(z,x)$ has a local singular representation of the form

$$M_1(z,x) = a_0(x) + a_2(x)\left(1 - \frac{z}{\rho(x)}\right) + a_3(x)\left(1 - \frac{z}{\rho(x)}\right)^{3/2} + \cdots$$

Corollary. Hwang's Quasi-Power-Theorem leads then to a **Central Limit Theorem**

Additional Parameters

Vertices of degree k in planar maps

$M(z,x,u)$ ... generating function for rooted planar maps, where z corresponds to the number of edges, x to the number of non-root faces of degree k, and u to the root-face degree

$$\begin{aligned} M(z,x,u)\left(1-z(x-1)u^{-k+2}\right) \\ &= 1+zu^2M(z,x,u)+zu\frac{uM(z,x,u)-M(z,x,1)}{u-1} \\ &\quad - z(x-1)u^{-k+2}G(z,x,M(z,x,1),u), \end{aligned}$$

where $G(z,x,y,u)$ is a polynomial of degree $k-2$ in u with coefficients that are analytic functions in (z,x,y) for $|z|\le 1/10$, $|x-1|\le 2^{1-k}$, and $|y|\le 2$.

Additional Parameters

Pure k-gons in planar maps

We say that a face is a pure k-gon ($k \geq 2$) if it is incident exactly to k different edges and k different vertices.

$P(z,x,u)$ ... generating function for rooted planar maps, where z corresponds to the number of edges, x to the number of non-root faces that are pure k-gons, and u to the root-face degree.

$$\begin{aligned} P(z,x,u) =& 1 + zu^2 P(z,x,u) + zu \frac{uP(z,x,u) - P(z,x,1)}{u-1} \\ & - z(x-1)u^{-k+2} \tilde{G}(z,x,P(z,x,1),u), \end{aligned}$$

where $\tilde{G}(z,x,y,u)$ is a polynomial of degree $k-2$ in u with coefficients that are analytic functions in (z,x,y) for $|z| \leq 1/10$, $|x-1| \leq 2^{1-k}$, and $|y| \leq 2$.

Additional Parameters

Vertices of degree k in simple planar maps

$S(z,x,u)$ ... generating function for simple rooted planar maps, where z corresponds to the number of edges, x to the number of non-root vertices of degree k, and u to the root-face degree.

$$\begin{aligned}
S(z,x,u) = &1 + zu^2 S(z,x,u) + zu\frac{uS(z,x,u)-S(z,x,1)}{u-1} \\
&- zuS(z,x,u)S(z,x,1) - (S(z,x,u)-1)(S(z,x,1)-1) \\
&+ (x-1)\Bigg(zu^{-k+2}S(z,x,u)G_1(z,x,S(z,x,1),u) \\
&\qquad - zuS(z,x,u)G_2(z,x,S(z,x,1)) \\
&\qquad - (S(z,x,u)-1)G_3(z,x,S(z,x,1))\Bigg),
\end{aligned}$$

where $G_1(z,x,y,u)$ is a polynomial of degree $k-2$ in u with coefficients that are analytic functions in (z,x,y) for $|z| \le 2/25$, $|x-1| \le 2^{-k-5}$, and $|y-1| \le 2/5$. Similarly properties hold for the functions $G_2(z,x,y)$ and $G_3(z,x,y)$.

Thank You!