Barrier resilience problems and crossing numbers of graphs

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Outline

Two independent parts

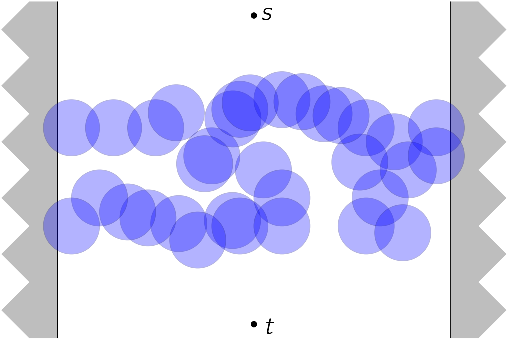
- Barrier resilience
- Crossing numbers of graphs

In common:

- Geometry
- Simple to state, interesting for general audience
- I have worked on them, I have something to explain

Original barrier resilience problem

[Kumar, Lai, Arora 2005], [Bereg, Kirkpatrick 2009]



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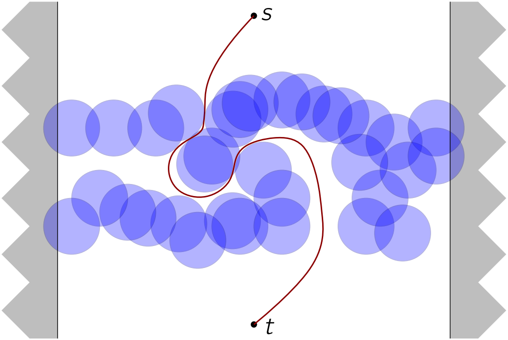

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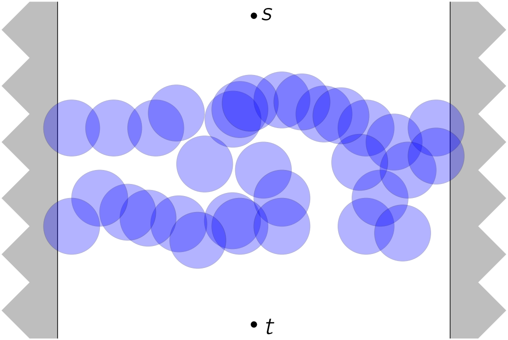

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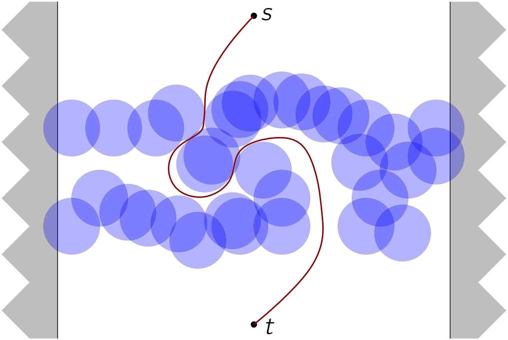

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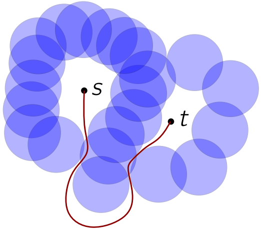

annulus

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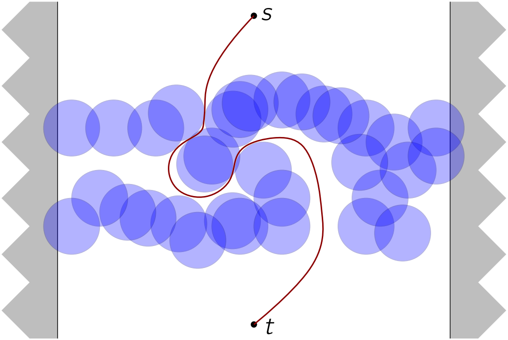

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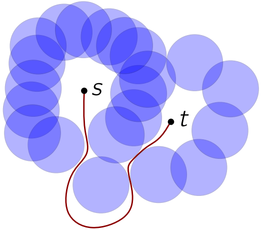

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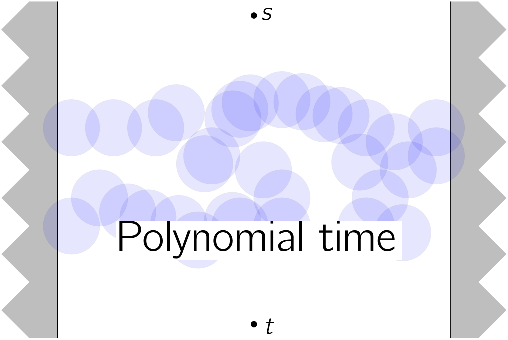

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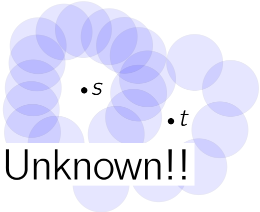
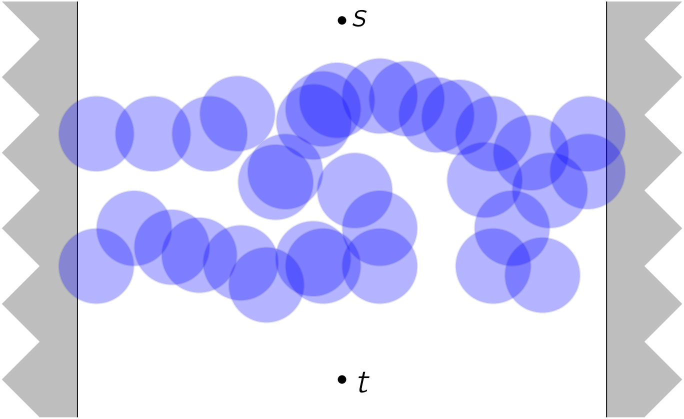


annulus

Barrier resilience - Rectangular domain in P

[Kumar, Lai, Arora 2005] via Menger's theorem



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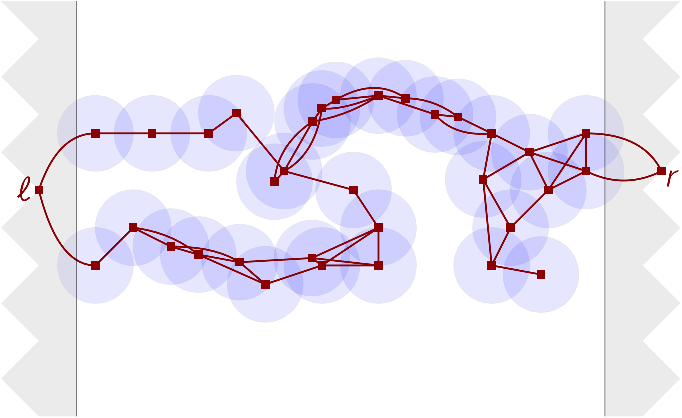

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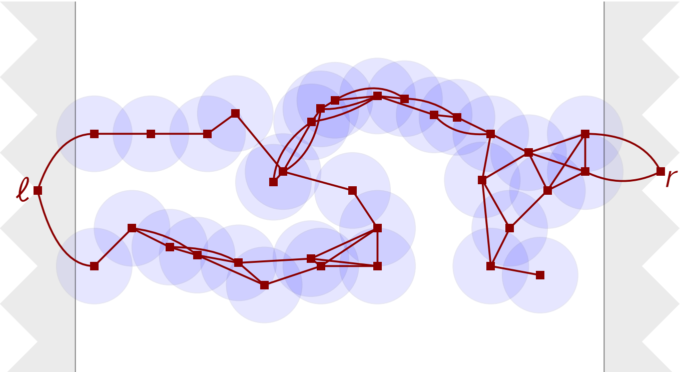

$$\begin{aligned}\text{OPT} &\geq \text{max nb } \ell\text{-}r \text{ vertex-disjoint paths}\\ &= \text{min size } \ell\text{-}r \text{ vertex cut}\\ &\geq \text{OPT}\end{aligned}$$

Barrier resilience - Rectangular domain in P

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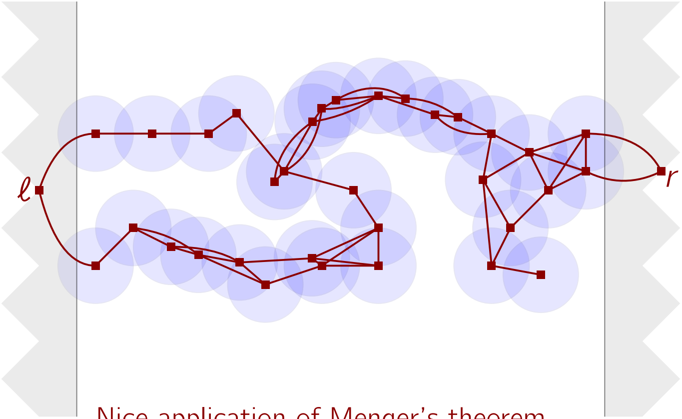

Nice application of Menger's theorem
No Menger-like theorem for the annular case

Related problems

If you cannot solve a problem, perturb it:

Related problems

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- other shapes instead of disks
  - the rectangular case can be solved in polynomial time for any reasonable shape; same argument
  - the annular case is NP-hard for (unit) segments or crossing rectangles

[Alt, C., Giannopoulos, Knauer 2017]
[Korman, Löffler, Silveira, Strash 2018]
[Tseng and Kirkpatrick 2011]

- the annular case is FPT wrt OPT for any connected shape

[Eiben and Lokshtanov 2020]

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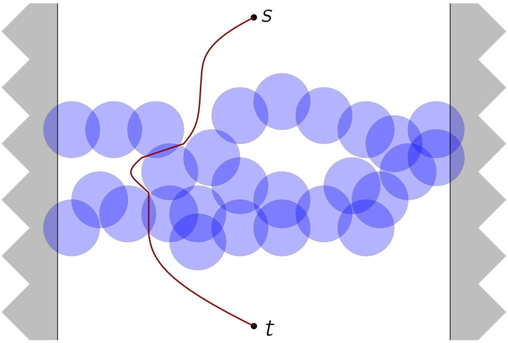
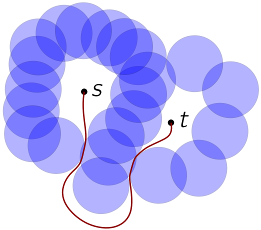
- the annular case is FPT wrt OPT for any connected shape
  [Eiben and Lokshtanov 2020]
- change the criteria
  - shrink the disks, instead of deleting them

Minimum shrinking problem

min $\sum$ shrinking [C., Jain, Lubiw, Mondal 2018]



rectangle



annulus

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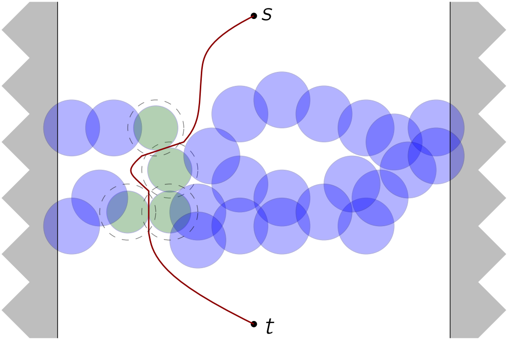

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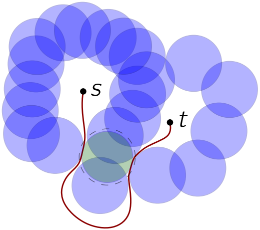

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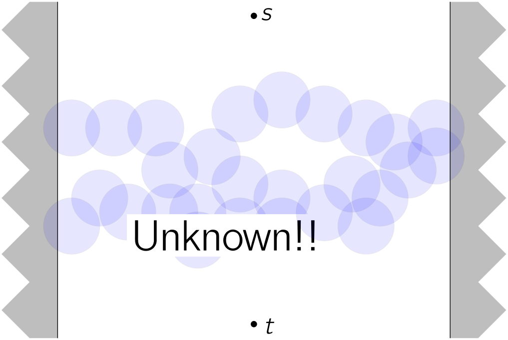

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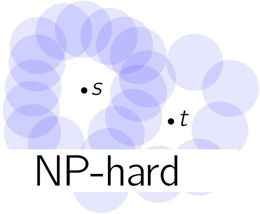

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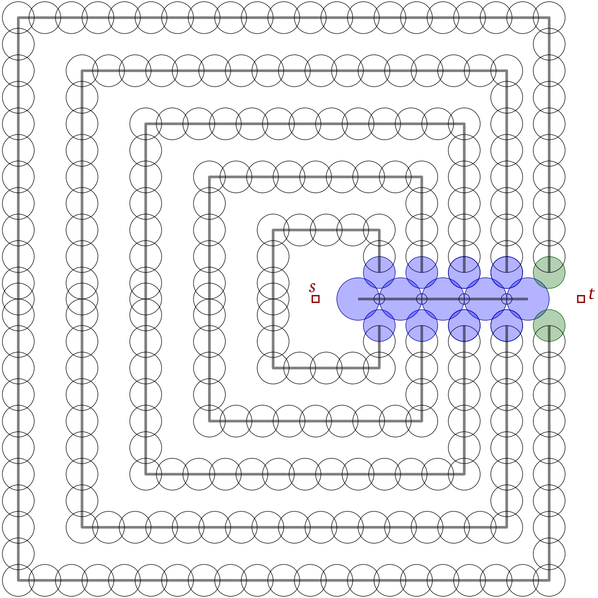
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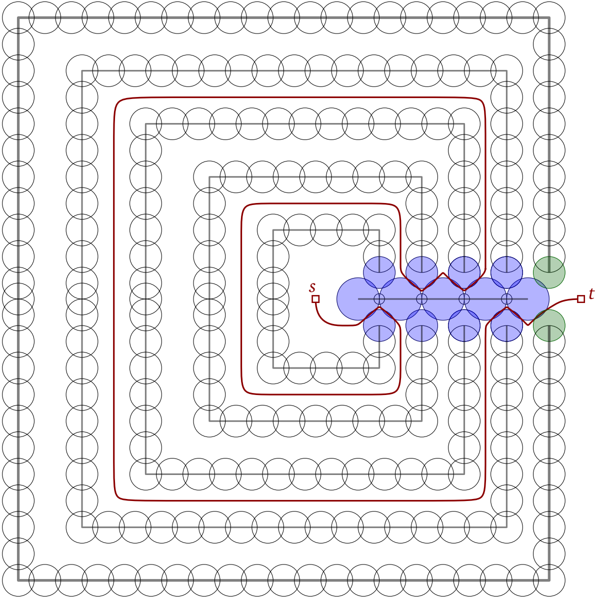

State of the art

	rectangular domain	annular domain
barrier problem total failure	polynomial, $\tilde{O}(n^{3/2})$ Menger's theorem max flow	unknown complexity FPT PTAS in some cases
shrinking barrier min $\sum$ shrinking	unknown complexity $(1+\varepsilon)$-approx in $O(n^5/\varepsilon^{2.5})$ time	(weakly!!) NP-hard different radii

unit disks vs. disks vs. pseudodisks

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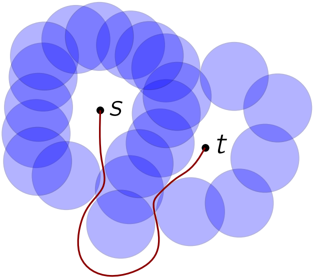

barrier

A short interlude

How to start a paper; two examples

A short interlude

How to start a paper; two examples

The Magical Number Seven, Plus or Minus Two...
George A. Miller (1956), Harvard University
Psychological Review, 63, 81-97

My problem is that I have been persecuted by an integer. For seven years this number has followed me around, has intruded in my most private data, and has assaulted me from the pages of our most public journals. This number assumes a variety of disguises, being sometimes a little larger and sometimes a little smaller than usual, but never changing so much as to be unrecognizable...

A short interlude

How to start a paper; two examples

On Hodge-Riemann Cohomology Classes
Julius Ross and Matei Toma
arXiv 2106.11285

Since the dawn of time, human beings have asked some fundamental questions: who are we? why are we here? is there life after death? Unable to answer any of these, in this paper we will consider cohomology classes on a compact projective manifold that have a property analogous to the Hard-Lefschetz Theorem and Hodge-Riemann bilinear relations.

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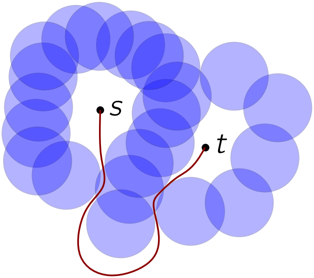


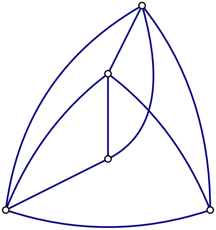
barrier

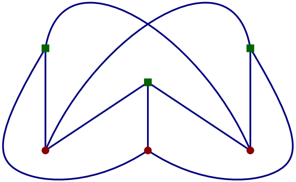
Drawing of a graph

Drawing D of a graph G (in the plane)

- each vertex one point - injectively
- each edge one continuous, simple curve
- endpoints of edge uv are points for u and v
- the interior of an edge does not contain other vertices
- no common point in the interior of three edges

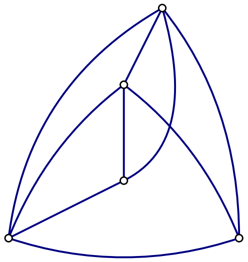
cr(D): number of crossings in drawing D
cr(G): mininum cr(D) over all drawings D of graph G

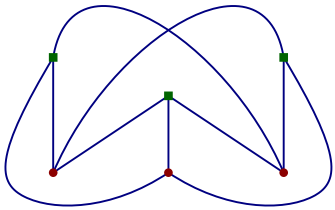




Crossing number 0

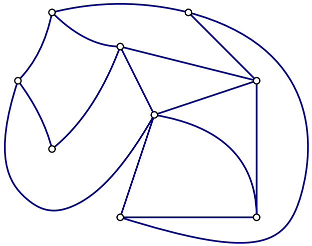
- $\text{cr}(G) = 0$ if and only if G planar
- Good understanding of planar graphs
- G planar $\Longleftrightarrow$ G does not contain a subdivision of K_5 or $K_{3,3}$
- Efficient algorithms to recognize planar graphs

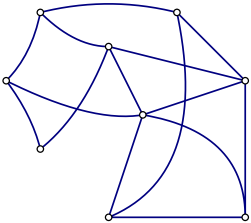




Why crossing number

Purchase, Cohen, James 1995: "Increasing the number of arc crossings in a graph decreases the understability of the graph".
Purchase, 1997: "reducing the number of edge crosses is by far the most important aesthetic".





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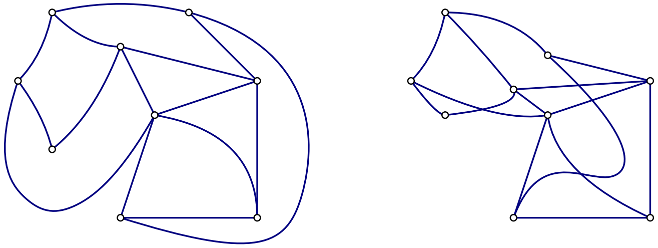
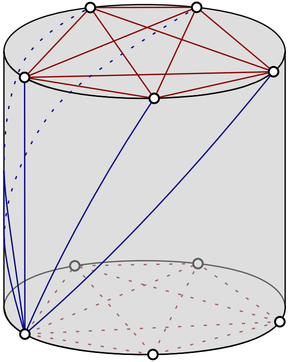


Huang, Eades, Hong, 2014: "The effect of crossing angles on human graph comprehension was validated."
Optimization, VLSI, Discrete Geometry

Main conjectures – Harary-Hill

The crossing number of complete graphs is

$$\mathrm{cr}(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor \approx \frac{n^4}{64} \approx \frac{\binom{|E(K_n)|}{2}}{8}$$

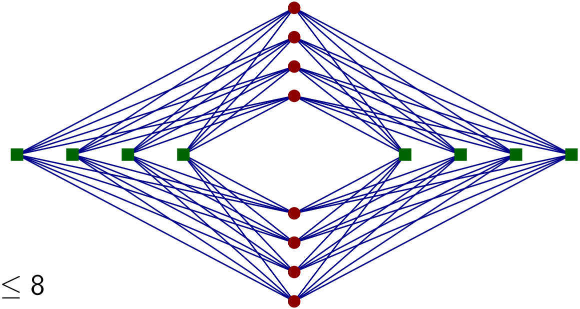


True for $n \leq 12$

Main conjectures – Zarankiewicz-Turán

The crossing number of complete bipartite graphs (here for balanced)

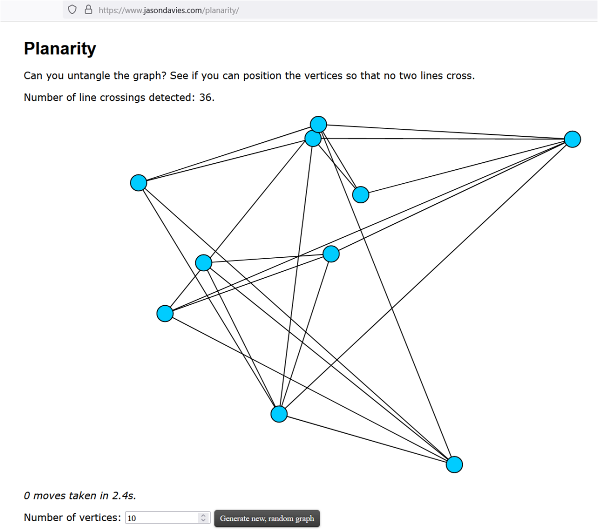
$$\mathrm{cr}(K_{n,n}) = \left\lfloor \frac{n}{2} \right\rfloor^2 \left\lfloor \frac{n-1}{2} \right\rfloor^2 \approx \frac{n^4}{16} \approx \frac{\binom{|E(K_{n,n})|}{2}}{8}$$



True for $n \leq 8$

Planarity game

https://www.jasondavies.com/planarity/



Rectilinear drawings

Each edge drawn as a straight line segment

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$\overline{\mathrm{cr}}(G) \dots$ rectilinear crossing number of G

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Each edge drawn as a straight line segment

$\overline{\mathrm{cr}}(G) \ldots$ rectilinear crossing number of G

- G planar $\iff \mathrm{cr}(G) = 0 \iff \overline{\mathrm{cr}}(G) = 0$

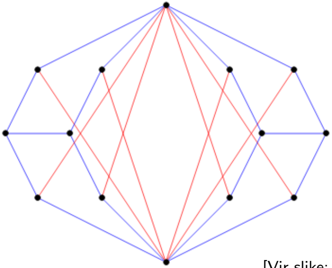
  [Wagner 1936, Fáry 1948]

Rectilinear drawings

Each edge drawn as a straight line segment

$\overline{\text{cr}}(G)\ldots$ rectilinear crossing number of G

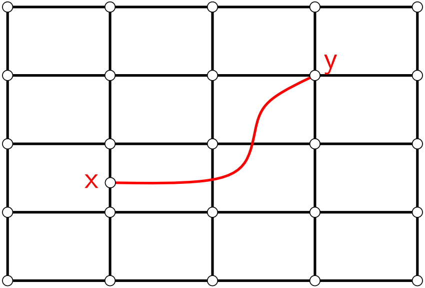
- G planar $\iff \text{cr}(G) = 0 \iff \overline{\text{cr}}(G) = 0$
  [Wagner 1936, Fáry 1948]
- $\text{cr}(G) = 1 \iff \overline{\text{cr}}(G) = 1$ [Bienstock, Dean 1993]
  $\text{cr}(G) = 2 \iff \overline{\text{cr}}(G) = 2$
  Claims without proof that $\text{cr}(G) = 3 \iff \overline{\text{cr}}(G) = 3$
  For each $k \geq 4$ there is G with $\text{cr}(G) = 4$ and $\overline{\text{cr}}(G) = k$



[Vir slike: Stackexchange]

Near-planar graphs

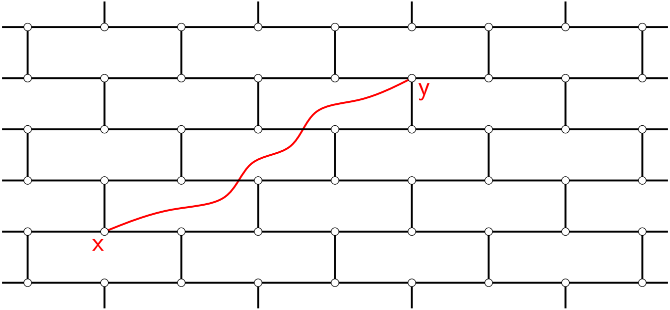
Non-planar H is near-planar if $H = G + xy$ for planar G



- weak relaxation of planarity
- near-planar $\subsetneq$ toroidal, apex

Near-planar – Riskin

- G planar, 3-connected, and 3-regular [Riskin 1996]
  - $cr(G+xy)$ attained by the following drawing: draw G planarly (unique) and insert xy minimizing crossings



Near-planar graphs are hard

[C., Mohar 2013]

Theorem

Computing $cr(G)$ for near-planar graphs is NP-hard.

Near-planar graphs are hard

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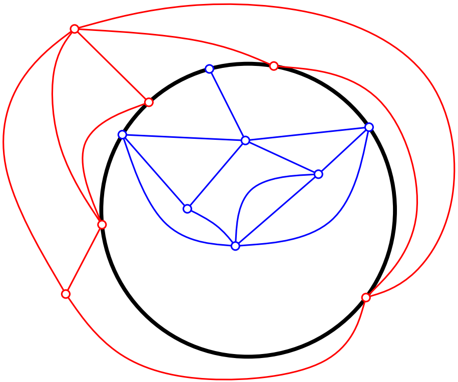
Theorem

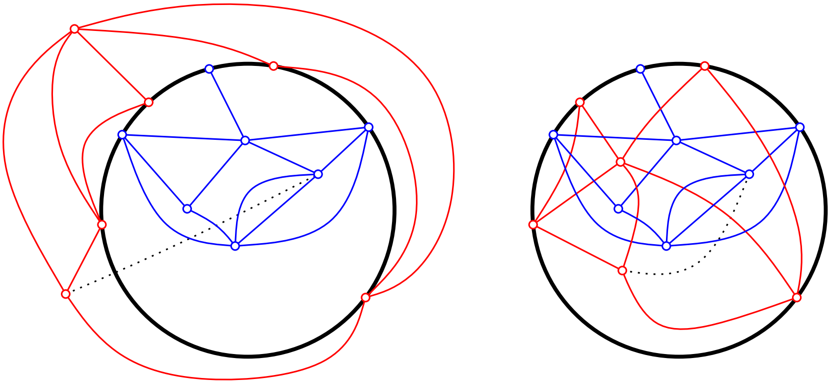
Computing $cr(G)$ for near-planar graphs is NP-hard.

- adding one edge makes a big mess
- crossing number of toroidal graphs hard
- new reduction from SAT
  - previous reductions were from Linear Ordering
- new problem: red-blue anchored drawings

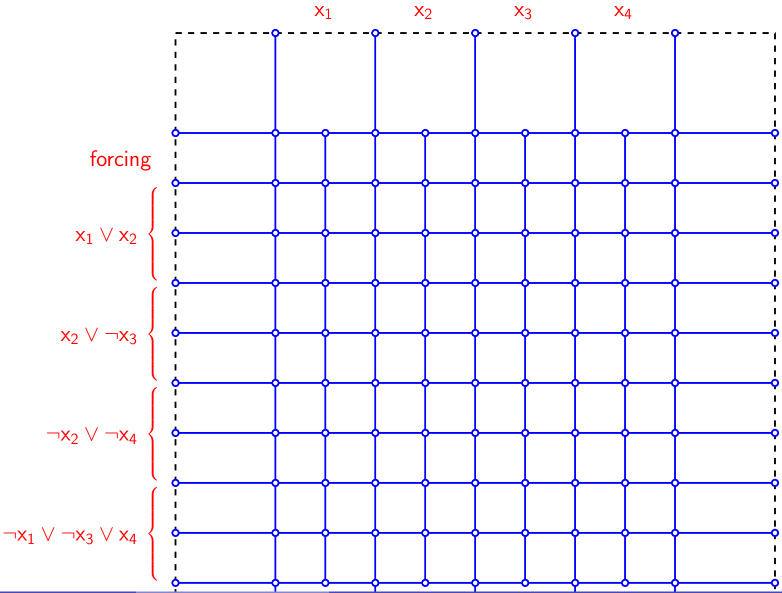
Red-blue anchored drawings?



Red-blue anchored drawings?



Crossing number of near-planar graphs



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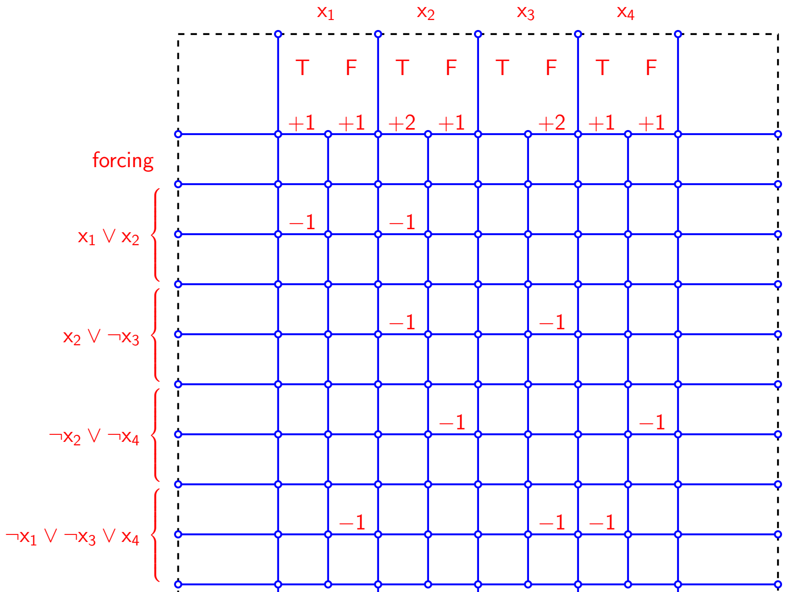

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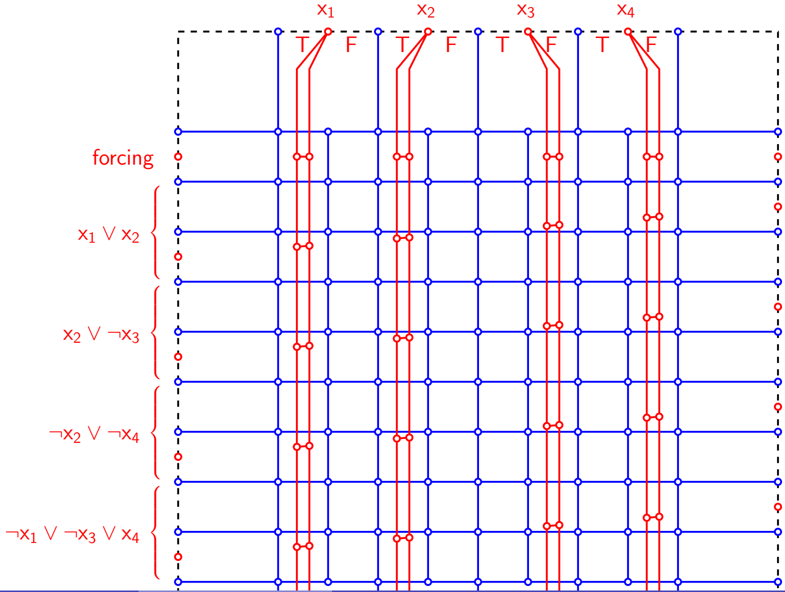

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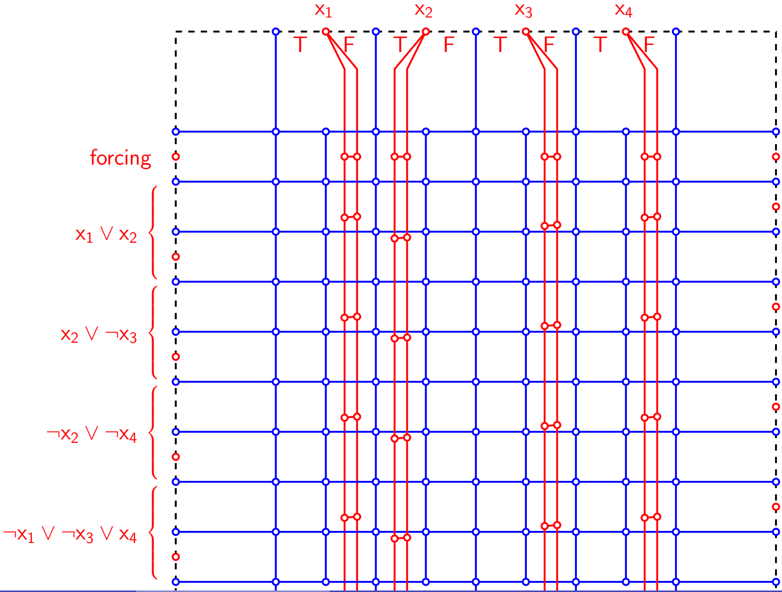

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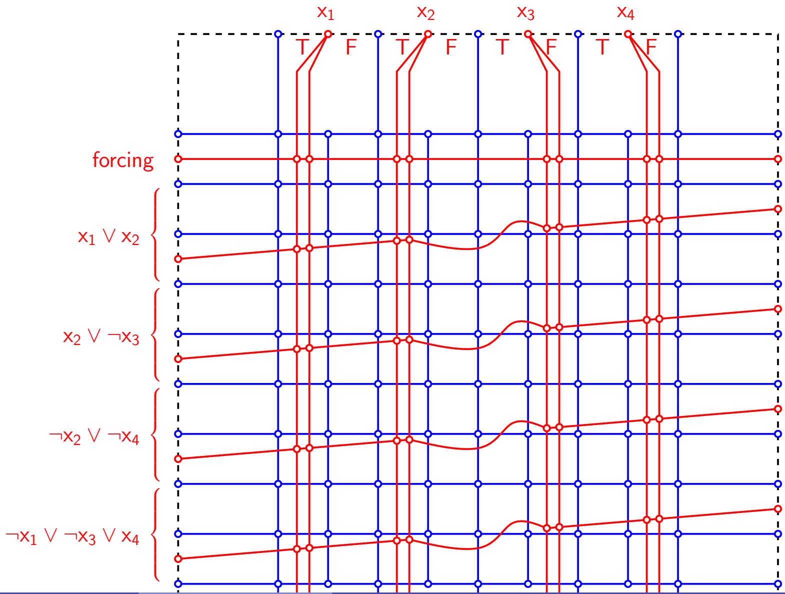

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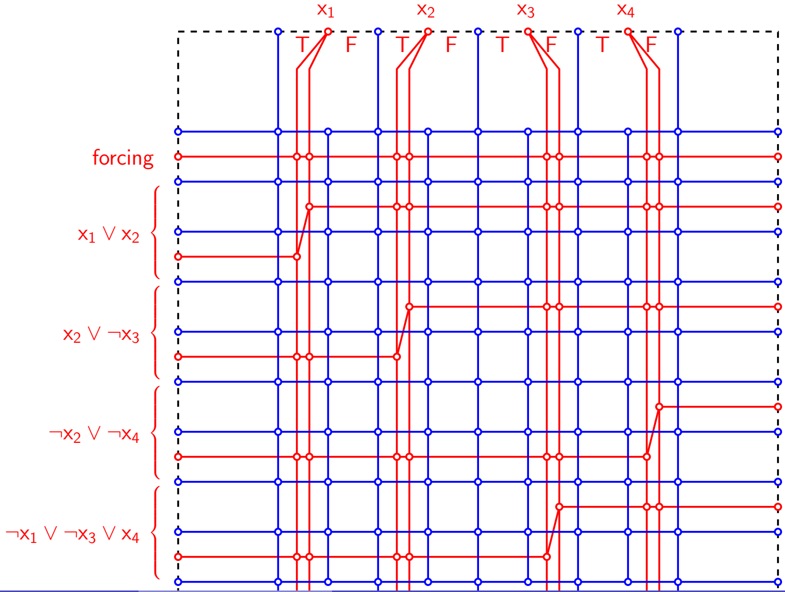
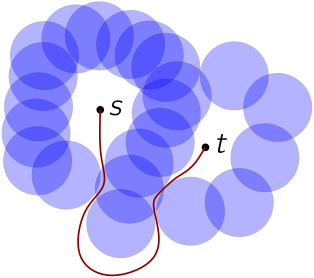


Crossing number of near-planar graphs

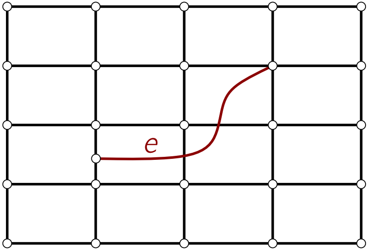
- adding one edge makes a big mess
- we need large degrees
  - three vertices of large degree suffice [Hliněný 2023]
- $\lfloor \Delta/2 \rfloor$-approximation [C., Mohar 2011]
  - number of edge-disjoint cycles separating x and y
  - number of vertex-disjoint cycles separating x and y
- Is it NP-hard for max degree 4?
- Research also on adding a vertex

- Similar proof for 1-planarity of near-planar graphs

Conclusions



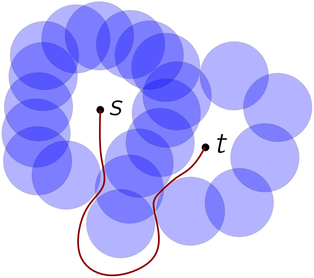
barrier



crossing number

near-planar

Conclusions



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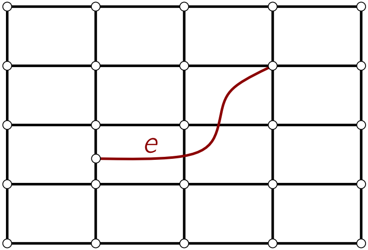

crossing number

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THANKS for your time!!