Coefficientwise Hankel-total positivity of the Laguerre polynomials

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May 28, 2024
Séminaire CALIN, LIPN

Based on joint work with

Alexander Dyachenko, Matthias Pétréolle, Alan Sokal

Statement

Define

$$\mathcal{L}_n^{(-1+\lambda)}(x) = \sum_{k=0}^{n} \binom{n}{k} (k+\lambda)(k+1+\lambda)\cdots(n-1+\lambda)x^k$$

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We provide a multivariate generalisation

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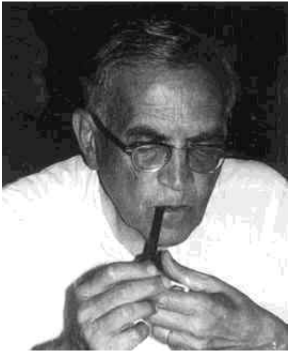
Will consider a matrix of polynomials soon!

Historical Note

First defined independently by two different groups in the 30s

(a) M.G. Krein (1907-1989)



(b) I.J. Schoenberg (1903-1990)

Source: MacTutor History of Mathematics Archive

We use Schoenberg's terminology.

Hankel Matrix

Given a sequence $a_0, a_1, \ldots$ the infinite matrix $H_\infty(\mathbf{a})$ whose ij^{th} entry is a_{i+j} is called the Hankel matrix of $(a_n)_{n\geq 0}$.

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We say that a sequence $(a_n)_{n\geq 0}$ is Hankel-totally positive (Hankel-TP in short) if its Hankel matrix is TP.
It implies that the sequence is log-convex but much stronger.

Fundamental Fact about Hankel-TP

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3. *There exists numbers $\alpha_0, \alpha_1, \ldots \geq 0$ such that*
$$\sum_{n=0}^{\infty} a_n t^n = \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \ddots}}}.$$

Many important combinatorial sequences are Stieltjes moment sequences

- Catalan numbers.
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- $n!$.
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- $(2n-1)!! = 1 \times 3 \times \cdots \times (2n-1)$.
  Have α_n 1,2,3,4,5,6,. . ..

Slide from talk of Elvey Price, Permutation Patterns 2023

GUESSING STIELTJES-NESS WITH OEIS

We ran the Euler-Viskovatov algorithm on all 304698 OEIS sequences with at least 15 terms (only considering terms a_n with $n \leq 150$ and $a_n \leq 10^{150}$).

For 6719 sequences the terms are consistent with being Stieltjes

$6719 - \epsilon$ **open questions:** Which of these sequences are really Stieltjes?

Refined results:

- In 1667 such cases, one of the terms $\alpha_j = 0$, so the generating function $A(t)$ is rational
- In 798 cases (including 328 rational cases), the coefficients α_j are all integers.
- For 7344 sequences the first 15 terms are consistent with being Stieltjes (625 of these not Stieltjes because of later terms)

Refined counting

We often count using polynomials rather than integers.
Example:

- Bell polynomials $B_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k$ count number of set partitions of the set $\{1, 2, \ldots, n\}$ by keeping track of the number of blocks.

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- Eulerian polynomials $\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k$. Count permutations of n letters with k descents.

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These polynomials can also be multivariate counting several statistics simultaneously.

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Coefficientwise TP of Hankel matrix of a sequence $(p_n(x))_{n\geq 0}$ implies its coefficientwise log-convex

Recall fact about Hankel-TP

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Coefficientwise Hankel TP from continued fractions

Theorem (Sokal(2014), Pétréolle–Sokal–Zhu (2023))

Let $\boldsymbol{\alpha} = \alpha_1, \alpha_2, \ldots$ *be a sequence of indeterminates and let* $S_n(\boldsymbol{\alpha})$ *be a polynomial defined by*

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Euler (1760) found the following continued fraction:

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Rising factorials

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Thus the sequence of rising factorials is coefficientwise Hankel TP.

Combinatorial theory of orthogonal polynomials

For a measure μ and sequence of monic polynomials $(p_n(x))_{n\geq0}$ with $\deg p_n(x) = x$, we say that $(p_n(x))_{n\geq0}$ is orthogonal with respect to μ if $\int p_n(x)p_m(x)d\mu(x) = 0$ for $m \neq n$.

Askey-scheme

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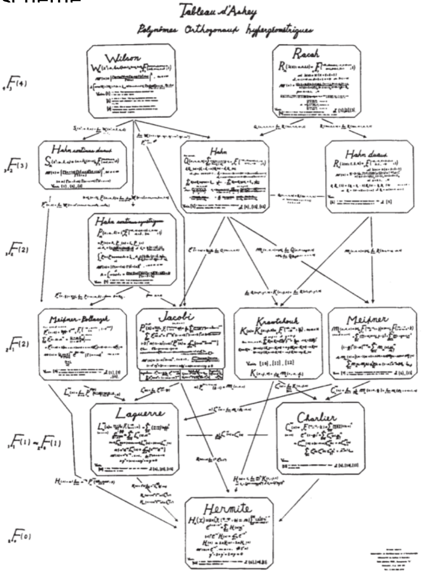

Askey scheme as proposed by Jacques Labelle at the first OPSFA meeting in Bar-Le-Duc (France) in 1984

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We will restrict to Laguerre polynomials

Laguerre polynomials

Laguerre polynomials are a sequence of orthogonal polynomials

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Orthogonal wrt measure $\mu(x) = x^\alpha e^{-x}$.

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Combinatorialists' Laguerre polynomials

$$\mathcal{L}_n^{(\alpha)}(x) = n! L_n^{(\alpha)}(-x) = \sum_{k=0}^{n} \binom{n}{k} (n+\alpha)(n-1+\alpha)\cdots(k+1+\alpha)x^k$$

Integral representation of Laguerre polynomials

For $\alpha \geq -1$ and $x \geq 0$, the Laguerre polynomials are a Stieltjes moment sequence

$$\mathcal{L}_n^{(\alpha)}(x) \;=\; e^{-x} x^{-\alpha/2} \int_0^\infty u^{n+\alpha/2}\, e^{-u}\, I_\alpha(2\sqrt{xu})\, du$$

where $I_\alpha(z)$ is the modified Bessel function

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Thus, these polynomials are themselves Stieltjes moment sequences.

Conjecture

Based on this integral representation, Corteel and Sokal (2017) conjectured

Conjecture

The sequence $\left(\mathcal{L}_n^{(-1+\lambda)}(x)\right)_{n\geq 0}$ *is coefficientwise Hankel-TP in* λ *and* x*.*

Statement of univariate result

Let

$$\mathrm{L} = \left(\binom{n}{k}(k+\lambda)(k+1+\lambda)\cdots(n-1+\lambda)\right)_{n,k\geq 0}$$

be the matrix of coefficients of the Laguerre polynomials.

Theorem (Zhu(2021,22), D.–Dyachenko–Pétréolle–Sokal('23))

(a) *The matrix* L *is totally positive.*

(b) *The sequence* $\left(\mathcal{L}_n^{(-1+\lambda)}(x)\right)_{n\geq 0}$ *is coefficientwise Hankel-TP.*

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We also provide a multivariate generalisation.

First need a combinatorial interpretation.

Laguerre digraph

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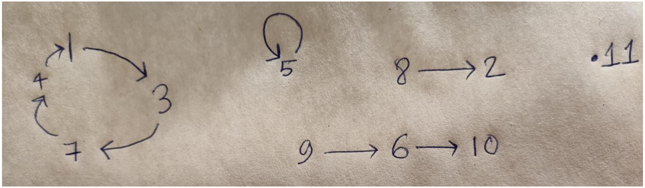
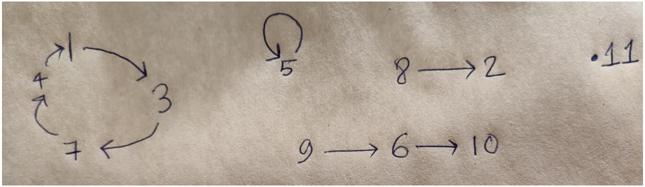
Example:



Connected components



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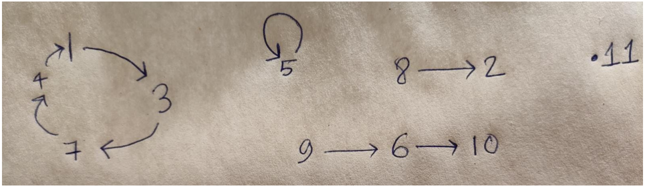

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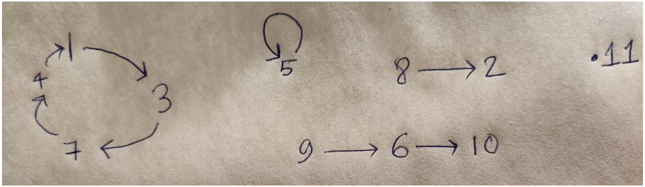

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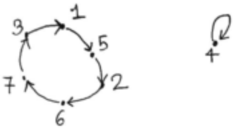

$\sigma = (1, 5, 2, 6, 7, 3)(4)$

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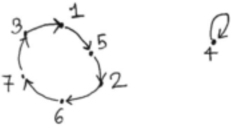
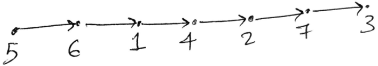


$\sigma = (1, 5, 2, 6, 7, 3)(4)$

2. One path, no cycles - linear structure of permutation



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$\mathrm{LD}_{n,k}$ - Set of Laguerre digraphs on n vertices with k paths

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In particular, $\mathrm{LD}_{n,k}$ *is enumerated by*

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Laguerre digraphs after Sokal (2022)

We have shown

$$\mathcal{L}_n^{(-1+\lambda)}(x) = \sum_{G \in \mathrm{LD}_n} \lambda^{\mathrm{cyc}(G)} x^{\mathrm{pa}(G)}$$

Classification of vertices

Let $G \in \mathrm{LD}_{n,k}$ and let i be a vertex of G. We define

- $p(i)$: the predecessor of i if it exists else $p(i) = 0$.
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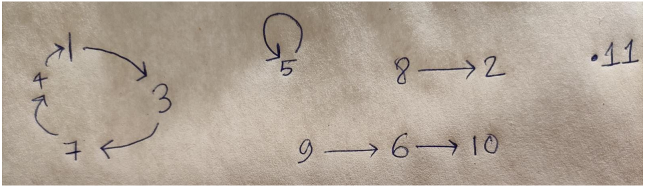
We classify the vertices $i \in [n]$ into five types:

- peak (p) if $p(i) < i > s(i)$;
- valley (v) if $p(i) > i < s(i)$;
- double ascent (da) if $p(i) < i < s(i)$;
- double descent (dd) if $p(i) > i > s(i)$;
- fixed point (fp) if $p(i) = i = s(i)$.

Illustration with example



Here

- Peaks $\{7, 10, 9, 8, 11\}$
- Valleys $\{1, 6\}$
- Double ascents $\{3\}$
- Double descents $\{2, 4\}$
- Fixed points (or loops) $\{5\}$

Multivariate Laguerre polynomials

Let $\mathrm{wt}(G) = y_{\mathrm{p}}^{\mathrm{p}(G)} y_{\mathrm{v}}^{\mathrm{v}(G)} y_{\mathrm{da}}^{\mathrm{da}(G)} y_{\mathrm{dd}}^{\mathrm{dd}(G)} y_{\mathrm{fp}}^{\mathrm{fp}(G)} \lambda^{\mathrm{cyc}(G)}$

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Define

$$\mathcal{L}_n^{(-1+\lambda)}(x; y_{\mathrm{p}}, y_{\mathrm{v}}, y_{\mathrm{da}}, y_{\mathrm{dd}}, y_{\mathrm{fp}}) = \sum_{G \in \mathrm{LD}_n} \mathrm{wt}(G)\, x^{\mathrm{pa}(G)}$$

Statement of multivariate result

Let

$$\mathrm{L} = \left(\frac{1}{{y_{\mathrm{p}}}^k} \sum_{G \in \mathrm{LD}_{n,k}} \mathrm{wt}(G) \right)_{n,k \geq 0}$$

Theorem (D.–Dyachenko–Pétréolle–Sokal('23))

Assume $\lambda y_{\mathrm{fp}} - \lambda y_{\mathrm{p}}, (y_{\mathrm{da}} + y_{\mathrm{dd}}) - (y_{\mathrm{p}} + y_{\mathrm{v}})$ *are non-negative. Then*

(a) *The matrix* L *is totally positive.*

(b) *The sequence* $\left(\mathcal{L}_n^{(-1+\lambda)}(x; y_{\mathrm{p}}, y_{\mathrm{v}}, y_{\mathrm{da}}, y_{\mathrm{dd}}, y_{\mathrm{fp}}) \right)_{n \geq 0}$ *is coefficientwise Hankel-TP.*

Proof uses the production-matrix method and Riordan arrays

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Hamburger moment sequences a la Flajolet (1980).

Guessing production matrices

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If A is lower-triangular with invertible diagonal entries, production matrix P can be computed

$$P = A^{-1}\Delta A$$

where $\Delta = (\delta_{i+1,j})_{i,j\geq 0}$.

Proof of result

The proof consists of two steps:

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The hardest part is usually to guess the production matrix.

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Turns out P is tridiagonal in our situation and $B_x^{-1} P B_x$ is quadridiagonal.

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   - Bijective proof. Gives finer control and a lot more statistics on Laguerre digraphs. Hope to extend to infinitely many statistics on Laguerre digraphs.
2. Prove that P and $B_x^{-1}PB_x$ are totally positive. Simple in the univariate case but difficult in the multivariate case.

The production matrices

The production matrix for the coefficient matrix L is

$$\begin{aligned}
p^{\mathsf{ob}}_{n,n+1} &= 1 \\
p^{\mathsf{ob}}_{n,n} &= (1+\alpha)y_{\mathrm{fp}} + n(y_{\mathrm{da}} + y_{\mathrm{dd}}) \\
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The production matrix for $B_x^{-1}\mathrm{L}B_x$ is

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\begin{aligned}
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p^{\flat}_{n,n-1} &= n(n+\alpha)y_{\mathrm{p}}y_{\mathrm{v}} + n(y_{\mathrm{da}} + y_{\mathrm{dd}})x \\
p^{\flat}_{n,n-2} &= n(n-1)y_{\mathrm{p}}y_{\mathrm{v}}x \\
p^{\flat}_{n,k} &= 0 \qquad \textit{if } k < n-2 \textit{ or } k > n+1
\end{aligned}
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Proof of production matrix: tridiagonal case

The production matrix P of L of factorises as $P = P_1 P_2$ where P_1 is a lower bidiagonal matrix and P_2 is an upper bidiagonal matrix.

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Let P be the production matrix for the matrix $B_x^{-1} \mathrm{L} B_x$.

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- Non-trivial result for the multivariate case.

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Very useful result for proving total positivity of tridiagonal matrices.

Theorem

Let L_1, L_2 be lower bidiagonal matrices, U be an upper bidiagonal matrix and D_1, D_2 be two diagonal matrices, all with nonnegative entries.

Total positivity of quadridiagonal matrices

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Let L_1, L_2 be lower bidiagonal matrices, U be an upper bidiagonal matrix and D_1, D_2 be two diagonal matrices, all with nonnegative entries. Then the matrix

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Proof via a difficult induction.
A non-trivial tridiagonal case is used to prove Hankel-total positivity of Schett polynomials

Final remarks

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- Tridiagonal production matrices have been considered for a long time as Jacobi-type continued fraction. The Laguerre polynomials are the first instance of a family of polynomials obtained using quadridiagonal production matrices. Another family are the Schett polynomials (D.–Sokal '23).