This presentation contains animations which require PDF browser which accepts JavaScript.

For best results use Acrobat Reader.

Robinson-**S**chensted-**K**nuth algorithm

Start with two empty tableaux. Read letters of the word one after another. With each letter proceed as follows:

1. start with the bottom row of the *insertion tableau P*,
2. insert the letter to the leftmost box in this row which contains a number which is **bigger** than the one which you want to insert,
3. if you had to bump some letter, this bumped letter must be inserted in to the next row according to the rule number 2,
4. if you inserted a letter to an empty box in the *insertion tableau P,* make a mark about the position of this box in the *recording tableau Q* and proceed to the next letter of the word.

74	99		
23	53	70	
16	37	41	82

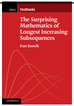
insertion tableau $P(w)$

8	9		
4	6	7	
1	2	3	5

recording tableau $Q(\mathbf{w})$

$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Further reading



Dan Romik
„*The Surprising Mathematics of Longest Increasing Subsequences*"

legal PDF file available on author's website

Want more? Visit

—> psniady.impan.pl/surprising

never have seen RSK in your life?

print the handout!

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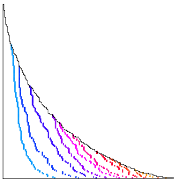
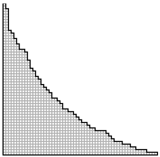
Poisson limit of bumping routes in the Robinson–Schensted correspondence

Piotr Śniady

IMPAN Toruń

joint work with Mikołaj Marciniak and Łukasz Maślanka



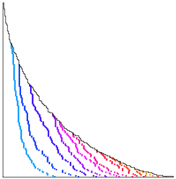


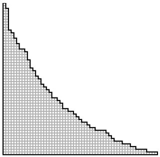
handout, slides $\longrightarrow$ psniady.impan.pl/bumping

what can you say about RSK with random input?

we apply Robinson–Schensted algorithm
to a very long random sequence;

- what can you say about
  bumping routes?
- what is the trajectory
  of your favorite number in the insertion tableau?





Robinson–Schensted–Knuth algorithm is a bijection. . .

input:

- sequence $\mathbf{w} = (w_1, \ldots, w_n)$

output:

- semistandard tableau P,
- standard tableau Q,

P and Q have the same shape with n boxes

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example:

w = (23, 53, 74, 16, 99, 70, 82, 37, 41)

74	99		
23	53	70	
16	37	41	82

insertion tableau P(w)

8	9		
4	6	7	
1	2	3	5

recording tableau Q(w)

Robinson–Schensted–Knuth algorith — the induction step

74	99		
23	53	70	
16	37	41	82

insertion tableau $P(\mathrm{w})$

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recording tableau $Q(\mathrm{w})$

$$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41)$$

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Robinson–Schensted–Knuth algorith — the induction step

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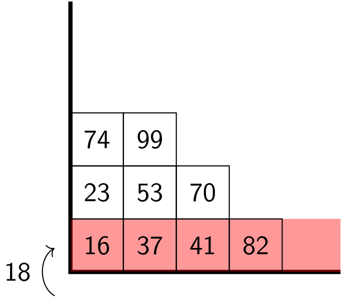
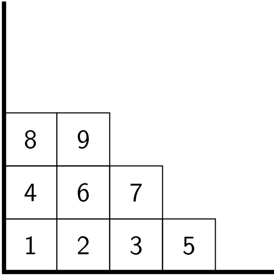
Robinson–Schensted–Knuth algorith — the induction step



insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

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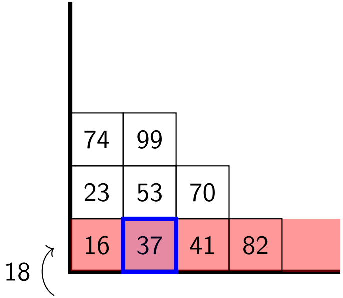
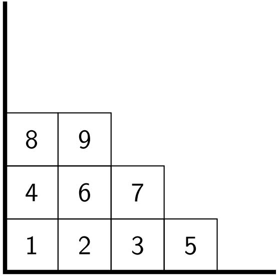
Robinson–Schensted–Knuth algorith — the induction step



insertion tableau $P(\mathrm{w})$



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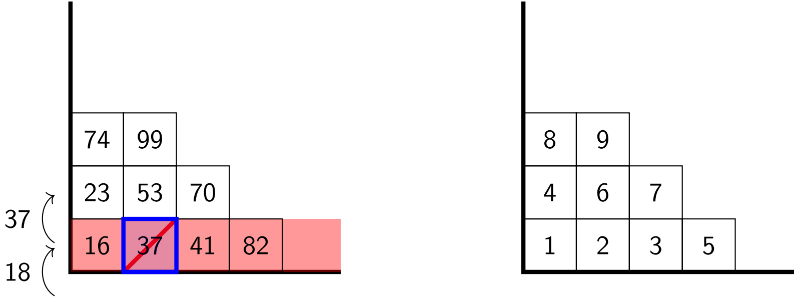
$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson–Schensted–Knuth algorith — the induction step



insertion tableau $P(\mathrm{w})$

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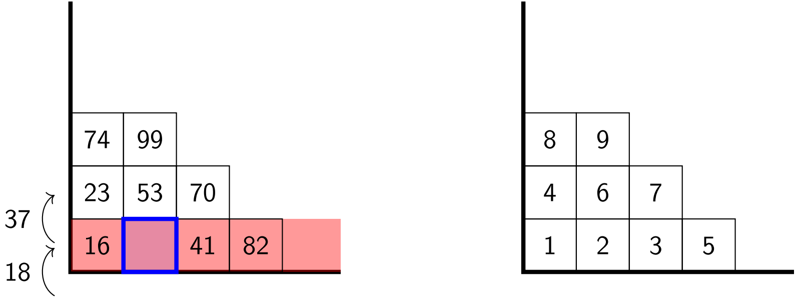
Robinson–Schensted–Knuth algorith — the induction step



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one page handout ⟶ `psniady.impan.pl/bumping`

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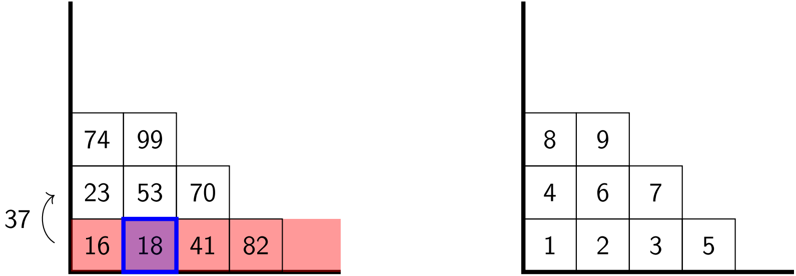

insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

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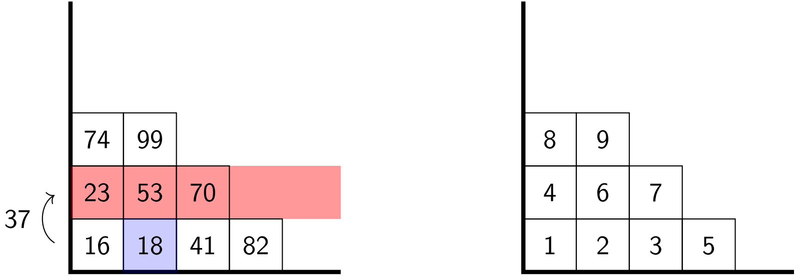
Robinson–Schensted–Knuth algorith — the induction step



insertion tableau $P(w)$ recording tableau $Q(w)$

$$w = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$$

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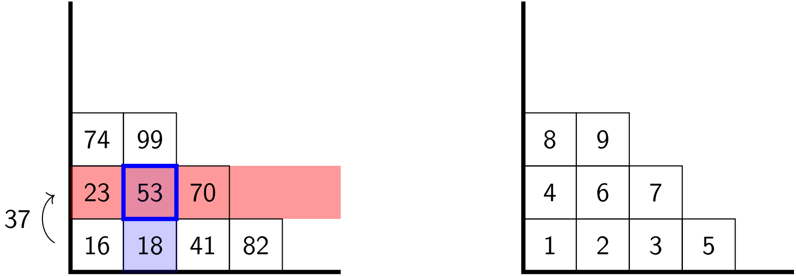
Robinson–Schensted–Knuth algorith — the induction step



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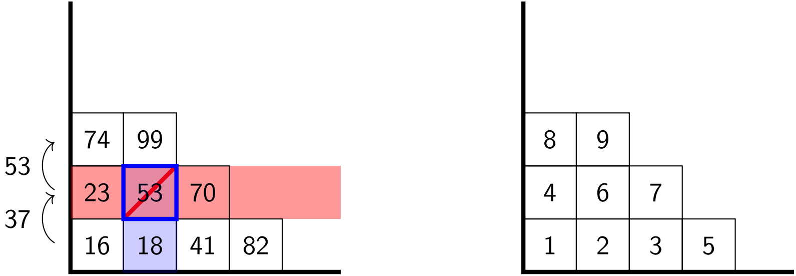

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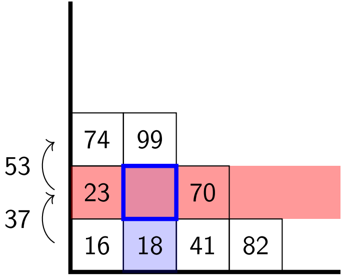
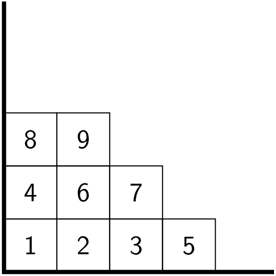


insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

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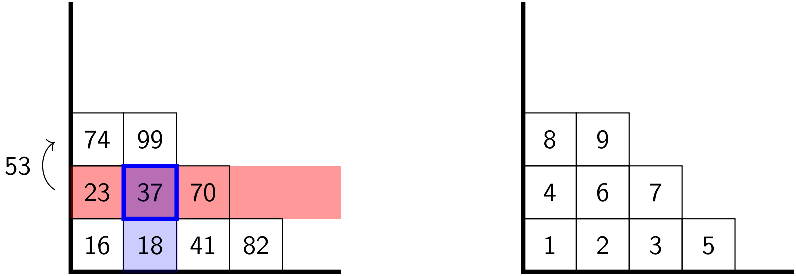
Robinson–Schensted–Knuth algorith — the induction step



insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

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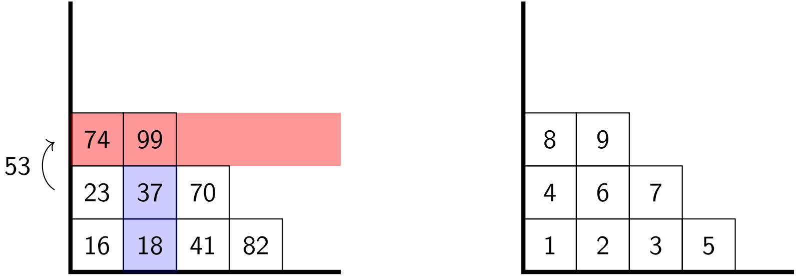

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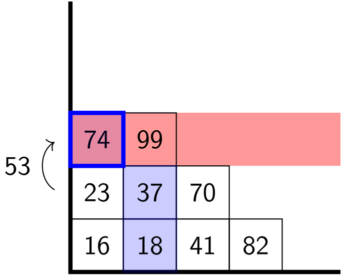

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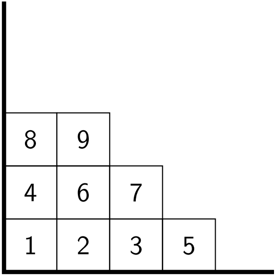

recording tableau $Q(\mathrm{w})$

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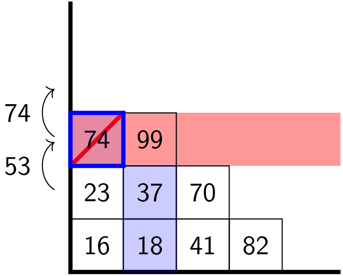

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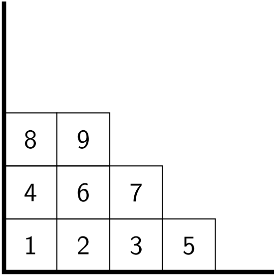

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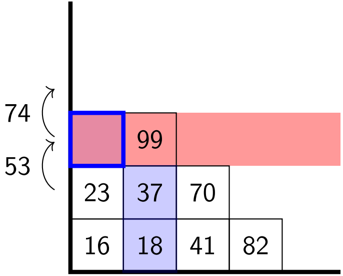

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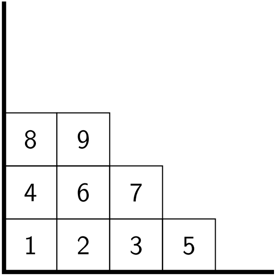

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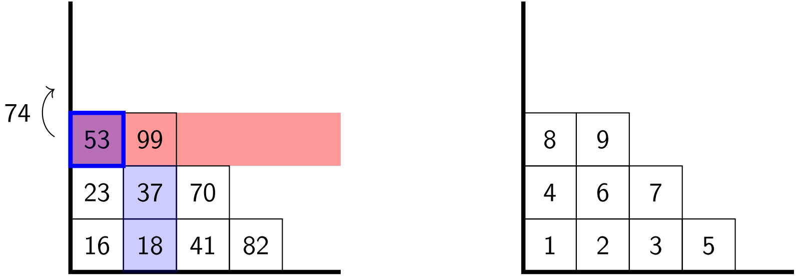

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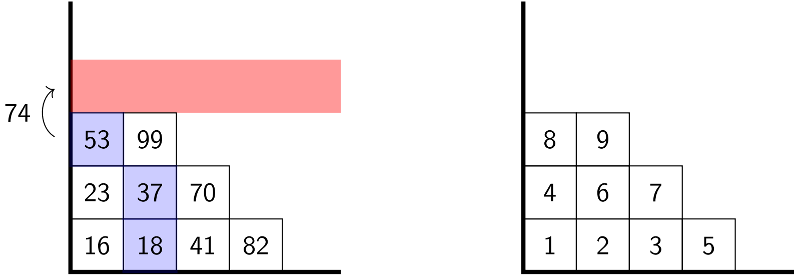

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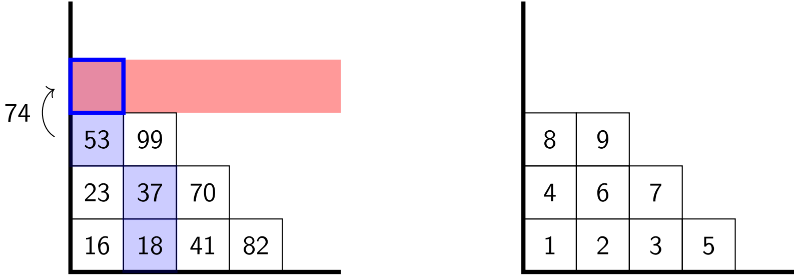

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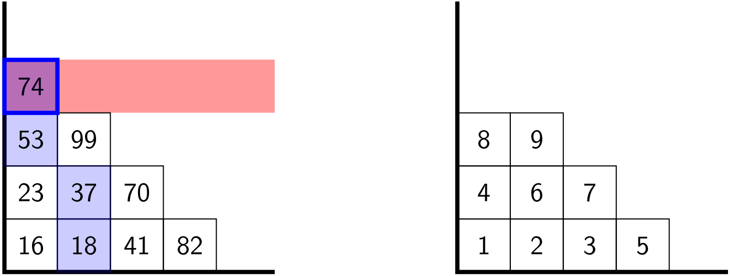

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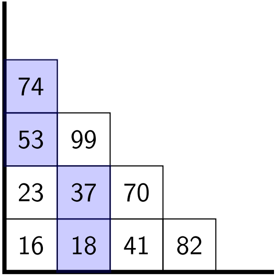
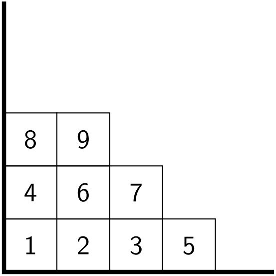
Robinson–Schensted–Knuth algorith — the induction step



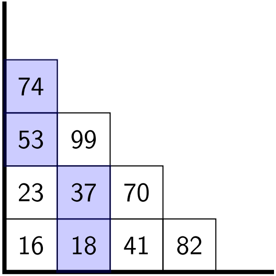
insertion tableau $P(\mathrm{w})$



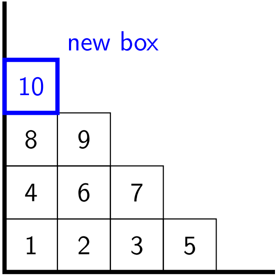
recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

Robinson–Schensted–Knuth algorith — the induction step



insertion tableau $P(\mathrm{w})$



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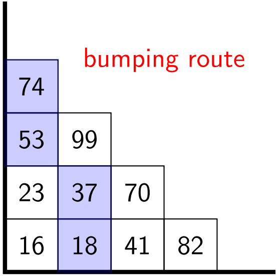
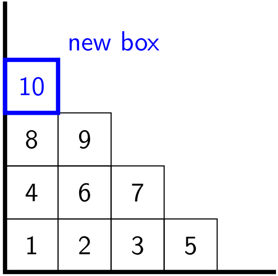


insertion tableau $P(w)$



recording tableau $Q(w)$

$w = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$

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Robinson–Schensted–Knuth algorith — the induction step

74			
53	99		
23	37	70	
16	18	41	82

insertion tableau $P(\mathrm{w})$

10			
8	9		
4	6	7	
1	2	3	5

recording tableau $Q(\mathrm{w})$

$$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$$

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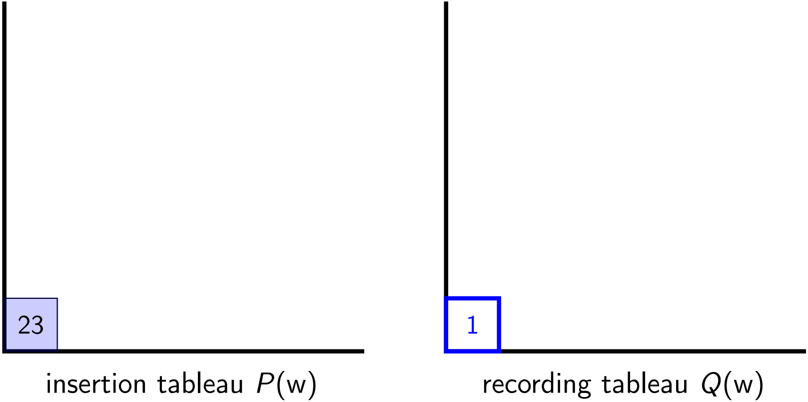
Robinson–Schensted–Knuth algorithm

insertion tableau $P(\mathrm{w})$

recording tableau $Q(\mathrm{w})$

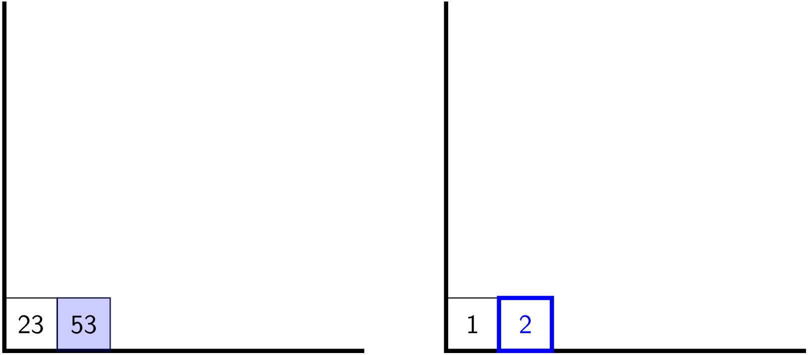
$\mathrm{w} = \emptyset$

Robinson–Schensted–Knuth algorithm



$\mathrm{w} = (23)$

Robinson–Schensted–Knuth algorithm

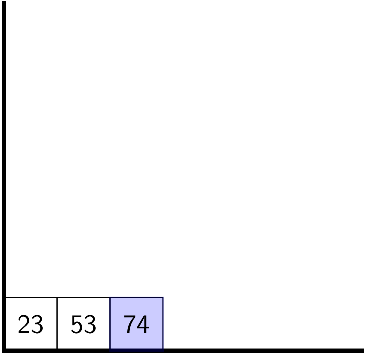


insertion tableau $P(\mathrm{w})$

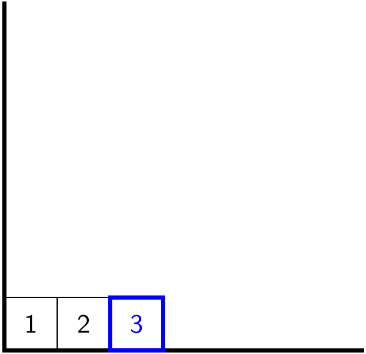
recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53)$

Robinson–Schensted–Knuth algorithm



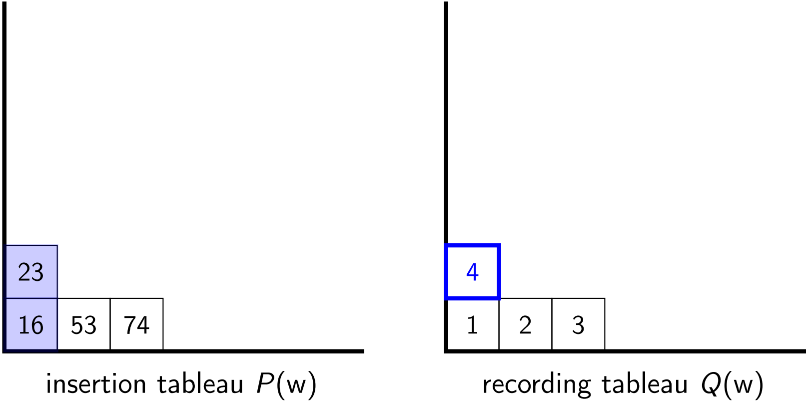
insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

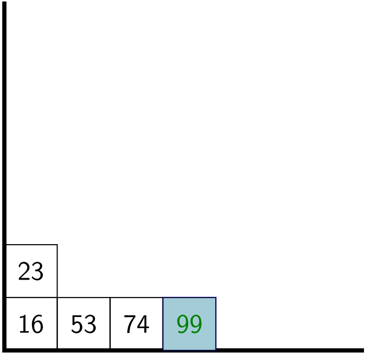
$\mathrm{w} = (23,\ 53,\ 74)$

Robinson–Schensted–Knuth algorithm

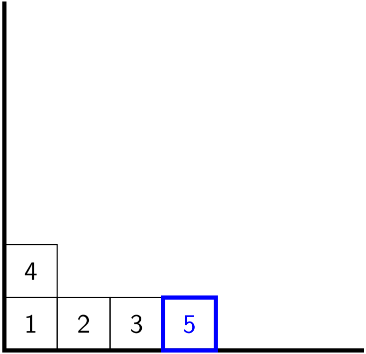


$\mathrm{w} = (23, 53, 74, 16)$

Robinson–Schensted–Knuth algorithm



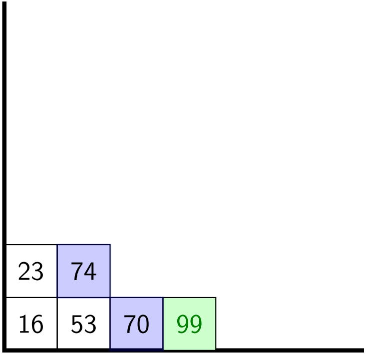
insertion tableau $P(\mathrm{w})$



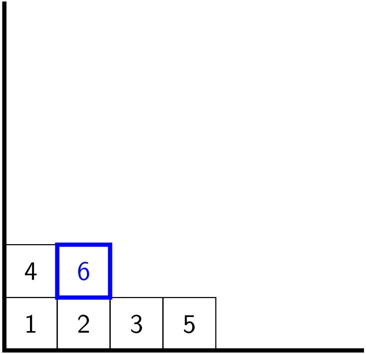
recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99)$

Robinson–Schensted–Knuth algorithm



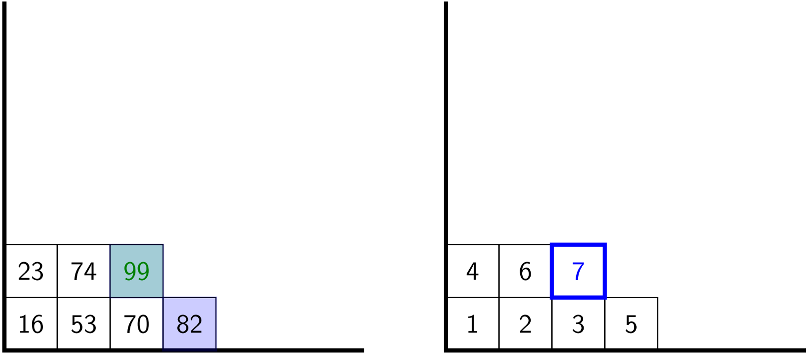
insertion tableau $P(\text{w})$



recording tableau $Q(\text{w})$

$\text{w} = (23, 53, 74, 16, 99, 70)$

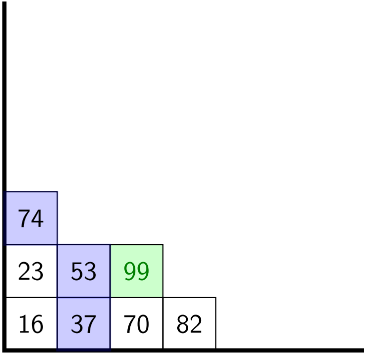
Robinson–Schensted–Knuth algorithm



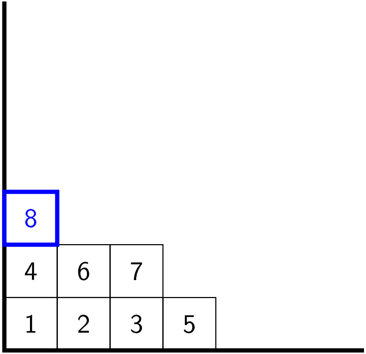
insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82)$

Robinson–Schensted–Knuth algorithm



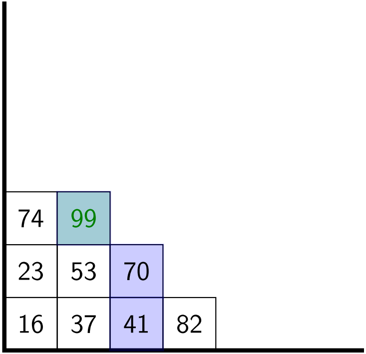
insertion tableau $P(\mathrm{w})$



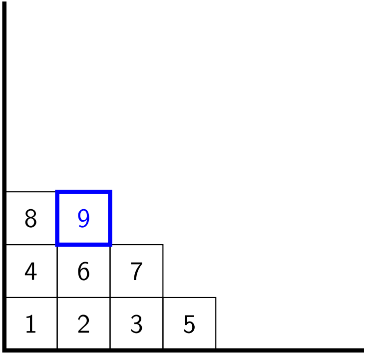
recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37)$

Robinson–Schensted–Knuth algorithm



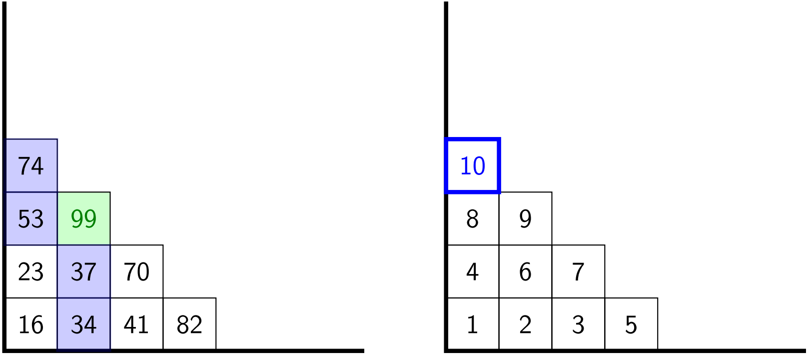
insertion tableau $P(w)$



recording tableau $Q(w)$

$w = (23, 53, 74, 16, 99, 70, 82, 37, 41)$

Robinson–Schensted–Knuth algorithm



insertion tableau $P(\mathrm{w})$ — recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34)$

Robinson–Schensted–Knuth algorithm

74			
53	99		
23	37	70	82
16	34	41	73

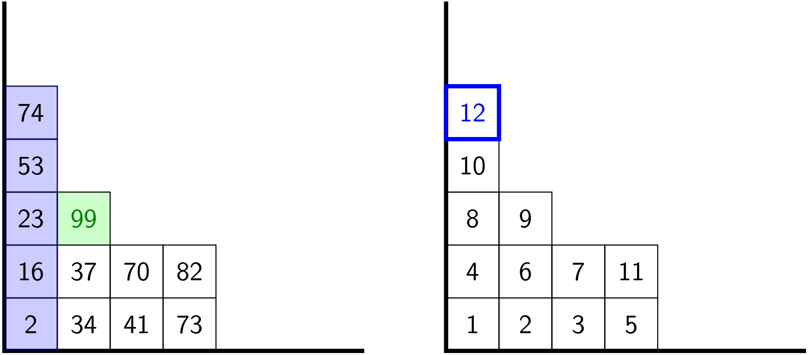
insertion tableau $P(\mathrm{w})$

10			
8	9		
4	6	7	11
1	2	3	5

recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73)$

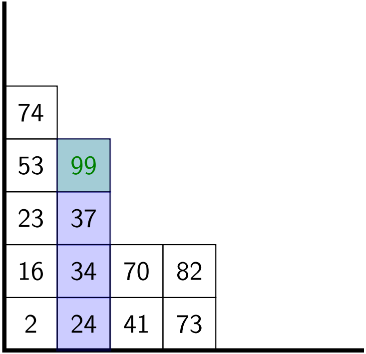
Robinson–Schensted–Knuth algorithm



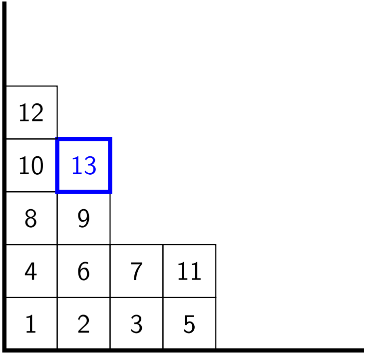
insertion tableau $P(\mathrm{w})$ recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73, 2)$

Robinson–Schensted–Knuth algorithm



insertion tableau $P(\mathrm{w})$



recording tableau $Q(\mathrm{w})$

$\mathrm{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73, 2, 24)$

why RSK?

- understanding irreducible representations of the symmetric groups,
- tool for Littlewood–Richardson coefficients,
- RSK applied to random inputs (of various types) produces lots of interesting random walks, famous random Young diagrams and Young tableaux,
- amazing bijection with lots of magic symmetries,

---

magic symmetry:

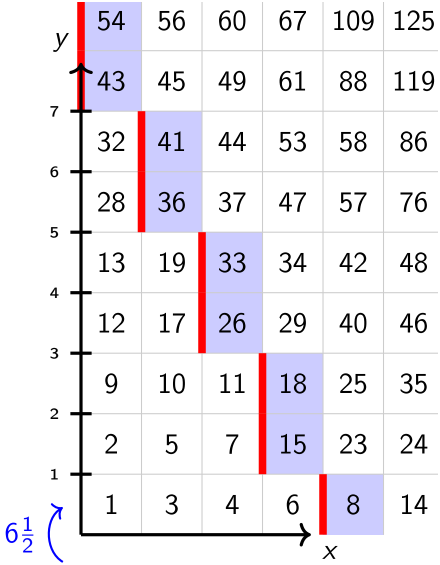
if $w_1, \dots, w_n$ are all different, then

$$\underbrace{P(w_n, \dots, w_1)}_{\text{the word } w \text{ read backwards}} = \Big[P(w_1, \dots, w_n) \Big]^{\text{transpose}}$$

let $\xi_1, \xi_2, \ldots$
be independent random variables
with the uniform distribution
on the unit interval $[0, 1]$

what can you say about
the infinite bumping route

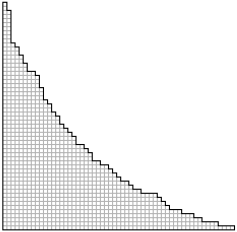
$$Q(\xi_1, \xi_2, \ldots) \longleftarrow m + \frac{1}{2} \;?$$



let $\xi_1, \xi_2, \ldots$ be independent random variables
with the uniform distribution on the unit interval $[0, 1]$

where is your favourite number ∞ in the insertion tableau

$$P(\xi_1, \ldots, \xi_m, \infty, \xi_{m+1}, \ldots, \xi_t)?$$



bumping route = trajectory

let $Q = Q(\xi_1, \xi_2, \dots)$ be a (finite or infinite) recording tableau; then

the bumping route related to the insertion $Q \leftarrow m + 1/2$

is equal to

the trajectory of ∞ in the sequence of insertions

$$P(\xi_1, \dots, \xi_m, \infty) \leftarrow \xi_{m+1} \leftarrow \xi_{m+2} \leftarrow \cdots$$

important probability distributions

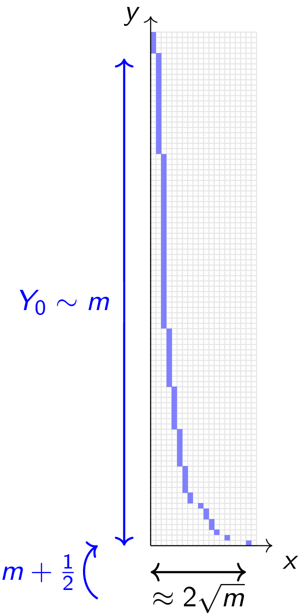
$\mathrm{Exp}(r)$ is the exponential distribution with parameter $r > 0$

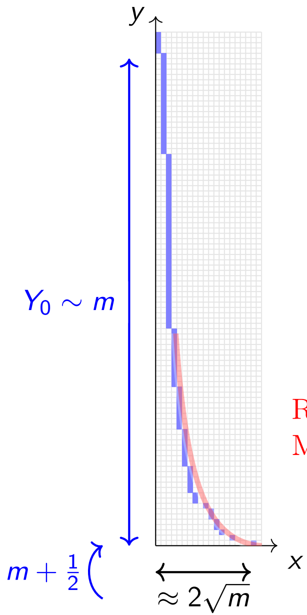
'time of waiting until the bus arrives'

$$\mathbb{E}\,\mathrm{Exp}(r) = \frac{1}{r}$$

$$\mathrm{Erlang}(4) = \mathrm{Exp}(1) + \mathrm{Exp}(1) + \mathrm{Exp}(1) + \mathrm{Exp}(1)$$

'time of waiting until the fourth bus arrives'

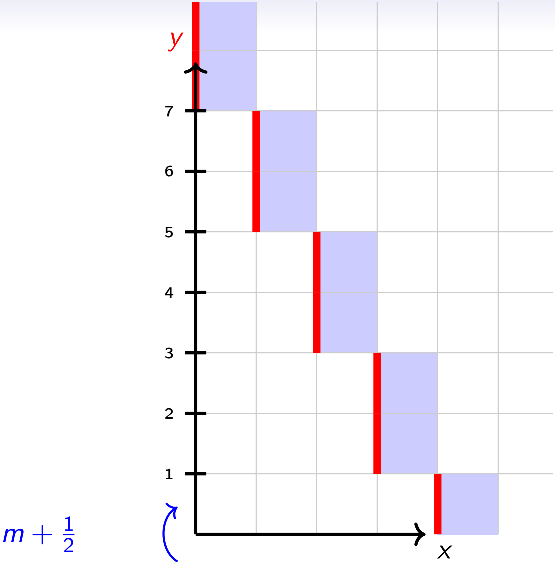

y
x
$Y_0 \sim m$
$m + \frac{1}{2}$
$\approx 2\sqrt{m}$

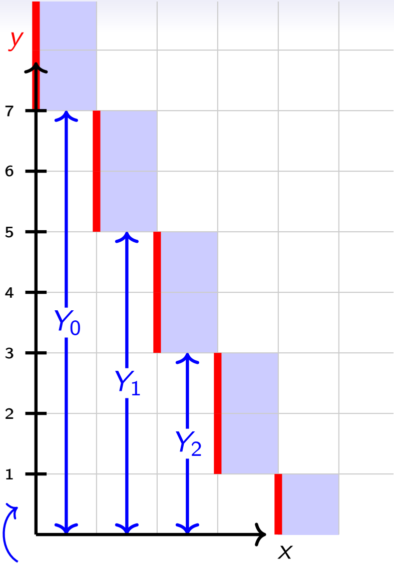


Romik and Śniady 2016
Marciniak 2020

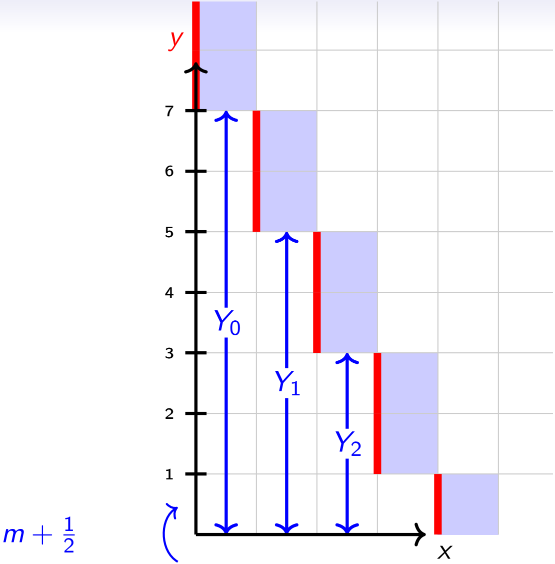
54	56	60	67	109	125
43	45	49	61	88	119
32	41	44	53	58	86
28	36	37	47	57	76
13	19	33	34	42	48
12	17	26	29	40	46
9	10	11	18	25	35
2	5	7	15	23	24
1	3	4	6	8	14

$m + \frac{1}{2} = 6\frac{1}{2}$

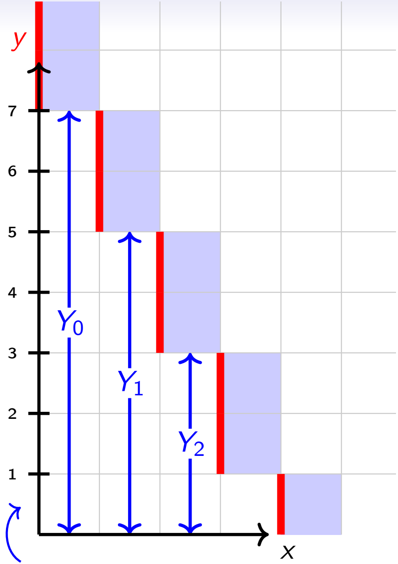

y
7
6
5
4
3
2
1
x
$m + \frac{1}{2}$



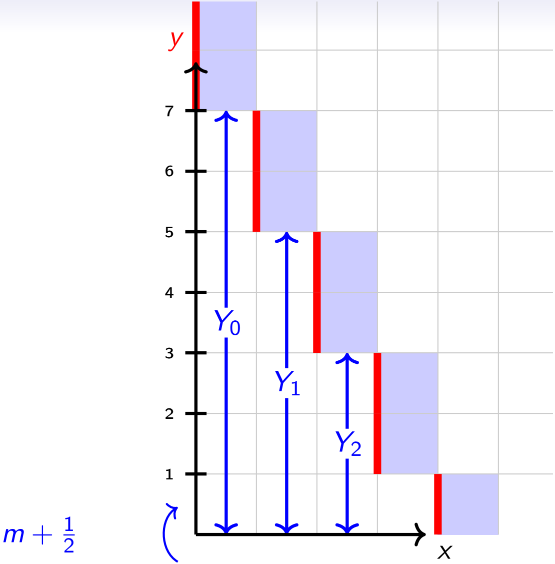
$m + \frac{1}{2}$



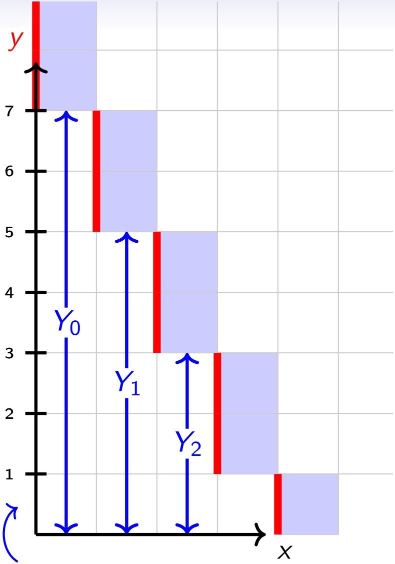
for each $m \geq 1$ $\quad \mathbb{P}(Y_0 < \infty) = 1$



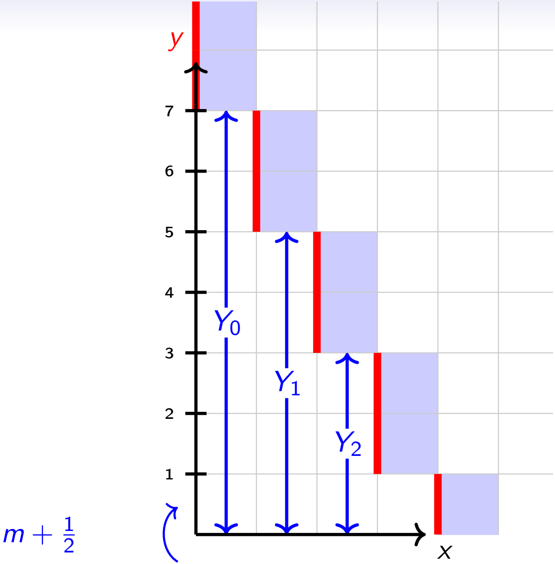
$m + \frac{1}{2}$



for each $m \geq 1$ $\mathbb{E} Y_0 = \infty$

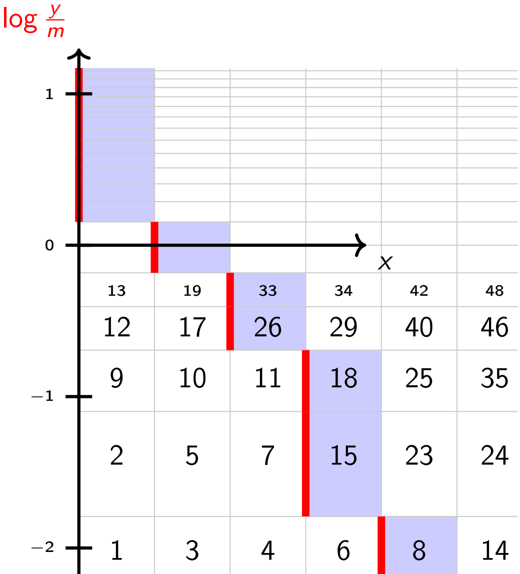


$m + \frac{1}{2}$

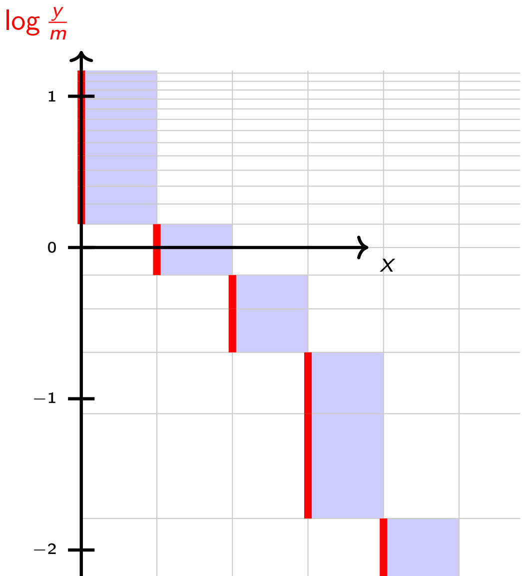


$$\left(\frac{Y_0}{m}, \frac{Y_1}{m}, \dots, \frac{Y_3}{m}\right) \xrightarrow[m\to\infty]{\text{dist}} ?$$

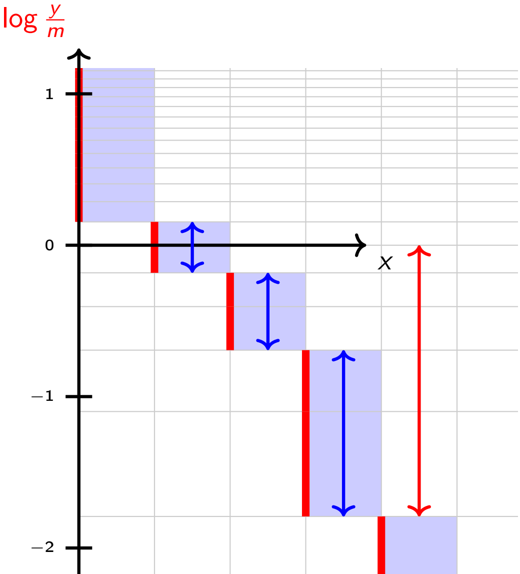
semi-logarithmic plot,



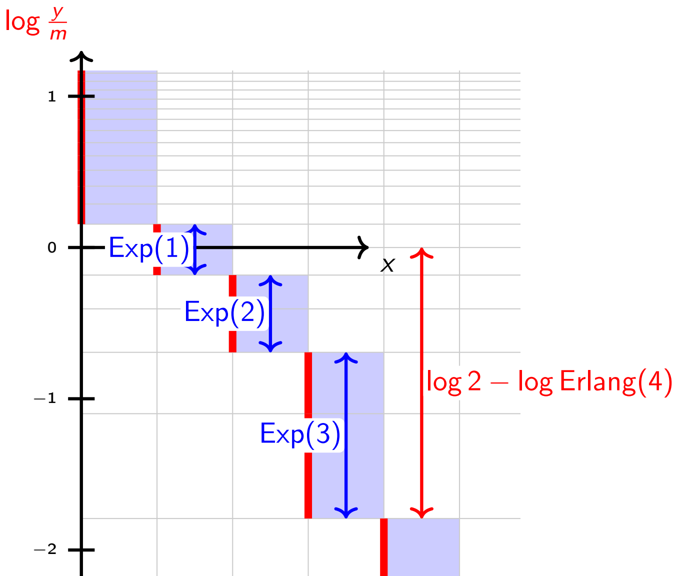
semi-logarithmic plot,



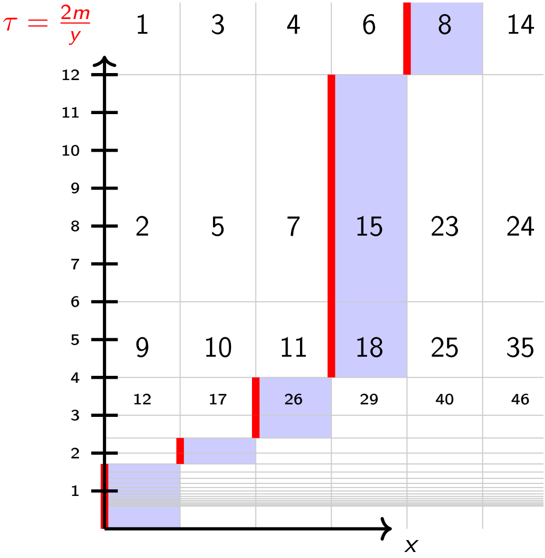
semi-logarithmic plot,



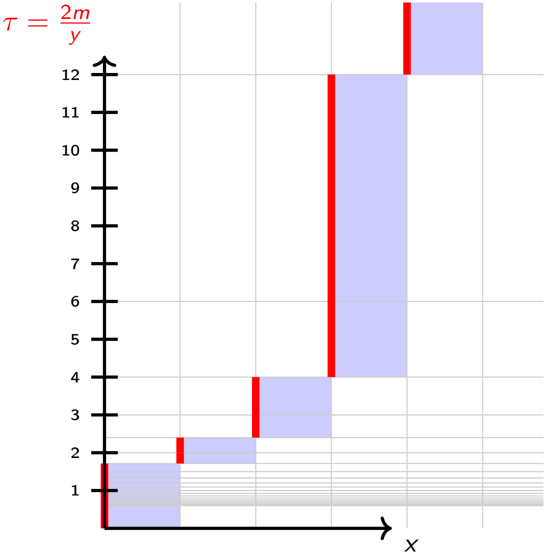
semi-logarithmic plot, $m \to \infty$



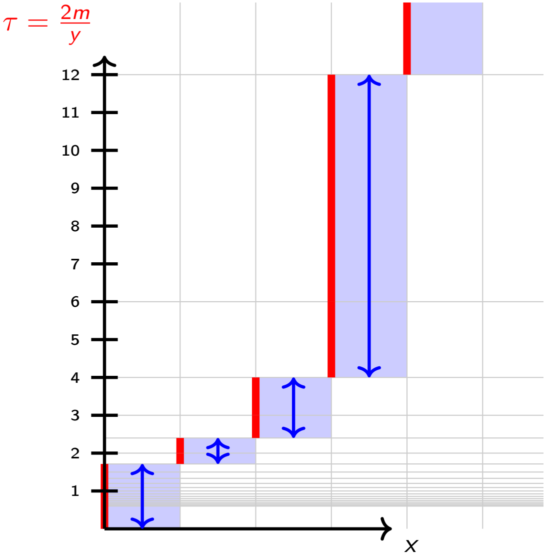
semi-projective plot,



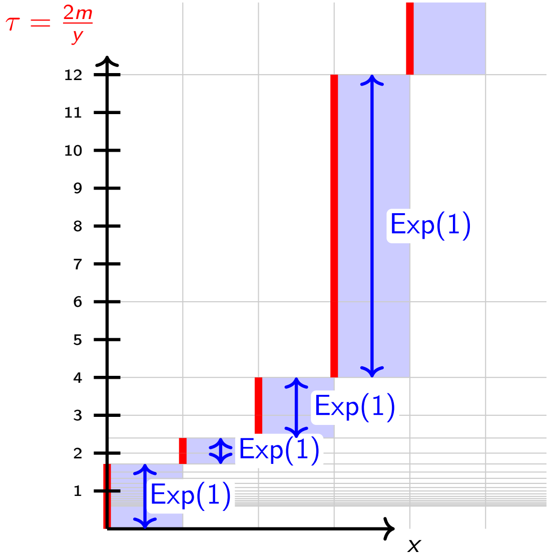
semi-projective plot,



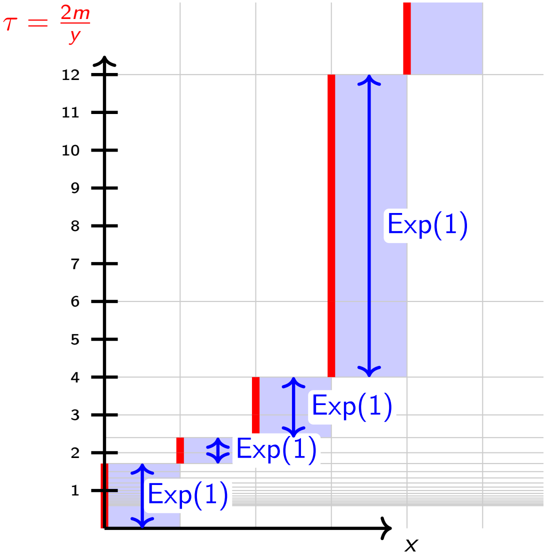
semi-projective plot,



semi-projective plot, $m \to \infty$

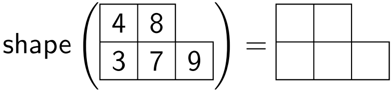


semi-projective plot, $m \to \infty$

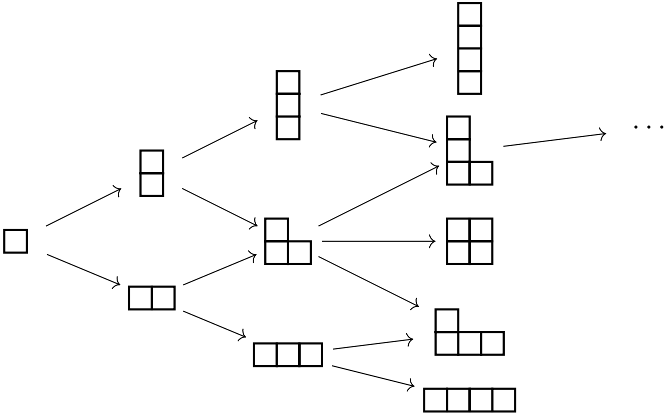


Corollary: the red line is a plot of the standard Poisson process

what do you see in an insertion tableau if you ignore the entries?

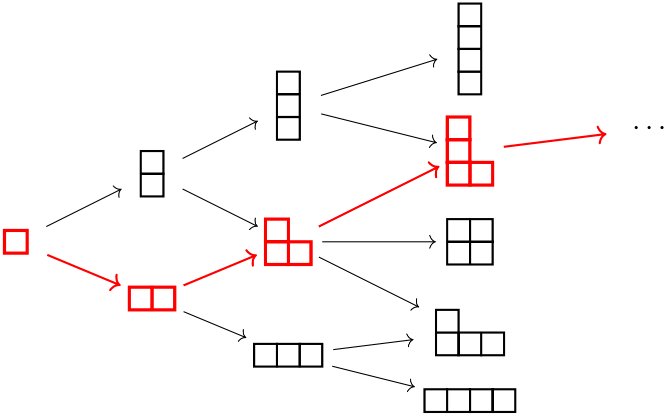


what do you see in an insertion tableau if you ignore the entries?



Plancherel growth process

$$\lambda^{(1)} \nearrow \lambda^{(2)} \nearrow \dots$$



define $\lambda^{(t)} = \text{shape}\, P(\xi_1, \dots, \xi_t)$ to be the shape of the insertion tableau related to the prefix of ξ

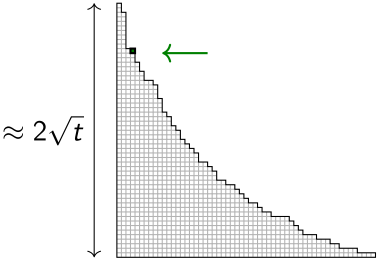
let $m = O\left(\sqrt{t}\right)$

let (x_t, y_t) be the coordinates of ∞ in the insertion tableau

$$P(\underbrace{\xi_1, \dots, \xi_m}_{m}, \infty, \xi_{m+1}, \dots, \xi_t)$$

$$y_t \approx 2\sqrt{t},$$

$$x_t = ?$$



let $m = O\left(\sqrt{n}\right)$

let (x, y) be the coordinates of ∞ in the insertion tableau

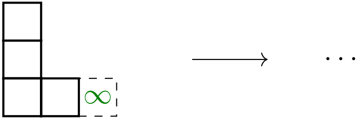
$$P(\underbrace{\xi_1, \dots, \xi_n}_{n}, \infty, \underbrace{\xi_{n+1}, \dots, \xi_{n+m}}_{m})$$

then

$$y \overset{\text{dist}}{\approx} \text{Pois}\left(\frac{m}{\sqrt{n}}\right)$$

this is a result about bottom rows in Plancherel growth process

$\longrightarrow$ improved version of a result of Aldous and Diaconis



let $m = O\left(\sqrt{n}\right)$

let (x, y) be the coordinates of ∞ in the insertion tableau

$$P(\underbrace{\xi_1, \dots, \xi_n}_{n}, \infty, \underbrace{\xi_{n+1}, \dots, \xi_{n+m}}_{m})$$

then

$$y \overset{\text{dist}}{\approx} \operatorname{Pois}\left(\frac{m}{\sqrt{n}}\right)$$

let $m = O\left(\sqrt{n}\right)$

let (x, y) be the coordinates of ∞ in the insertion tableau

$$P(\underbrace{\xi_1, \ldots, \xi_n}_{n}, \infty, \underbrace{\xi_{n+1}, \ldots, \xi_{n+m}}_{m})$$

then

$$y \overset{\text{dist}}{\approx} \text{Pois}\left(\frac{m}{\sqrt{n}}\right)$$

Hint: read the word backwards? RSK gives the transpose!

let $m = O\left(\sqrt{n}\right)$

let (x, y) be the coordinates of ∞ in the insertion tableau

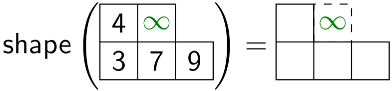
$$P(\underbrace{\xi_1, \dots, \xi_m}_{m}, \infty, \underbrace{\xi_{n+1}, \dots, \xi_{n+m}}_{n})$$

then

$$x \overset{\text{dist}}{\approx} \text{Pois}\left(\frac{m}{\sqrt{n}}\right)$$

Hint: read the word backwards? RSK gives the transpose!

what you do see in an insertion tableau if you ignore the entries, except for ∞?

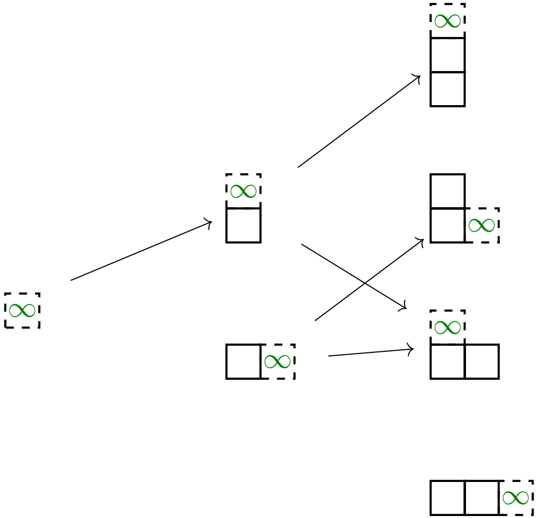


augmented Plancherel growth process

$$\Lambda^{(t)} = \text{shape}\, P(\xi_1, \dots, \xi_m, \infty, \xi_{m+1}, \dots, \xi_t)$$

is a Markov chain

what you do see in an insertion tableau if you ignore the entries, except for ∞?



component 1: probability distribution of $\Lambda^{(t)}$

$$\Lambda^{(t)} \overset{\text{dist}}{\approx} \text{Pois}\left(\frac{m}{\sqrt{t}}\right) \times \text{Plancherel}(t)$$

bad news: the result of Aldous and Diaconis is not enough

component 2: transition probabilities

suppose that Markov chain Λ at time t
has probability distribution

$$\Lambda^{(t)} \overset{\text{dist}}{\approx} \text{Pois}\left(\frac{m}{\sqrt{t}}\right) \times \text{Plancherel}(t)$$

then for $u > t$

$$\Lambda^{(u)} \overset{\text{dist}}{\approx} \text{Pois}\left(\frac{m}{\sqrt{u}}\right) \times \text{Plancherel}(u)$$

component 2: transition probabilities

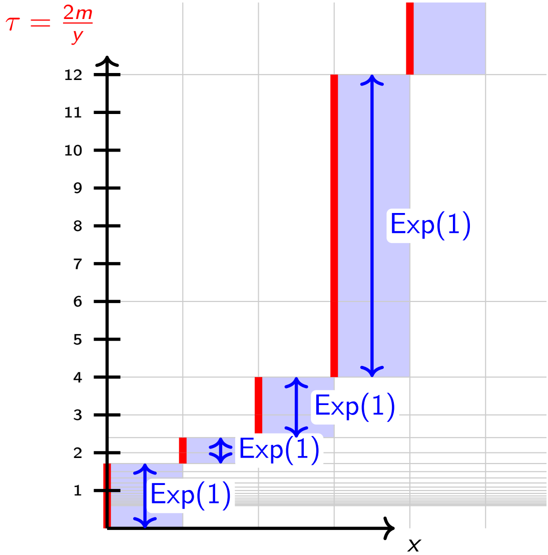
suppose that Markov chain Λ at time t
has probability distribution

$$\Lambda^{(t)} \overset{\text{dist}}{\approx} \delta_x \times \text{Plancherel}(t)$$

then for $u > t$

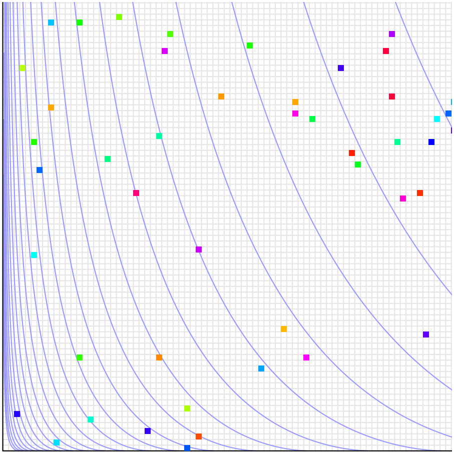
$$\Lambda^{(u)} \overset{\text{dist}}{\approx} \text{Binom}\left(x, \sqrt{\frac{t}{u}}\right) \times \text{Plancherel}(u)$$

semi-projective plot, $m \to \infty$



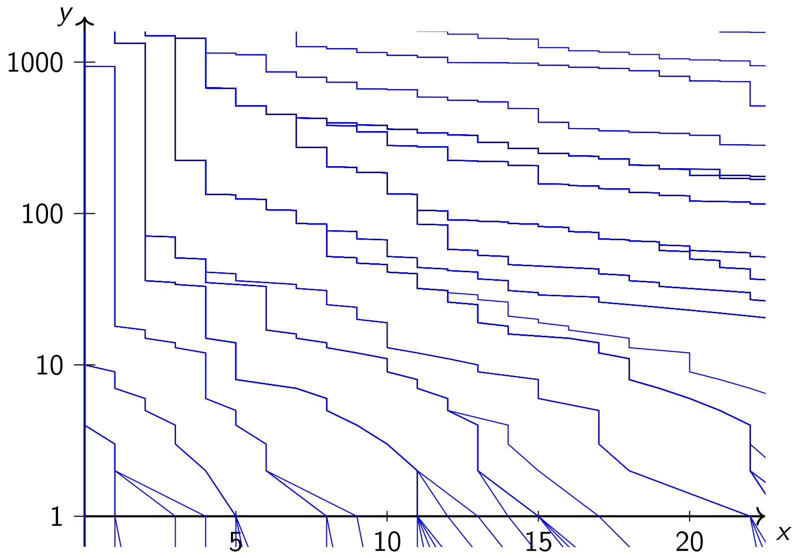
Corollary: the red line is a plot of the standard Poisson process

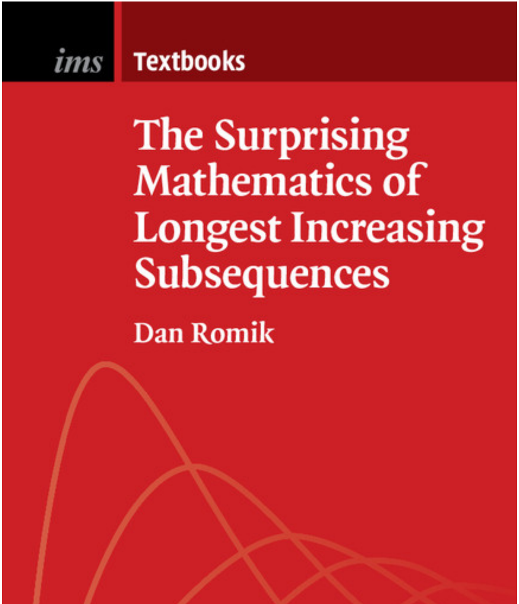
hydrodynamics of the insertion tableau $P(w)$



$m = 1.6 \cdot 10^4$

open problems: bumping forest





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- Mikołaj Marciniak, Łukasz Maślanka, Piotr Śniady
  Poisson limit theorems for the Robinson–Schensted correspondence and the Hammersley multi-line process
  arXiv:2005.13824

- Mikołaj Marciniak, Łukasz Maślanka, Piotr Śniady
  Poisson limit of bumping routes in the Robinson–Schensted correspondence
  arXiv:2005.14397

- Dan Romik, Piotr Śniady.
  Limit shapes of bumping routes in the Robinson–Schensted correspondence.
  Random Structures & Algorithms 48 (2016), no. 1, 171–182