Quantum mechanics of ribbon graphs: a lattice interpretation of the Kronecker coefficient

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joint on work with S. Ramgoolam (QMUL)
"Quantum mechanics of bipartite ribbon graphs: Integrality, Lattices and Kronecker coefficients"
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+ Ann. Inst. HP D Comb. Phys. Interact. **1**, 77-138 2014

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Outline

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1. Introduction
   - Counting graphs, permutations, and algebras
   - Summary
   - Goals
2. Review: $\mathcal{K}(n)$ an algebra and Hilbert space
3. Step1: Integrality and Hamiltonians
   - Integrality structure in the product of $\mathcal{K}(n)$
   - The centre of $\mathcal{K}(n)$ and reconnection operators $T_k^{(i)}$
   - Fourier subspace of a Young diagram triple as eigenspace of $T_k^{(i)}$
4. Step2: Square of Kronecker coefficients and ribbon graph sub-lattices
5. Step3: Kronecker coefficient and sub-lattices of ribbon graphs
6. Conclusion

Introduction: Combinatorics and representation theory

R P Stanley [Positivity Problems and Conjectures in Algebraic Combinatorics '99]:
"The theory of symmetric functions is rife with positivity results and problems, stemming from the possibility of expanding a symmetric function in terms of a number of possible bases. If the coefficients in such an expansion are real numbers (respectively, polynomials with real coefficients), then we can ask whether they are nonnegative (respectively, have nonnegative coefficients). Often these coefficients will have a representation-theoretic interpretation, such as the multiplicity of an irreducible representation within some larger representation. Sometimes the only known proof of positivity will be such an interpretation, and the problem will be to find a combinatorial proof."

Pb10:

$$s_\lambda * s_\nu = \sum_{\nu} C(\lambda, \nu, \mu) s_\mu \tag{1}$$

s_λ, s_ν are Schur symmetric functions, $C(\lambda, \nu, \mu)$ is the Kronecker coefficient.

Introduction: Combinatorics and representation theory

Barcelo-Ram [Combinatorial Representation Theory, arXiv:math/9707221[math.RT]]
"In January 1997, (...) Gil Kalai, in his usual fashion, began asking very pointed questions about exactly what all the combinatorial representation theorists were doing their research on. After several unsuccessful attempts at giving answers that Gil would find satisfactory, (...). in the end, Arun gave two talks at MSRI in which he tried to clear up the situation. (...) After some arm twisting it was agreed that, Arun and Helene would write such a paper on combinatorial representation theory. What follows is our attempt to define the field of combinatorial representation theory, describe the main results and main questions and give an update of its current status.

Of course this is wholly impossible. Everybody in the field has their own point of view and their own preferences of questions and answers. (...)

What do we mean by 'combinatorial representation theory'? First and foremost, combinatorial representation theory is representation theory. The adjective 'combinatorial' will refer to the way in which we answer representation theoretic questions."

Introduction: “Combinatorial, have you said Combinatorial?”

$\rightarrow$ In this talk: address a question with an RT input but, in the end, has a combinatorial answer.

Introduction: tensors, colored/ribbon graphs

From mathematical physics to combinatorics [BG, Ramgoolam, '14, '17]

• Permutation methods and algebras: Role in the enumeration of observables of tensor models. [BG, Ramgoolam, '14].

• Tensor observable $\equiv$ a contraction of tensors $\equiv$ classical Lie group invariant $(U(N), O(N), Sp(N))$ $\equiv$ a colored graph.

Illustration in rank 3: complex tensors T_{abc}, $a, b, c = 1...N$, with distinguished indices

- Counting 3-regular 3-edged colored graphs is counting permutation triples $(\sigma_1, \sigma_2, \sigma_3) \in (S_n \times S_n \times S_n)$ up to the equivalence

$$(\sigma_1, \sigma_2, \sigma_3) \sim (\gamma_1 \sigma_1 \gamma_2, \; \gamma_1 \sigma_2 \gamma_2, \; \gamma_1 \sigma_3 \gamma_2) , \qquad \gamma_i \in S_n . \qquad (2)$$

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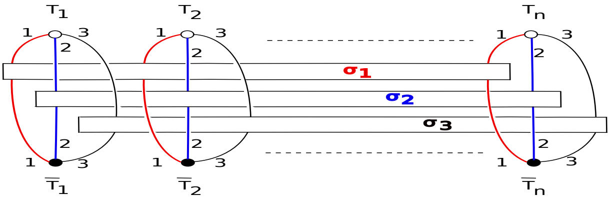

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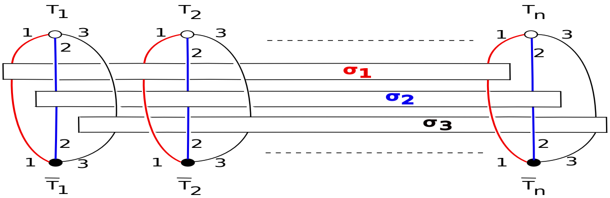

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[BG, Ramgoolam, '14]

- Gauge-fixed version of the same counting

$$(\sigma_1, \sigma_2) \sim (\gamma\sigma_1\gamma^{-1}, \gamma\sigma_2\gamma^{-1}) \quad (3)$$

- Counting bi-partite ribbon graphs with n edges and at most n white and n black vertices (shortly called "bi-partite ribbon graphs with n edges").

- Construction of the ribbon graph:

- cycles of s_1 define vertices of the black type; draw the labels in the cyclic order they appear in their cycle, and all with a given and unique orientation;
- cycles of s_2 define vertices of white type; draw the labels in the cyclic order they appear in their cycle, and all with the same orientation as above;
- the edges are labelled from $1...n$ and connect the labels

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Illustrations: $n = 2$

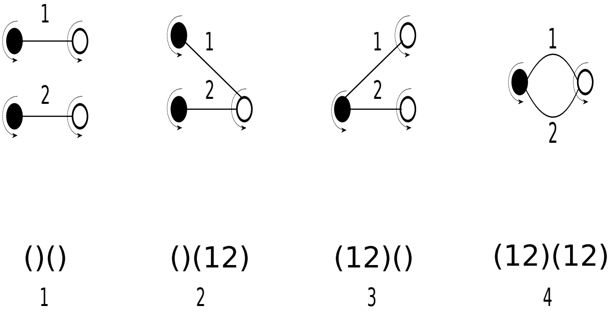
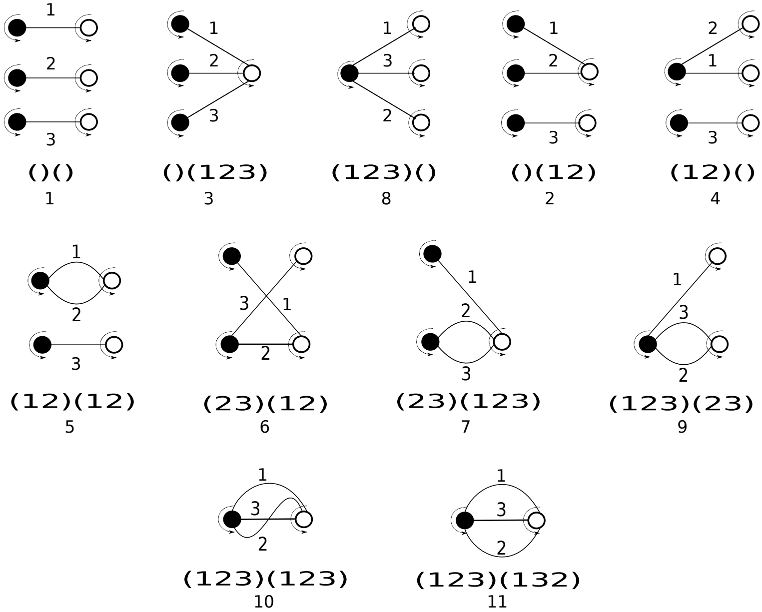


Figure: Bipartite ribbon graphs with $n = 2$ edges

Illustrations: $n = 3$



Introduction: Counting orbits

[BG, Ramgoolam, '14]

- Counting cosets $(S_n \times S_n)/\mathrm{Diag}(S_n)$.
- Burnside's lemma

$$|G/H| = \frac{1}{|H|} \sum_{h \in H} \sum_{g \in G} \delta(hgh^{-1}g^{-1}) \tag{4}$$

- Number of bi-partite ribbon graphs with n edges

$$Z(n) = \frac{1}{n!} \sum_{\sigma_1, \sigma_2 \in S_n} \sum_{\gamma \in S_n} \delta(\gamma\sigma_1\gamma^{-1}\sigma_1^{-1})\delta(\gamma\sigma_2\gamma^{-1}\sigma_2^{-1}) \tag{5}$$

$\rightarrow$ Programming in Gap, and Mathematica [OEIS: A110143 (isomorphism of graph coverings)] Illustration at rank $d = 3$

$$1; 4; 11; 43; 161; 901; 5579; 43206; 378360; 3742738, \ldots \tag{6}$$

$\rightarrow$ All the above generalizes at any rank d.

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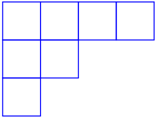
Revisiting the counting under a different light,
that of representation theory of the symmetric group.

Basics of representation of the symmetric group

- Irreps of symmetric group S_n are labelled by Young diagrams or $R \vdash n$ partition of n.

$$n = 7, \quad R = (1,2,4) = \tag{7}$$



- $D^R_{ij}(\sigma) = \langle R, j|\sigma|R, i\rangle$ the real matrix representation of σ in the irrep $R \vdash n$ (dimension $d(R)$)

$$\text{Orthogonality}: \quad \sum_{\sigma \in S_n} D^R_{ij}(\sigma) D^S_{kl}(\sigma) = \frac{n!}{d(R)}\, \delta^{RS}\, \delta_{ik}\delta_{jl}\ ;$$

$$\text{Clebsch} - \text{Gordan}: \quad \sum_{\sigma \in S_n} D^{R_1}_{i_1 j_1}(\sigma) D^{R_2}_{i_2 j_2}(\sigma) D^{R_3}_{i_3 j_3}(\sigma) = \frac{n!}{d(R_3)} \sum_{\tau} C^{R_1,R_2;R_3,\tau}_{i_1,i_2;i_3} C^{R_1,R_2;R_3,\tau}_{j_1,j_2;j_3}$$

$$\tau \in [\![1, \mathrm{C}(R_1, R_2, R_3)]\!]$$

- Expansion of the delta

$$\delta(\sigma) = \sum_R \frac{d(R)}{n!} \chi^R(\sigma)\,, \qquad d(R) = \frac{n!}{h(R)}\,, \quad h(R) = \prod_{i,j} \text{hook} - \text{Length}_{i,j}$$

$$\sum_{\sigma \in S_n} \chi^R(\sigma)\chi^S(\sigma) = n!\, \delta_{RS}\ ; \tag{8}$$

Introduction: Revisiting the counting

[BG, Ramgoolam, '17]

- A small calculation

$$
\begin{aligned}
Z(n) &= \frac{1}{n!}\sum_{\sigma_i\in S_n}\sum_{\gamma_1,\gamma_2\in S_n}\delta(\gamma_1\sigma_1\gamma_2^{-1}\sigma_1^{-1})\delta(\gamma_1\sigma_2\gamma_2^{-1}\sigma_2^{-1})\delta(\gamma_1\sigma_3\gamma_2^{-1}\sigma_3^{-1})\\
&= \frac{1}{(n!)^2}\sum_{\gamma_i\in S_n}\sum_{R_i\vdash n}\chi^{R_1}(\gamma_1)\chi^{R_1}(\gamma_2)\chi^{R_2}(\gamma_1)\chi^{R_2}(\gamma_2)\chi^{R_3}(\gamma_1)\chi^{R_3}(\gamma_2)\\
&= \sum_{R_1,R_2,R_3\vdash n}(C(R_1,R_2,R_3))^2 \qquad (9)
\end{aligned}
$$

where the symbol

$$C(R_1,R_2,R_3) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi^{R_1}(\sigma)\chi^{R_2}(\sigma)\chi^{R_3}(\sigma) \qquad (10)$$

is the Kronecker coefficient.

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&= \frac{1}{(n!)^2} \sum_{\gamma_i \in S_n} \sum_{R_i \vdash n} \chi^{R_1}(\gamma_1) \chi^{R_1}(\gamma_2) \chi^{R_2}(\gamma_1) \chi^{R_2}(\gamma_2) \chi^{R_3}(\gamma_1) \chi^{R_3}(\gamma_2) \\
&= \sum_{R_1, R_2, R_3 \vdash n} (\mathsf{C}(R_1, R_2, R_3))^2 \qquad (9)
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- Counts

$\to$ multiplicity of the one-dimensional (trivial) representation in the tensor product $R_1 \otimes R_2 \otimes R_3$.

$\to$ Find a combinatorial rule to characterize them in general (Murnaghan '38, Stanley '99)

$\to$ Ikenmeyer, Burgisser, Walter, Pak, Panova.... : C not like the Littlewood-Richardson coefficients;

$\to$ Littlewood-Richardson (LR) coefficients:

- a combinatorial interpretation in terms of fillings of skew tableaux, or filling of the hive model in relationship with Knutson and Tao saturation conjecture (see [King, Tollu, Toumazet '06]);

$\to$ Combinatorial interpretation and scaling limits of C: focus in the ordinary fashion of fillings tableaux, using other numbers (Kostka, LR) the interpretation is known, or reduced YD (3 hooks, 2 hooks and 1 rectangle, 3 rectangles...).

Proposition (Combinatorial interpretation of the sum of C^2)

$$\sum_{R_1,R_2,R_3\vdash n} (C(R_1,R_2,R_3))^2 = \#\ \textit{bi-partite ribbon graphs with } n \textit{ edges}$$
$$= \#\ \textit{3-regular 3-edge colored bi-partite graphs with } 2n-\textit{vertices} \tag{11}$$

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Giving a sense to $\sum_{R_1,R_2,R_3\vdash n}(C(R_1,R_2,R_3))^2$
from the representation theoretic base of an algebra.

$\mathcal{K}(n)$, the graph algebra

- Group algebra $\mathbb{C}(S_n)$, i.e. an element of which writes $a = \sum_{\sigma \in S_n} \lambda_\sigma \sigma, \;\; \lambda_\sigma \in \mathbb{C}$
- Back to coset formulation: Consider the orbits

$$(\sigma_1, \sigma_2) \sim (\gamma \sigma_1 \gamma^{-1}, \gamma \sigma_2 \gamma^{-1}) \tag{12}$$

- Define $\mathcal{K}(n) \subset \mathbb{C}(S_n)^{\otimes 2}$ is the vector space over $\mathbb{C}$

$$\mathcal{K}(n) = \mathrm{Span}_{\mathbb{C}} \Big\{ \sum_{\gamma \in S_n} \gamma \sigma_1 \gamma^{-1} \otimes \gamma \sigma_2 \gamma^{-1}, \; \sigma_1, \sigma_2, \in S_n \Big\} \tag{13}$$

$\rightarrow$ Fact 1: $\dim_{\mathbb{C}} \mathcal{K}(n) = Z(n)$.
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$\rightarrow$ Fact 2: a ribbon graph r is 1-1 with an orbit $\mathrm{Orb}(r)$ and is 1-1 with a base vector E_r of $\mathcal{K}(n)$.

$\mathcal{K}(n)$, the graph algebra

- Take a base element of $\mathcal{K}(n)$

$$A_{\sigma_1,\sigma_2} = \sum_{\gamma \in S_n} \gamma\sigma_1\gamma^{-1} \otimes \gamma\sigma_2\gamma^{-1} \tag{14}$$

- Associative multiplication

$$A_{\sigma_1,\sigma_2} A_{\sigma_3,\sigma_4} = coeff. \sum_{\tau \in S_n} A_{\sigma_1\tau\sigma_3\tau^{-1},\, \sigma_2\tau\sigma_4\tau^{-1}} \tag{15}$$

- There is a pairing

$$\boldsymbol{\delta}_2(\otimes_{i=1}^2 \sigma_i; \otimes_{i=1}^2 \sigma_i') = \prod_{i=1}^{2} \delta(\sigma_i \sigma_i'^{-1}) \tag{16}$$

that extends by linearity to $\mathcal{K}(n)$ and that is non-degenerate.

Theorem (BG, Ramgoolam '17)

$\mathcal{K}(n)$ is an associative unital semi-simple algebra. There is a Fourier representation theoretic base $\{Q^{R_1,R_2,R_3}_{a,b}\}$, $R_i \vdash n$, $a, b \in [\![1, \mathrm{C}(R_1, R_2, R_3)]\!]$ that makes the Wedderburn-Artin decomposition manifest.

$\rightarrow$ Fact 3: By the Wedderburn-Artin theorem $\mathcal{K}(n)$ decomposes as a direct sum of matrix sub-algebras.

$\rightarrow$ Fact 4: The base $\{Q^{R_1,R_2,R_3}_{a,b}\}$ decomposes $\mathcal{K}(n)$ in direct blocks.

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Summary

$\rightarrow$ Bi-partite ribbon graphs with n edges (and ....) are in 1-1 corresp. with the orbits of $S_n \times S_n$ on which S_n acts by conjugation.

$\rightarrow \mathrm{Rib}(n) \subset S_n \times S_n$ is the set of orbits representing bi-partite ribbon graphs with n edges.

$\rightarrow$ We construct the subspace $\mathcal{K}(n) \subset \mathbb{C}(S_n)^{\otimes 2}$ with base vectors that are in 1-1 corresp. with elements of $\mathrm{Rib}(n)$. We call these geometric base vectors.

$\rightarrow \mathcal{K}(n)$ is an associative unital semi-simple algebra and its Wedderburn-Artin base is given by $\{Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}\}$, where the matrix blocks are of size $C(R_1,R_2,R_3)^2$ for each triple of $R_i \vdash n$.

Goal: Forming a lattice combinatorial interpretation of the Kronecker coefficient

Step 1: Building integral operators (an integrable quantum mechanical system)

For $i = 1, 2, 3$, and $k = 2, 3, \ldots, \tilde{k}(n) \leq n$

- Show that there are operators $T_k^{(i)}$ acting on $\mathcal{K}(n)$ such that $\{Q_{\tau_1,\tau_2}^{R_1,R_2,R_3}\}$ forms an eigenbasis of $T_k^{(i)}$ with eigenvalue given by normalized characters $\chi_{R_i}(T_k)/d(R_i)$ and the multiplicity of each eigenvalue is precisely given by $\mathrm{C}(R_1, R_2, R_3)^2$. INPUT from RT.

- $\frac{\chi_{R_1}(\sigma)}{d(R_1)}$ is combinatorial: $\chi_{R_1}(\sigma)$ can be computed by the Murnaghan-Nakayama rule (see Wiki); $d(R_1)$ by the hook-length formula.

- Show that, in the geometric base of $\mathcal{K}(n)$, the $T_k^{(i)}$'s have matrices $\mathcal{M}_k^{(i)}$ with positive integer coefficients.

- By product: Show that $T_k^{(i)}$ are mutually commuting and Hermitian with respect to a sesquilinear form making $\mathcal{K}(n)$ Hilbert space. This makes the framework as a quantum mechanical (integrable) model where $T_k^{(i)}$ are seen as Hamiltonians.

Goal: Forming a lattice combinatorial interpretation of the Kronecker coefficient

Step 2: Getting $C(R_1, R_2, R_3)^2$

- Stacking the matrices, for a fixed triple R_1, R_2, R_3, seek all ribbon graph vectors that solve the system:

$$\begin{bmatrix} \mathcal{M}_2^{(1)} - \frac{\chi_{R_1}(T_2)}{d(R_1)} id \\ \vdots \\ \mathcal{M}_{\widetilde{k}}^{(1)} - \frac{\chi_{R_1}(T_{\widetilde{k}})}{d(R_1)} id \\ \mathcal{M}_2^{(2)} - \frac{\chi_{R_2}(T_2)}{d(R_2)} id \\ \vdots \\ \mathcal{M}_{\widetilde{k}}^{(2)} - \frac{\chi_{R_2}(T_{\widetilde{k}})}{d(R_2)} id \\ \mathcal{M}_2^{(3)} - \frac{\chi_{R_3}(T_2)}{d(R_3)} id \\ \vdots \\ \mathcal{M}_{\widetilde{k}}^{(3)} - \frac{\chi_{R_3}(T_{\widetilde{k}})}{d(R_3)} id \end{bmatrix} \cdot v = \mathcal{L}_{R_1,R_2,R_3} \cdot v = \mathbf{0} \tag{17}$$

id is the identity matrix of size $Z(n) \times Z(n)$. (This is the counting problem.)

- Use the so-called Hermite normal form (HNF) procedure to extract the dimension of the null space of the system.
- Then, the null space is generated by integer linear combinations of basis vectors of $\mathcal{K}(n)$ and forms a sub-lattice in $\mathbb{Z}^{|\mathrm{Rib}(n)|}$ of dimension $C(R_1, R_2, R_3)^2$. (Theorem 1)

Goal: Forming a lattice combinatorial interpretation of the Kronecker coefficient

Step 3: Getting $C(R_1, R_2, R_3)$

- Use an involution $S : \mathcal{K}(n) \to \mathcal{K}(n)$, $S^2 = id$, such that

$$\mathcal{K}(n) \equiv V^{|\mathrm{Rib}(n)|} = \bigoplus_{R_1,R_2,R_3} \left(V_{S=1}^{\mathrm{Rib}(n):R_1,R_2,R_3} \oplus V_{S=-1}^{\mathrm{Rib}(n):R_1,R_2,R_3} \right) \tag{18}$$

- Show that

$$\begin{aligned} \mathrm{Dim}\left(V_{S=-1}^{\mathrm{Rib}(n):R_1,R_2,R_3} \right) &= \frac{C(R_1,R_2,R_3)(C(R_1,R_2,R_3)-1)}{2} \\ \mathrm{Dim}\left(V_{S=+1}^{\mathrm{Rib}(n):R_1,R_2,R_3} \right) &= \frac{C(R_1,R_2,R_3)(C(R_1,R_2,R_3)+1)}{2} \end{aligned} \tag{19}$$

(counting problem) and, using the HNF algorithm, show these correspond also to the dimensions of 2 sublattices in $\mathbb{Z}^{|\mathrm{Rib}(n)|}$ formed by integer linear combinations of vectors in $\mathcal{K}(n)$.

- By choosing an injection from the smaller sub-lattice into the larger sub-lattice, we get a constructive interpretation of $C(R_1, R_2, R_3)$ as the dimension of a sub-lattice of $\mathbb{Z}^{|\mathrm{Rib}(n)|}$. (Theorem 2)

Outline

$\mathcal{K}(n)$, the graph algebra

$$\mathcal{K}(n) = \mathrm{Span}_{\mathbb{C}}\Big\{ \sum_{\gamma\in S_n} \gamma\tau_1\gamma^{-1} \otimes \gamma\tau_2\gamma^{-1},\ \tau_1,\tau_2 \in S_n \Big\} \tag{20}$$

• A non-degenerate symmetric bilinear pairing $\boldsymbol{\delta}_2 : \mathbb{C}(S_n)^{\otimes 2} \times \mathbb{C}(S_n)^{\otimes 2} \to \mathbb{C}$ where

$$\boldsymbol{\delta}_2(\otimes_{i=1}^2 \sigma_i ; \otimes_{i=1}^2 \sigma_i') = \prod_{i=1}^{2} \delta(\sigma_i \sigma_i'^{-1}) \tag{21}$$

which extends to linear combinations with complex coefficients.
• $\mathcal{K}(n)$ is a semi-simple algebra admits a Wedderburn-Artin decomposition in matrix blocks.

$\mathcal{K}(n)$, the graph algebra

• A base of $\mathcal{K}(n)$: pick a ribbon graph (etc) labeled by $r = 1, ..., |\mathrm{Rib}(n)|$, in $\mathrm{Orb}(r)$ there is a representative $(\tau_1^{(r)}, \tau_2^{(r)})$, we define the base vector

$$\begin{aligned} E_r &= \frac{1}{n!}\sum_{\mu \in S_n} \mu\tau_1^{(r)}\mu^{-1} \otimes \mu\tau_2^{(r)}\mu^{-1} = \frac{|\mathrm{Aut}(r)|}{n!}\sum_{a\in \mathrm{Orb}(r)} \tau_1^{(r)}(a) \otimes \tau_2^{(r)}(a) \\ &= \frac{1}{|\mathrm{Orb}(r)|}\sum_{a\in \mathrm{Orb}(r)} \tau_1^{(r)}(a) \otimes \tau_2^{(r)}(a) \end{aligned} \tag{22}$$

- $\mathrm{Aut}(\tau_1^{(r)}, \tau_2^{(r)}) =$ the subgroup of S_n which leaves fixed the pair $(\tau_1^{(r)}, \tau_2^{(r)})$, and $|\mathrm{Aut}(\tau_1^{(r)}, \tau_2^{(r)})| = |\mathrm{Aut}(r)|$.
- Orbit-stabilizer theorem: $\mathrm{Orb}(r) \equiv S_n/\mathrm{Aut}(\tau_1^{(r)}, \tau_2^{(r)})$.

• $\delta_2(E_r, E_s) = \frac{1}{|\mathrm{Orb}(r)|}\delta_{rs}$. Thus $\{E_r\}$ is orthogonal.

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Fourier base of $\mathcal{K}(n)$

Introduce the Fourier basis of $\mathbb{C}(S_n)$

$$Q^R_{ij} = \frac{\kappa_R}{n!} \sum_{\sigma \in S_n} D^R_{ij}(\sigma)\sigma \tag{23}$$

- Fourier base [BR, Ramgoolam, '17]

$$\underbrace{\sum_{i_l, j_l} C^{R_1,R_2;R_3,\tau}_{i_1,i_2;i_3} C^{R_1,R_2;R_3,\tau'}_{j_1,j_2;j_3}}_{\text{Make it legs/momentum invariant}} \underbrace{\sum_{\gamma} \rho_L(\gamma)\rho_R(\gamma)}_{\text{Make it invariant}} \underbrace{Q^{R_1}_{i_1 j_1} \otimes Q^{R_2}_{i_2 j_2}}_{\text{Ordinary base of } C(S_n)^{\otimes 2}} = Q^{R_1,R_2,R_3}_{\tau,\tau'} \tag{24}$$

$$Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} = \kappa_{R_1,R_2} \sum_{\sigma_1,\sigma_2 \in S_n} \sum_{i_1,i_2,i_3,j_1,j_2} C^{R_1,R_2;R_3,\tau_1}_{i_1,i_2;i_3} C^{R_1,R_2;R_3,\tau_2}_{j_1,j_2;j_3} D^{R_1}_{i_1 j_1}(\sigma_1) D^{R_2}_{i_2 j_2}(\sigma_2)\, \sigma_1 \otimes \sigma_2$$

$\kappa_{R_1,R_2} = \frac{d(R_1)d(R_2)}{(n!)^2}$.

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The basis $Q^{R_1,R_2,R_3}_{\tau,\tau'}$

- The set $\{Q^{R_1,R_2,R_3}_{\tau,\tau'}\}$ forms an invariant orthogonal matrix base of $\mathcal{K}(n)$.

Invariance	$(\gamma\otimes\gamma)\cdot Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}\cdot(\gamma^{-1}\otimes\gamma^{-1}) = Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}$
Multiply like matrices	$Q^{R,S,T}_{\tau_1,\tau_2}Q^{R',S',T'}_{\tau_2',\tau_3} = \delta^{RR'}\delta^{SS'}\delta^{TT'}\delta_{\tau_2\tau_2'}Q^{R,S,T}_{\tau_1,\tau_3}$
Orthogonality	$\boldsymbol{\delta}_2(Q^{R_1,R_2,R_3}_{\tau_1,\tau_1'};Q^{R_1',R_2',R_3'}_{\tau_2,\tau_2'}) = \kappa_{R_1,R_2}\,d(R_3)\,\delta_{R_1R_1'}\delta_{R_2R_2'}\delta_{R_3R_3'}\delta_{\tau_1\tau_2}\delta_{\tau_1'\tau_2'}$.

- At fixed $[R_1,R_2,R_3]$, $Q^{R_1,R_2,R_3}_{\tau,\tau'}$ is a matrix with $\mathrm{C}(R_1,R_2,R_3)^2$ entries.

$\rightarrow$ This is the Wedderburn-Artin basis for $\mathcal{K}(n)$.

$\mathcal{K}(n)$ is a Hilbert space

• Inner product g on $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$, using the basis of permutation pairs and extend it by linearity. $\forall \alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in S_n \times S_n$

$$g(\alpha, \beta) = g(\alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2) = \delta(\alpha_1^{-1}\beta_1)\delta(\alpha_2^{-1}\beta_2) \tag{25}$$

and that extends to a sesquilinear form on $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$ as

$$g(\sum_i a_i \alpha_{1i} \otimes \alpha_{2i}, \sum_j b_j \beta_{1j} \otimes \beta_{2j}) = \sum_{i,j} \bar{a}_i b_j \; \delta(\alpha_{1i}^{-1}\beta_{1j})\delta(\alpha_{2i}^{-1}\beta_{2j}) \tag{26}$$

where $a_i, b_i \in \mathbb{C}$.

• Involution, the conjugation S:

$$S(\sum_i c_i \sigma_i) := \sum_i c_i \sigma_i^{-1}, \qquad S(\sigma_1 \otimes \sigma_2) = \sigma_1^{-1} \otimes \sigma_2^{-1}$$

$$\text{extends by linearity} \qquad S(\sum_i a_i \; \sigma_{1i} \otimes \sigma_{2i}) = \sum_i a_i \sigma_{1i}^{-1} \otimes \sigma_{2i}^{-1} \tag{27}$$

$S^2 = id$.

• Note the following relation

$$g(\alpha, \beta) = \delta_2(\overline{S(\alpha)}\beta). \tag{28}$$

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Outline

Integrality structure in the product of $\mathcal{K}(n)$

- The product of two base elements

$$E_r E_s = \sum_{t=1}^{|\mathrm{Rib}(n)|} C_{rs}^t E_t \tag{29}$$

$\rightarrow$ For notational convenience,

$$\sigma^{(r)} = \sigma_1^{(r)} \otimes \sigma_2^{(r)}\,, \qquad \mu\sigma^{(r)}\mu^{-1} = \mu\sigma_1^{(r)}\mu^{-1} \otimes \mu\sigma_2^{(r)}\mu^{-1}$$

$$E_r = \frac{1}{n!}\sum_{\mu\in S_n} \mu\sigma^{(r)}\mu^{-1} = \frac{1}{|\mathrm{Orb}(r)|}\sum_{a\in\mathrm{Orb}(r)} \sigma^{(r)}(a) \tag{30}$$

Integrality structure of $\mathcal{K}(n)$

$$
\begin{aligned}
E_r E_s &= \frac{1}{n!}\frac{1}{|\mathrm{Orb}(r)|}\sum_{\mu\in S_n}\sum_{a\in\mathrm{Orb}(r)} \mu\sigma^{(r)}(a)\sigma^{(s)}\mu^{-1} \\
&= \frac{1}{|\mathrm{Orb}(r)|}\sum_{a\in\mathrm{Orb}(r)}\frac{1}{|\mathrm{Orb}(\sigma^{(r)}(a)\sigma^{(s)})|}\sum_{b\in\mathrm{Orb}(\sigma^{(r)}(a)\sigma^{(s)})}\sigma(b) \\
&= \sum_t \frac{1}{|\mathrm{Orb}(r)|}\sum_{b\in\mathrm{Orb}(t)}\frac{\sigma^{(t)}(b)}{|\mathrm{Orb}(t)|}\sum_{a\in\mathrm{Orb}(r)}\delta(\mathrm{Orb}(t),\mathrm{Orb}(\sigma^{(r)}(a)\sigma^{(s)})) \\
&= \sum_t \frac{1}{|\mathrm{Orb}(r)|}E_t\sum_{a\in\mathrm{Orb}(r)}\delta(\mathrm{Orb}(t),\mathrm{Orb}(\sigma^{(r)}(a)\sigma^{(s)}))\,, \qquad (31)
\end{aligned}
$$

where $\delta(\mathrm{Orb}(s),\mathrm{Orb}(t)) = \delta_{st}$.
We have thus expressed the product of C_{rs}^t in terms of $\frac{1}{|\mathrm{Orb}(r)|}$ times the non-negative integer

$$
\begin{aligned}
&\sum_{a\in\mathrm{Orb}(r)}\delta(\mathrm{Orb}(t),\mathrm{Orb}(\sigma^{(r)}(a)\sigma^{(s)})) \\
&= \text{Number of times the multiplication of elements from orbit } r \\
&\text{ with a fixed element in orbit } s \text{ to the right produces an element in orbit } t
\end{aligned}
\qquad (32)
$$

Hamiltonian operators $T_k^{(i)}$ in the centre of $\mathcal{K}(n)$

- For $k \in \mathbb{N}$, such that $2 \leq k \leq n$,
$\mathcal{C}_k =$ conjugacy class of permutations $\sigma \in S_n$ made of a single cycle of length k and remaining cycles of length 1.
Ex: for $n = 3$, $k = 2$, $\mathcal{C}_2 = \{(1,2)(3), (2,3)(1), (1,3)(2)\}$.

- Define T_k as the sum

$$T_k = \sum_{\sigma \in \mathcal{C}_k} \sigma \tag{33}$$

$|T_k|$ will refer to as the number of terms.
- T_k are central element in $\mathbb{C}(S_n)$.

Proposition (Kemp and Ramgoolam '20)

For any n, there exists $2 \leq k_*(n) \leq n$ such that the set $\{T_2, \cdots, T_{k_*(n)}\}$ generates the centre $\mathcal{Z}(\mathbb{C}(S_n))$ of $\mathbb{C}(S_n)$.

$$\begin{aligned}
k_*(n) &= 2 \text{ for } n \in \{2,3,4,5,7\} \\
k_*(n) &= 3 \text{ for } n \in \{6,8,9 \cdots, 14\} \\
k_*(n) &= 4 \text{ for } n \in \{15,16,\cdots,23,25,26\} \\
k_*(n) &= 5 \text{ for } n \in \{24,27,\cdots,41\} \\
k_*(n) &= 6 \text{ for } n \in \{42,\cdots,78,79,81\}
\end{aligned} \tag{34}$$

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Hamiltonian operators $T_k^{(i)}$ in the centre of $\mathcal{K}(n)$

Lemma ($TQ = \hat{\chi} Q$)

For any n, for any $2 \leq k_*(n) \leq n$, $T_k Q^R_{ij} = \frac{\chi_R(T_k)}{d(R)}\, Q^R_{ij}$

Proof: Use the Schur lemma on $\sum_{\sigma \in \mathcal{C}_k} D^R_{li}(\sigma) = D^R_{li}(T_k) = \alpha \delta_{li}$ and then determine $\sum_i D^R_{ii}(T_k) = \chi_R(T_k) = \alpha d(R)$.

- At any $n \geq 2$, we will define elements in $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$

$$\begin{aligned} T_k^{(1)} &= T_k \otimes 1 = \sum_{\sigma \in \mathcal{C}_k} \sigma \otimes 1\,, \\ T_k^{(2)} &= 1 \otimes T_k = \sum_{\sigma \in \mathcal{C}_k} 1 \otimes \sigma\,, \\ T_k^{(3)} &= \sum_{\sigma \in \mathcal{C}_k} \sigma \otimes \sigma\,. \end{aligned} \tag{35}$$

- $T_k^{(i)}$ cut and join vertices of ribbon graphs.

Hamiltonian operators $T_k^{(i)}$ in the centre of $\mathcal{K}(n)$

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Hamiltonian operators $T_k^{(i)}$ in the centre of $\mathcal{K}(n)$

Proposition

1) $T_k^{(i)}$ commute with $\gamma \otimes \gamma$ and so are in $\mathcal{K}(n)$ (in fact in its centre).
2) $T_k^{(i)}$'s act as linear operators on $\mathcal{K}(n)$ by left multiplication (left regular representation)
Denote their matrices $T_k^{(i)} E_s = \sum_s (\mathcal{M}_k^{(i)})_s^t E_t$.
3) The matrix elements $(\mathcal{M}_k^{(i)})_r^s$ are non-negative integers.

Proof:
1) is obvious from the fact that T_k is central. 2) Ok.
3) $T_k^{(i)} \propto E_r$ obtained by summing over diagonal conjugations of permutations of the form $\sigma \otimes 1, 1 \otimes \sigma, \sigma \otimes \sigma$, where σ is a cyclic permutation of a subset of k numbers from $\{1, 2, \cdots, n\}$.
Each $T_k^{(i)}$ corresponds to a ribbon graph, with some label r which we will call $r(k,i)$.
The proportionality constant is given as

$$\begin{aligned} & T_k^{(i)} = |\mathrm{Orb}(r(k,i))| \ E_{r(k,i)} \\ & T_k^{(i)} E_s = |\mathrm{Orb}(r(k,i))| \ E_{r(k,i)} E_s \\ & = \frac{|\mathrm{Orb}(r(k,i))|}{|\mathrm{Orb}(r(k,i))|} \sum_t E_t \sum_{a \in \mathrm{Orb}(r(k,i))} \delta(\mathrm{Orb}(t), \mathrm{Orb}(\sigma^{(r(k,i))}(a) \sigma^{(s)})) \end{aligned} \tag{36}$$

Hamiltonian operators $T_k^{(i)}$ in the centre of $\mathcal{K}(n)$

$$(\mathcal{M}_k^{(i)})_s^t = \text{ Number of times the multiplication of elements in the sum } T_k^{(i)} \text{ with a fixed element in orbit } s \text{ to the right produces an element in orbit } t. \tag{37}$$

Proposition

$T_k^{(i)}$ are hermitian operators on $\mathcal{K}(n)$ in the inner product g

$$g(E_s, T_k^{(i)} E_r) = g(T_k^{(i)} E_s, E_r)\,. \tag{38}$$

Proof: First observe that E_s and $T_k^{(i)} E_r$ have real coeffs then
$g(E_s, T_k^{(i)} E_r) = g(T_k^{(i)} E_r, E_s)$
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Hamiltonian operators $T_k^{(i)}$ in the centre of $\mathcal{K}(n)$

- Quantum evolution: $\forall t \in \mathbb{R}$

$$E_r(t) = e^{-itT_k^{(i)}} E_r \tag{39}$$

- Construct interesting Hamiltonians $H = \sum_{i,k} a_{i,k} T_k^{(i)}$.

Common eigenspace of operators $T_k^{(i)}$

Proposition

For all $k \in \{2, 3, \cdots n\}$, $\{R_i \vdash n : i \in \{1, 2, 3\}\}$, $\tau_1, \tau_2 \in [\![1, C(R_1, R_2, R_3)]\!]$, the Fourier basis elements $Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}$ are eigenvectors of $T_k^{(i)}$:

$$T_k^{(1)} Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} = (\sum_{\sigma \in \mathcal{C}_k} \sigma \otimes 1) Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} = \frac{\chi_{R_1}(T_k)}{d(R_1)} Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} \,, \tag{40}$$

$$T_k^{(2)} Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} = (\sum_{\sigma \in \mathcal{C}_k} 1 \otimes \sigma) Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} = \frac{\chi_{R_2}(T_k)}{d(R_2)} Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} \,, \tag{41}$$

$$T_k^{(3)} Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} = (\sum_{\sigma \in \mathcal{C}_k} \sigma \otimes \sigma) Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} = \frac{\chi_{R_3}(T_k)}{d(R_3)} Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} \,. \tag{42}$$

Proof: $T_k^{(i)}$ is formed by T_k and $Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}$ formed by Q^R_{ij}.

Hamiltonian operators $T_k^{(i)}$ in the centre of $\mathcal{K}(n)$

Proposition

For any $\widetilde{k}_* \in \{k_*(n), k_*(n)+1, \cdots, n\}$ the list of eigenvalues of the Hamiltonian operators $\{T_2^{(1)}, T_3^{(1)}, \cdots, T_{\widetilde{k}_*}^{(1)}; T_2^{(2)}, T_3^{(2)}, \cdots, T_{\widetilde{k}_*}^{(2)}; T_2^{(3)}, T_3^{(3)}, \cdots, T_{\widetilde{k}_*}^{(3)}\}$ uniquely determines the Young diagram triples (R_1, R_2, R_3).

Proof: [Kemp, Ramgoolam '19] $\square$

Conclusion: the common eigenspace of operators in the list $\{T_2^{(1)}, T_3^{(1)}, \cdots, T_{\widetilde{k}_*}^{(1)}; T_2^{(2)}, T_3^{(2)}, \cdots, T_{\widetilde{k}_*}^{(2)}; T_2^{(3)}, T_3^{(3)}, \cdots, T_{\widetilde{k}_*}^{(3)}\}$ generates the space spanned by $\{Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}\}$ that is of dimension $\mathrm{C}(R_1, R_2, R_3)^2$.

Integrality, null space

• The vectors in the Fourier subspace for a triple (R_1, R_2, R_3) solve the following matrix equation $\mathcal{K}(n) \to \mathcal{K}(n)^{\times 3(\widetilde{k}_* - 1)}$ in the basis of ribbon graph vectors:

$$\begin{bmatrix} \mathcal{M}_2^{(1)} - \frac{\chi_{R_1}(T_2)}{d(R_1)} \\ \vdots \\ \mathcal{M}_{\widetilde{k}_*}^{(1)} - \frac{\chi_{R_1}(T_{\widetilde{k}})}{d(R_1)} \\ \mathcal{M}_2^{(2)} - \frac{\chi_{R_2}(T_2)}{d(R_2)} \\ \vdots \\ \mathcal{M}_{\widetilde{k}_*}^{(2)} - \frac{\chi_{R_2}(T_{\widetilde{k}_*})}{d(R_2)} \\ \mathcal{M}_2^{(3)} - \frac{\chi_{R_3}(T_2)}{d(R_3)} \\ \vdots \\ \mathcal{M}_{\widetilde{k}_*}^{(3)} - \frac{\chi_{R_3}(T_{\widetilde{k}_*})}{d(R_3)} \end{bmatrix} \cdot v = \mathbf{0} \tag{43}$$

Integrality, null space

• The normalized characters $\frac{\chi_{R_i}(T_{\widetilde{k}})}{d(R_i)}$ are known, from the Murnaghan-Nakayama construction, to be rational numbers.

• Multiplying the rectangular matrix by an integer to clear the denominators $\rightsquigarrow$ integer rectangular matrix, which we will denote as $\mathcal{L}_{R_1,R_2,R_3} \cdot v = \mathbf{0}$.

• Null spaces of integer matrices have integer null vector bases. These can be interpreted in terms of lattices and can be constructed using integral algorithms.

• End of Step 1.

Outline

HNF: Null-vectors of integer matrices and lattices

• The null space of $\mathcal{L}_{R_1,R_2,R_3}$ has dimension $\mathrm{C}(R_1, R_2, R_3)^2$.

• Theory of **Hermite normal forms (HNF)**: Null spaces of integer matrices have bases given as integer vectors and have an interpretation in terms of sub-lattices.

$$\mathbb{Z}^{|\mathrm{Rib}(n)|} \tag{44}$$

[Refs: H. Cohen, "A Course in Computational Algebraic Number Theory," Springer Science & Business Media, Springer, 2000.
D. Micciancio, "Basic algorithms", available at
`http://cseweb.ucsd.edu/classes/wi10/cse206a/lec2.pdf`.]

• For a fix (R_1, R_2, R_3), let us write $X = \mathcal{L}_{R_1,R_2,R_3}$

HNF of an integer matrix

Let A be a matrix with integer coefficient (integer matrix for short)

- $A = Uh$ where

$\rightarrow U$ is a unimodular, i.e. $\in GL_n(Z)$

$\rightarrow h$ is upper triangular and is unique

- any rows of zeros are located below any other row
- the leading coefficient (pivot) of a non zero row is always strictly to the right of the leading coefficient of the row above it; moreover it is positive
- the element below the pivot are zero and elements above pivot are non negative and strictly smaller than the pivot.

$$\begin{matrix} 1 & 2 & 2 & 1 & 1 & 1 \\ 0 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

- Sequences of allowed operations (unimodular transformation):

- swop of rows
- multiply a row by -1
- add an integer multiple of a row to another row of the matrix.

- The HNF can be computed by usual softwares (Sage, Gap...).

HNF of a integer matrix

$\rightarrow$ Suppose that we want to solve $Xv = 0$, equivalently $v^t X^t = 0$.

$\rightarrow$ X^t transformed to $h = UX^t$

Note that

$\rightarrow$ A null row of h is such that $\forall j$,

$$h_{ij} = 0 = \sum_k U_{ik} (X^t)_{kj} = \sum_k X_{jk} U_{ik} \tag{45}$$

$\rightarrow$ The row vectors of U corresponding to indices i of vanishing rows of h are null vectors of X.

Conclusion:

$$\begin{aligned} &\text{Dim null space of } X = \#\text{ null rows of } h \\ &\text{null space of } X = \text{Span}\{U_i,\ i | h_i = 0\} \end{aligned} \tag{46}$$

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End of Step 2

Theorem (1)

For every triple of Young diagrams (R_1, R_2, R_3) with n boxes, the lattice

$$\mathbb{Z}^{|\mathrm{Rib}(n)|} \tag{47}$$

of integer linear combinations of the geometric basis vectors E_r of $\mathcal{K}(n)$ contains a sub-lattice of dimension $(C(R_1, R_2, R_3))^2$ spanned by a basis of integer null vectors of the operator $\mathcal{L}_{R_1,R_2,R_3}$.

$\rightarrow$ Constructive procedure: treatment of rows of X^T is combinatorial.

$\square$s

Outline

Conjugation operation S

- S satisfies $S^2 = id$.

- Either $S(E_r) = E_r$ (self-conjugate), or distinct pairs $E_s \neq E_t$ are related by $S(E_s) = E_t; S(E_t) = E_s$ (conjugate pair).

- Under the conjugation action

$$S(Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}) = Q^{R_1,R_2,R_3}_{\tau_2,\tau_1} \tag{48}$$

(Proof based on the reality of the D^R_{ij}.)

- Consider $\mathcal{K}(n)$ as an underlying vector space $V^{|\mathrm{Rib}(n)|}$:

$$\begin{aligned} V^{\mathrm{Rib}(n)} &= V^{\mathrm{Rib}(n)}_{S=1} \oplus V^{\mathrm{Rib}(n)}_{S=-1} \\ &= \mathrm{Span}\,\{E^{(s)}_r, (E^{(n)}_r + E^{(\bar{n})}_r)\} \oplus \mathrm{Span}\{E^{(n)}_r - E^{(\bar{n})}_r\} \end{aligned} \tag{49}$$

Using the Wedderburn-Artin decomposition of $\mathcal{K}(n)$ we also have

$$V^{\mathrm{Rib}(n)} = \bigoplus_{R_1,R_2,R_3} V^{\mathrm{Rib}(n):\, R_1,R_2,R_3}\,, \qquad \mathrm{Dim}\, V^{\mathrm{Rib}(n):\, R_1,R_2,R_3} = C(R_1,R_2,R_3)^2 \tag{50}$$

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Conjugation operation S

- S satisfies $S^2 = id$.

- Either $S(E_r) = E_r$ (self-conjugate), or distinct pairs $E_s \neq E_t$ are related by $S(E_s) = E_t; S(E_t) = E_s$ (conjugate pair).

- Under the conjugation action

$$S(Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}) = Q^{R_1,R_2,R_3}_{\tau_2,\tau_1} \tag{48}$$

(Proof based on the reality of the D^R_{ij}.)

- Consider $\mathcal{K}(n)$ as an underlying vector space $V^{|\mathrm{Rib}(n)|}$:

$$\begin{aligned} V^{\mathrm{Rib}(n)} &= V^{\mathrm{Rib}(n)}_{S=1} \oplus V^{\mathrm{Rib}(n)}_{S=-1} \\ &= \mathrm{Span}\,\{E_r^{(s)}, (E_r^{(n)} + E_r^{(\bar{n})})\} \oplus \mathrm{Span}\{E_r^{(n)} - E_r^{(\bar{n})}\} \end{aligned} \tag{49}$$

Using the Wedderburn-Artin decomposition of $\mathcal{K}(n)$ we also have

$$V^{\mathrm{Rib}(n)} = \bigoplus_{R_1,R_2,R_3} V^{\mathrm{Rib}(n):\, R_1,R_2,R_3}\,, \qquad \mathrm{Dim}\, V^{\mathrm{Rib}(n):\, R_1,R_2,R_3} = \mathrm{C}(R_1,R_2,R_3)^2 \tag{50}$$

Refined counting S

- Projecting on $S = +1$-space

$$V_{S=1}^{\mathrm{Rib}(n):\ R_1,R_2,R_3} = \mathrm{Span}\{Q_{\tau,\tau}^{R_1,R_2,R_3} : 1 \le \tau \le C(R_1,R_2,R_3)\} \oplus \mathrm{Span}\{Q_{\tau_1,\tau_2}^{R_1,R_2,R_3} + Q_{\tau_2,\tau_1}^{R_1,R_2,R_3} : 1 \le \tau_1 < \tau_2 \le C(R_1,R_2,R_3)\} \tag{51}$$

$$\mathrm{Dim}\left(V_{S=+1}^{\mathrm{Rib}(n):R_1,R_2,R_3}\right) = \frac{C(R_1,R_2,R_3)(C(R_1,R_2,R_3)+1)}{2} \tag{52}$$

- On $S = -1$-space

$$V_{S=-1}^{\mathrm{Rib}(n):\ R_1,R_2,R_3} = \mathrm{Span}\{Q_{\tau_1,\tau_2}^{R_1,R_2,R_3} - Q_{\tau_2,\tau_1}^{R_1,R_2,R_3} : 1 \le \tau_1 < \tau_2 \le C(R_1,R_2,R_3)\}$$

$$\mathrm{Dim}\left(V_{S=-1}^{\mathrm{Rib}(n):R_1,R_2,R_3}\right) = \frac{C(R_1,R_2,R_3)(C(R_1,R_2,R_3)-1)}{2} \tag{53}$$

End of step 3: Projections and HNF

- Projection $\mathcal{K}(n) \to V^{\mathrm{Rib}(n):R_1,R_2,R_3}_{S=\pm 1}$:

$$\begin{bmatrix} \mathcal{L}_{R_1,R_2,R_3} \\ S \pm 1 \end{bmatrix} v = 0 \tag{54}$$

- HNF construction of $V^{\mathrm{Rib}(n):R_1,R_2,R_3}_{S=\pm 1}$ determines sub-lattices of $\mathbb{Z}^{|\mathrm{Rib}(n)|}$.

- $C(R_1, R_2, R_3)$: Injection $V^{\mathrm{Rib}(n):R_1,R_2,R_3}_{S=-1} \hookrightarrow V^{\mathrm{Rib}(n):R_1,R_2,R_3}_{S=+1}$ a constructive interpretation of $C(R_1, R_2, R_3)$.

Theorem (2)

For every triple of Young diagrams (R_1, R_2, R_3) with n boxes, there are three constructible sub-lattices of $\mathbb{Z}^{|\mathrm{Rib}(n)|}$ of respective dimensions $C(R_1, R_2, R_3)(C(R_1, R_2, R_3) + 1)/2$, $C(R_1, R_2, R_3)(C(R_1, R_2, R_3) - 1)/2$, and $C(R_1, R_2, R_3)$.

- Reminiscent of a result of Burgisser and Ikenmeyer [2008]: `Kron` in `GapP`, i.e. expresses as a difference of two counting problems in $\#\mathrm{P}$.

Outline

Conclusion

• The Kronecker coefficient = dimension of a constructible sub-lattice of ribbon graphs in $\mathbb{Z}^{|\mathrm{Rib}(n)|}$.

• Proof relies on integral hermitian operators with integral eigenvalues acting on a Hilbert space and algebra of ribbon graphs (QM). The HNF algorithm offers the lattice interpretation.

• The method can be generalized to $\mathsf{C}(R_1, R_2, \ldots, R_d)$ and even applied to the LR coefficient (so yet another combinatorial intepretation !).

• A way of exploring quantum supremacy.

• Open problems:

1) Contact with existing literature on the combinatorial intepretation of the Kronecker for particular cases (rectangular, hook shapes).

2) Part of our proof relied on RT: $T_k Q^R_{ij} = \hat{\chi}_R(T_k) Q^R_{ij}$.
$\rightarrow$ Provide a proof that the integer eigenvalues of $T_k : \mathbb{C}(S_n) \rightarrow \mathbb{C}(S_n)$ satisfy Murnaghan-Nakayama rule $\Rightarrow$ Provide an entire combinatorial setting to caracterize the Kronecker.

Exploring quantum supremacy: Vanishing of the Kronecker

- Assume we prepare our ribbon graph states in the lab.
- Total Hamiltonian

$$\mathcal{H} = \sum_{k=2}^{k_*(n)} \sum_{i=1}^{3} a_{i,k} \; T_k^{(i)} \tag{55}$$

where $a_{i,k}$ are integer coefficients such that $\mathcal{H}$ have eignevalues

$$\omega_{R_1,R_2,R_3} = \sum_{i,k} a_{i,k} \frac{\chi_{R_i}(T_k)}{d(R_i)} \tag{56}$$

that distinguish all triples of R_1, R_2, R_3.

- Time evolved ribbon graph state

$$E_r(t) = e^{-i\mathcal{H}t} E_r$$
$$Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} \to e^{-i\mathcal{H}t} Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} \equiv Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}(t) = e^{-it\sum_{k,i} a_{i,k} \widehat{\chi}_{R_i}(T_k)} Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} \tag{57}$$

Exploring quantum supremacy: Vanishing of the Kronecker

- Choose a state Ψ is to take a linear combination of $T_k^{(1)}$

$$\Psi = \sum_{k=2}^{k_*(n)} b_k T_k^{(1)} \qquad \text{such that} \qquad \sum_{k=2}^{k_*(n)} b_k \widehat{\chi}_R(T_k) \neq 0 \quad \forall R \tag{58}$$

b_k can be taken, for example, to be square roots of distinct prime numbers, e.g. $\{\sqrt{2}, \sqrt{3}, \cdots\}$. No cancellations between contributions from different values of k.

- To detect if a Kronecker is vanishing

$$(Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}, \Psi) = \Theta(\mathrm{C}(R_1, R_2, R_3)) \; \frac{d(R_1)d(R_2)d(R_3)}{n!} \sum_k b_k \widehat{\chi}_{R_1}(T_k) \tag{59}$$

is therefore not zero for any non-zero Fourier basis state.

- Observe that the time evolved overlap

$$(E_r(t), \Psi) \tag{60}$$

as $E_r(t)$ expands in terms of Q-states, we could a priori tune b_k to detect a single, could have a component of the form

$$e^{-it \sum_{k,i} a_{i,k} \widehat{\chi}_{R_i}(T_k)} \tag{61}$$

if and only if $\mathrm{C}(R_1, R_2, R_3)$ is non-zero.