A complete characterization of (f_0, f_1)-pairs of 6-polytopes

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Outline

Motivations

- The f-vector of a d-polytope P is the vector $\left(f_0, f_1, \cdots, f_{d-1}\right)$ where faces of P of dimension 0, 1, 2, $d-2$ and $d-1$ are called vertices, edges, subfacets (or ridges), and facets of P, respectively.
- For example the f-vector of a tetrahedron T (a 3-simplex) is $f(T) = (4, 6, 4)$ and the f-vector of the octahedron is $(6, 12, 8)$.
- For a simplicial complex Δ of dimension d, its f-vector is $(f_0(\Delta), \cdots, f_d(\Delta))$; $f_d(\Delta) = 1$. The h-vector is $(h_0(\Delta), \cdots, h_d(\Delta))$ where

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{d+1-k} f_{i-1};$$

$\forall\ k = 0, \cdots, d+1.$

Motivations

- We set $f_{-1} = 1$. The f-vector and the h-vector uniquely determine each other through the linear relation

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{k=0}^{d} h_k t^{d-k}.$$

- The g-theorem says that the h-vector increases until the middle $(g_i = h_i - h_{i-1} \geq 0)$ and $h_i = h_{d-i}$. The h-vector of the tetrahedron is $(1,1,1,1)$ and for the octahedron is $(1,3,3,1)$ and is palindromic. $f_0 - 1$, $f_1 - (f_0 - 1)$, $f_2 - f_1 + f_0 - 1 = 1$. Euler formula $\sum_{i=0}^{d}(-1)^i f_i = 1 - (-1)^d$.

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$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 6 & & & \\
 & & 1 & & 5 & & 12 & & \\
 & 1 & & 4 & & 7 & & 8 & \\
\text{h=} \quad 1 & & 3 & & 3 & & 1 & &
\end{array}$$

Motivations

- In 1980 Billera & Lee and Stanley have proved the characterization of the f-vectors of simplicial and of simple polytopes conjectured by McMullen in 1971 through the famous "g-theorem".
- Grünbaum, Barnette and Barnette-Reay have characterized for any $0 \leq i < j \leq 3$ the following sets:
$$\left\{ (f_i, f_j) : P \text{ is a 4-polytope } \right\}.$$
- Steinitz found the characterization for $d = 3$, $\mathcal{E}^3 = \left\{ (f_0, f_1) : \frac{3}{2} f_0 \leq f_1 \leq 3f_0 - 6 \right\}$.

Motivations

- For $d = 4, 5$ the results is given by the set $S = \left\{(f_0, f_1) : \frac{d}{2} f_0 \leq f_1 \leq \binom{f_0}{2}\right\}$ from which some exceptions have been removed.
- Grünbaum proved the case $d = 4$ by removing four exceptions: $(6, 12)$, $(7, 14)$, $(8, 17)$ and $(10, 20)$.
- The case $d = 5$ becomes more complicated and has been proved in two different ways by G. Pineda-Villavicencio, J. Ugon and D. Yost, and more recently by T. Kusunoki and S. Murai. For this case, exceptions are infinitely many:

$$\begin{aligned}\mathcal{E}^5 \quad = \quad & \left\{(f_0, f_1) : \frac{5}{2} f_0 \leq f_1 \leq \binom{f_0}{2}\right\} \setminus \\ & \left(\left\{\left(f_0, \left\lfloor \frac{5}{2} f_0 + 1 \right\rfloor\right) : f_0 \geq 7\right\} \cup \left\{(8, 20), (9, 25), (13, 35)\right\}\right), \qquad (1)\end{aligned}$$

where $\lfloor r \rfloor$ denotes the integer part of a rational number r.

Definitions and background

- The excess degree or excess $\Sigma(P)$ of a d-polytope P is defined as the sum of the excess degrees of its vertices and given by $\epsilon(P) = 2f_1 - df_0 = \sum_u (deg(u) - d)$.
- (Proposition 1 G. Pineda-Villavicencio, J. Ugon and D. Yost) Let P be a d-polytope. Then the smallest values of $\Sigma(P)$ are 0 and $d-2$.
- We set for all d-dimensional polytopes $\phi(v,d) = \frac{1}{2}dv + \frac{1}{2}(v-d-1)(2d-v)$.
- (Proposition 2) Let P be a d-polytope. If $f_0(P) \leq 2d$, then $f_1(P) \geq \phi(f_0(P), d)$. If $d \geq 4$, then $(f_0(P), f_1(P)) \neq (d+4, \phi(d+4,d)+1)$.
- D. W. Barnette proved that for all d-dimensional simplicial polytope the following inequality holds: $f_{d-1} \geq (d-1)f_0 - (d+1)(d-2)$.

(f_0, f_1)-vectors pairs for 6-polytopes

- If P is a 6-polytope having a simple vertex v and Q the 6-polytope obtained from P by truncating the vertex v then

$$f_0(Q) = f_0(P) + 5 \text{ and } f_1(Q) = f_1(P) + 15.$$

  We can prove that if for a 6-polytope P we have $f_1(P) \leq \frac{7}{2} f_0(P)$ then P has at least one simple vertex.

- (Theorem 1) The set of (f_0, f_1)-vectors pairs for 6-polytopes is given by

$$\mathcal{E}^6 = \left\{ (f_0, f_1) : 3f_0 \leq f_1 \leq \binom{f_0}{2} \right\} \setminus \left(\left\{ \left(f_0, 3f_0 + 1 \right) : f_0 \geq 7 \right\} \cup \left\{ (8,24); (9,27); (9,29); (10,30); (10,32); (10,34); (11,33); (12,38); (12,39); (13,39); (14,42); (14,44); (15,47); (18,54); (19,57); (17,53); (20,62) \right\} \right).$$

Proof

- If P is a d-polytope with $d > 4$, then

$$f_1(P) \neq \left\lfloor \frac{d}{2} f_0(P) + 1 \right\rfloor. \tag{3}$$

- Assume that $f_1(P) = \left\lfloor \frac{d}{2} f_0(P) + 1 \right\rfloor$. If $f_0(P)$ is even then $2f_1(P) - df_0(P) = 2$ and $0 < 2 < d - 2$ which is impossible since from Proposition 1, $\Sigma(P)$ can not take any value between 0 and $d - 2$. If $f_0(P)$ is odd then $2f_1(P) - df_0(P) = 1$ and $0 < 1 < d - 2$ which is also impossible.
- The following relations hold $(8, 24); (9, 27); (9, 29); (10, 30); (10, 32); (10, 34); (11, 33) \notin \mathcal{E}^6$ and $\left(f_0 + 1, \left\lfloor \frac{7}{2} f_0 + 1 \right\rfloor\right) \notin \mathcal{E}^6$ for $f_0 = 7, 8, 9$.

Proof

- The fact that $(10, 34) \notin \mathcal{E}^6$ is given by Proposition 2 (2) and all the remaining are given by Proposition 2 (1).
- (Lemma 1) The following result is obtained from pyramids over 5-polytopes

$$\left\{(f_0, f_1) : \frac{7}{2}f_0 - \frac{7}{2} \leq f_1 \leq \binom{f_0}{2}\right\} \setminus \left(\left\{\left(f_0 + 1, \left\lfloor \frac{7}{2}f_0 + 1 \right\rfloor\right) : f_0 \geq 7\right\} \cup \left\{(9, 28), (10, 34), (14, 48)\right\}\right) \subset \mathcal{E}^6. \quad (4)$$

- There is no 6-polytope with 11 vertices and 36 edges and no 6-polytope with 12 vertices and 38 edges.

Proof

- The following exist $(13, 43), (14, 48) \in \mathcal{E}^6$.
- There is no 6-polytope with 12 vertices and 39 edges.
- (Lemma 2) For an odd integer $f_0 \geq 12$ we have $\left(f_0 + 1, \left\lfloor \frac{7}{2} f_0 + 1 \right\rfloor\right) \in \mathcal{E}^6$. Furthermore if $\left(f_0 + 1, \left\lfloor \frac{7}{2} f_0 + 1 \right\rfloor\right) \in \mathcal{E}^6$, then $\left(f_0 + 7, \left\lfloor \frac{7}{2}(f_0 + 6) + 1 \right\rfloor\right) \in \mathcal{E}^6$.
- Suppose that f_0 is odd. If $f_0 \geq 12$ then $f_0 - 4 \geq 8$ and from Lemma 1, $(f_0 - 4, \left\lfloor \frac{7}{2}(f_0 - 4) \right\rfloor) \in \mathcal{E}^6$. Also $\left\lfloor \frac{7}{2}(f_0 - 4) \right\rfloor < \frac{7}{2}(f_0 - 4)$ as $f_0 - 4$ is odd then $\left(f_0 + 1, \left\lfloor \frac{7}{2} f_0 + 1 \right\rfloor\right) \in \mathcal{E}^6$ by truncation of simple vertex.

- Let P be a 6-polytope with (f_0, f_1)-pairs equal to $\left(f_0(P)+1, \left\lfloor \frac{7}{2} f_0(P)+1 \right\rfloor\right) \in \mathcal{E}^6$ then after the truncation of a simple vertex of P and a pyramid over a simplex facet of the resulting polytope we obtain a 6-polytope Q with $f_0(Q) = f_0(P) + 7$ and $f_1(Q) = \left\lfloor \frac{7}{2}(f_0(P)+6)+1 \right\rfloor$.
- For any integer f_0 satisfying $f_0 \geq 12$, $\left(f_0+1, \left\lfloor \frac{7}{2} f_0 + 1 \right\rfloor\right) \in \mathcal{E}^6$.
- Assume that $f_0 \geq 12$. From Lemma 2 it is enough to check the result for $f_0 = 12, 13, 14, 15, 16, 17$. The cases $f_0 = 12, 13, 15, 17$ come from Lemma 2. We now consider $f_0 = 14, 16$ which are $(15, 50)$ and $(17, 57)$.

- Consider the 6-polytope P with $f_0(P) = 10$ and $f_1(P) = 35$ obtained from a pyramid over a 5-polytope Q.
- If we assume that P has no simple vertex then each of its vertices has degree 7 since $\sum_{v \in P} deg(v) = 70$ and this is impossible since taking a pyramid over Q implies that P has a vertex of degree 9.
- Then P has a simple vertex which truncation gives a polytope P' with $f_0(P') = 15$ and $f_1(P) = 50$. Hence $(15, 50) \in \mathcal{E}^6$.

  Let R be a 6-polytope with $f_0(R) = 12$ and $f_1(P) = 42$ obtained from a pyramid over a 5-polytope. The same procedure as above gives $(17, 57) \in \mathcal{E}^6$.
- The following polytopes pairs do not exist:

$$(13, 39); (14, 42); (14, 44); (15, 47); (18, 54), (19, 57) \notin \mathcal{E}^6.$$

- Consider the case $(19, 57)$ which is a simple polytope if it exists. Let P be such polytope. The dual $P^\star$ of P is a simplicial polytope with f-vector sequence $(f_0, f_1, f_2, f_3, f_4, f_5)$ where $f_4 = 57$ and $f_5 = 19$.
- For all d-dimensional simplicial polytope the following inequality holds: $f_{d-1} \geq (d-1)f_0 - (d+1)(d-2)$. Then $f_5 \geq 5f_0 - 28$ implies that $f_0 = 8$ or $f_0 = 9$.
- The g-theorem for simplicial polytopes says that the sequence of integers $(h_0, \cdots, h_7)$ is the h-vector of $P^\star$. We also have $h_i = h_{7-i}\ \forall\ i = 0, \cdots, 7$ and now compute the numbers $h_i's$ and obtain:
- $$\begin{aligned} h_1 &= -7 + f_0, \\ h_2 &= 21 - 6f_0 + f_1, \\ h_3 &= -35 + 15f_0 - 5f_1 + f_2, \\ h_4 &= 35 - 20f_0 + 10f_1 - 4f_2 + f_3, \\ h_5 &= -21 + 15f_0 - 10f_1 + 6f_2 - 3f_3 + f_4, \\ h_6 &= 7 - 6f_0 + 5f_1 - 4f_2 + 3f_3 - 2f_4 + f_5. \end{aligned} \tag{5}$$

- From $h_1 = h_6$ and $h_2 = h_5$ we get $f_2 = \frac{1}{2}(28 - 14f_0 + 6f_1 + f_4 - f_5)$ and the system of equations $h_3 = h_4$; $h_2 = h_5$ also gives $f_2 = \frac{1}{9}(168 - 84f_0 + 34f_1 + f_4)$.
- Equaling these two expressions of f_2 we get $f_1 = \frac{1}{14}(-84 + 42f_0 + 7f_4 - 9f_5)$ which is not an integer for $f_0 = 8, 9$. In conclusion $(19, 57) \notin \mathcal{E}^6$.
- The following pairs are possible:

$$(15,45); (15,49); (16,48); (17,54); (19,59); (23,69); (24,72); (27,83); (35,107) \in \mathcal{E}^6.$$

- We set

$$X' = \Big\{(8,24); (9,27); (9,29); (10,30); (10,32); (10,34); (11,33); (11,36); (12,38); (12,39); (13,39); (14,42); (14,44); (15,47); (18,54); (17,53); (19,57); (20,62)\Big\}$$

- For $f_0 \geq 7$; if $(f_0, f_1) \notin X'$ and $f_1 \in \{3f_0\} \cup \left]3f_0 + 1, \frac{7}{2}f_0 - \frac{7}{2}\right[$ then $(f_0, f_1) \in \mathcal{E}^6$.
- The cases $(17, 53); (20, 62)$ are unfeasible and $(22, 68); (25, 77); (30, 92) \in \mathcal{E}^6$ holds. $\square$
- Let $\mathcal{E}^d_{>3d-10}$ be the set of d-polytopes whose excess degree is larger than $3d - 10$. For $d = 4$, the set $\mathcal{E}^4_{>2}$ of 4-polytopes whose excess degree is larger than 2 is given by:

$$\mathcal{E}^4_{>2} = \left\{ (f_0, f_1) : 1 + 2f_0 < f_1 \leq \binom{f_0}{2} \right\}.$$

The case $d = 7$

- In the same way for $d = 5, 6$ we obtain:

$$\mathcal{E}^5_{>5} = \left\{ (f_0, f_1) : \frac{5}{2} + \frac{5}{2} f_0 < f_1 \leq \binom{f_0}{2} \right\},$$

  and

$$\mathcal{E}^6_{>8} = \left\{ (f_0, f_1) : 4 + 3f_0 < f_1 \leq \binom{f_0}{2} \right\}.$$

- (Theorem 2) Let $\mathcal{E}^7$ be the set of (f_0, f_1)-pairs of 7-polytopes. For $v = (p, q)$ such that $p \geq 8$ and $\frac{7}{2}p \leq q \leq \binom{p}{2}$, if $v \notin \mathcal{E}^7$ then $\epsilon_7(v) \leq 4 \times 7 - 10 = 11$. In other words the set of (f_0, f_1)-vector pairs for 7-polytopes with excess strictly larger than 11 is given by

$$\mathcal{E}^7_{>11} = \left\{ (f_0, f_1) : \frac{7}{2} f_0 + \frac{11}{2} < f_1 \leq \binom{f_0}{2} \right\}.$$

  With $\epsilon_d(v) = 2q - dp$.

The case $d = 7$

- From the previous section we had

$$\mathcal{E}^6 = \left\{(f_0, f_1) : 3f_0 \leq f_1 \leq \binom{f_0}{2}\right\} \setminus \Bigg(\left\{\Big(f_0, 3f_0+1\Big) : f_0 \geq 7\right\} \cup \left\{(8,24); (9,27); (9,29); (10,30); (10,32); (10,34); (11,33); (12,38); (12,39); (13,39); (14,42); (14,44); (15,47); (18,54); (19,57); (17,53); (20,62)\right\}\Bigg) \tag{6}$$

- A pyramid over the 6-polytopes gives:

$$\left\{(f_0, f_1) : 4f_0 - 4 \leq f_1 \leq \binom{f_0}{2}\right\} \setminus \Bigg(\left\{\Big(f_0+1, 4f_0+1\Big) : f_0 \geq 7\right\} \cup \left\{(9,32); (10,36); (10,38); (11,40); (11,42); (11,44); (12,44); (13,50); (13,51); (14,52); (15,56); (15,58); (16,62); (18,70); (19,72); (20,76); (21,82)\right\}\Bigg) \subset \mathcal{E}^7.$$

Proof

- A direct computation shows that $\epsilon_7((f_0+1, 4f_0+1)) > 11$ if and only if $f_0 > 17$. Assume that $f_0 > 17$ and let us prove that $(f_0-6, 3f_0-14) \in \mathcal{E}^6$.
- We have $\epsilon_6((f_0-6, 3f_0-14)) = 8$ and if $(f_0-6, 3f_0-14) \notin \mathcal{E}^6$ then $(f_0-6, 3f_0-14) = (10, 34)$, because $(10, 34)$ is the only vector not in $\mathcal{E}^6$ with excess equal to 8.
- Therefore we get $f_0 = 16$ which is a contradiction. In conclusion for $f_0 > 17$ there is a 6-polytope P with (f_0, f_1)-pair $(f_0-6, 3f_0-14)$; and a pyramid over P give a 7-polytope Q having (f_0, f_1)-vector which is equal to $(f_0-5, 4f_0-20)$.
- As $4(f_0-5) < (4f_0-20)+1$ the polytope Q has a simple vertex whose truncation gives a 7-polytope having (f_0, f_1)-pair equals $(f_0+1, 4f_0+1)$.

Proof

- We can conclude that all the 7-polytopes with excess greater than 11 and with (f_0, f_1)-pairs in $\left\{\left(f_0+1, 4f_0+1\right) : f_0 \geq 7\right\}$ exist.
- Let us focus on the set

$$L = \Big\{(9,32); (10,36); (10,38); (11,40); (11,42); (11,44); (12,44); (13,50); (13,51); (14,52); (15,56); (15,58); (16,62); (18,70); (19,72); (20,76); (21,82)\Big\}.$$

- The only vectors $v = (p, q) \in L$ with $\epsilon_7(v) > 11$ are

$$v = (p, q) = (16,62); (18,70); (20,76); (21,82); (23,90); (26,102); (31,123).$$

Proof

- For $v = (p, q) = (16, 62); (20, 76); (21, 82); (26, 102); (31, 123)$, we compute $v' = (p - 8, q - p - 20) = (8, 26); (12, 36); (13, 41); (18, 56); (23, 72) \in \mathcal{E}^6$.
- Then their exist 6-polytopes $P_{v'}$ whose (f_0, f_1)-pairs are equal to v'. A pyramid over them give 7-polytopes having (f_0, f_1)-pairs equal to $(p - 7, q - 28) = (9, 34); (13, 48); (14, 54); (19, 74); (24, 95)$.
- In each case we observe that $q - 28 < 4(p - 7)$ which means that each of them has a simple vertex whose truncation give 7-polytopes with (f_0, f_1)-pairs equal to $(p - 1, q - 7)$. As truncations of simple vertices generate simplex facets then pyramids on these give the result.

Proof

- Consider $v = (18, 70)$. There is a 6-polytope R with $(f_0, f_1) = (10, 35)$. A pyramid over R gives a 7-polytope R' having (f_0, f_1)-vector equal to $(11, 42)$. As $42 < 4 \times 11$ then R' has a simple vertex whose truncation gives a 7-polytopes R'' with $(f_0(R''), f_1(R'')) = (17, 63)$.
- The truncation of a simple vertex in R'' with generate a simplex facet F and a pyramid other F gives a 7-polytope with (f_0, f_1)-vector equal to $(18, 70)$. The same method works for $(23, 90)$.
- We now turn to the pair $v = (f_0, f_1)$ with $f_0 \geq 8$ and $f_1 \in]\frac{7}{2}f_0, 4f_0 + 1[$. The condition $\epsilon_7(v) > 11$ implies that $f_1 \geq \frac{11}{2} + \frac{7}{2}f_0$ and then we need to discuss two cases: $\frac{11}{2} + \frac{7}{2}f_0 > 4f_0 - 4$ and $\frac{11}{2} + \frac{7}{2}f_0 < 4f_0 - 4$.

Proof

- If $\frac{11}{2} + \frac{7}{2} f_0 > 4f_0 - 4$ then there is nothing else to prove as we end up in the pyramid case. Suppose that $\frac{11}{2} + \frac{7}{2} f_0 < 4f_0 - 4$ i.e. $f_0 > 19$ and set for k, $X_k^7 = \{(k, f_1); \frac{11}{2} + \frac{7}{2} k < f_1 < 4k - 4\}$.
- We can prove by truncation that if $X_k^7 \subset \mathcal{E}_{>11}^7$, then $X_{k+6}^7 \subset \mathcal{E}_{>11}^7$. To prove that each vector (f_0, f_1) satisfying this condition defines a 7-polytope it is sufficient to show that $X_k^7 \subset \mathcal{E}_{>11}^7$ for $k = 8, \cdots, 13$. Which have already been solved.
- Finally we conclude that all the pairs (p, q) with $p \geq 8$, $\epsilon_7(v) > 11$ and $\frac{7}{2} p \leq q \leq \binom{p}{2}$, characterize 7-polytopes. In other words the set of (f_0, f_1)-vectors pair for 7-polytopes with excess strictly larger than 11 is given by

$$\mathcal{E}_{>11}^7 = \left\{ (f_0, f_1) : \frac{7}{2} f_0 + \frac{11}{2} < f_1 \leq \binom{f_0}{2} \right\}.$$

Conjecture

Let $d \geq 4$ be an integer and $\mathcal{E}^d$ be the set of (f_0, f_1)-pairs of d-polytopes. For $v = (p, q)$ such that $p \geq d + 1$ and $\frac{d}{2} p \leq q \leq \binom{p}{2}$, if $v \notin \mathcal{E}^d$ then $2q - dp \leq 4d - 10$. In other words the set of (f_0, f_1)-pairs for d-polytopes; $d \geq 4$ with excess strictly larger than $3d - 10$ is given by

$$\mathcal{E}^d_{>3d-10} = \left\{ (f_0, f_1) : \frac{d}{2} f_0 + \frac{3d - 10}{2} < f_1 \leq \binom{f_0}{2} \right\}.$$