Toward Uniform Random Generation in 1-safe Petri Nets

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Advisor: Jean Mairesse, Samy Abbes

2019 Apr 23 - LIPN

Motivation

- Complexity and scale in software systems are increasing.
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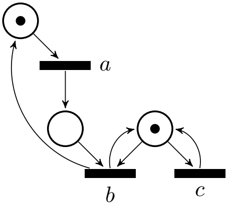
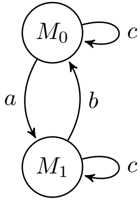
- Approach : Statistical model checking
  $\rightarrow$ probabilistic framework in a trace monoid
- Goal : Random generation for concurrent systems
  $\rightarrow$ 1-safe Petri nets

Concurrent models - 1-safe Petri nets



M_0



Reachability graph

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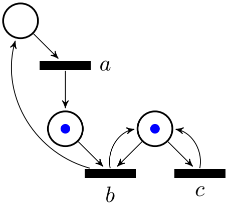
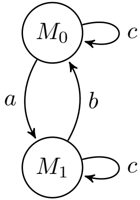


M_1



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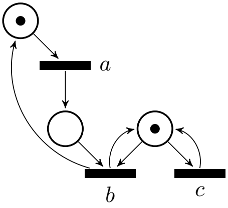

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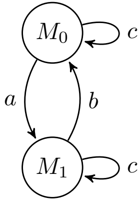

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- Concurrency :
- Casuality :
- Conflit :

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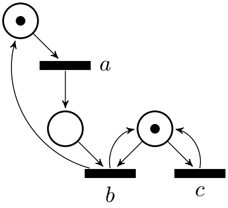

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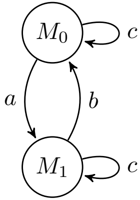

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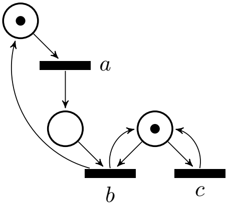

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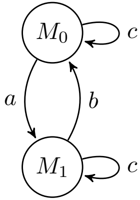

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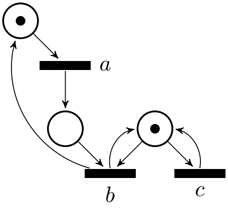

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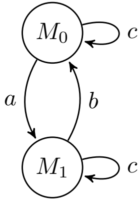

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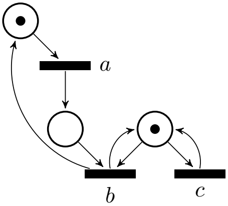

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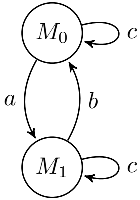

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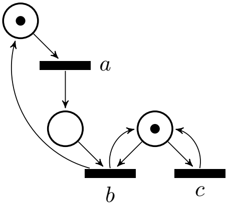

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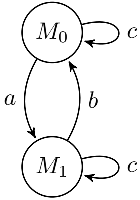


Reachability graph

- From M_0, *abacb* is a valid firing sequence.
- We lost the feature of concurrency by viewing the firing sequences as the sequential executions.
  ex : $abacb = abcab$

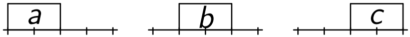
Concurrent models - trace monoids

Trace monoid $\mathcal{M}$

- Alphabet : $\Sigma = \{a, b, c\}$
- Independent relation : $\mathcal{I} = \{(a, c)\}$

Heap of pieces

- Pieces:



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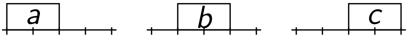
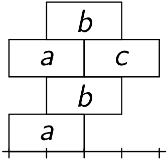


- Example of heap :



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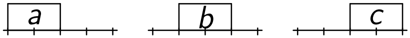

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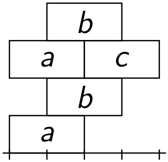

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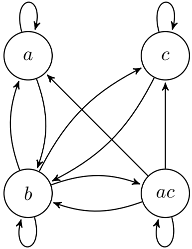

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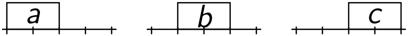

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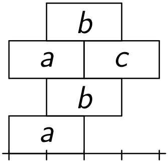

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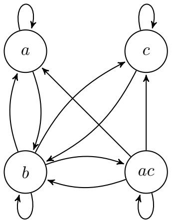
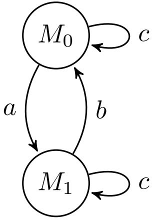
Framework- random sampling from a Markov chain

- Take account of "concurrency" and "states"



traces in a trace monoid



words in an automaton

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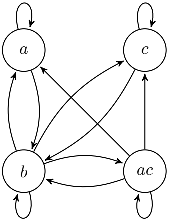

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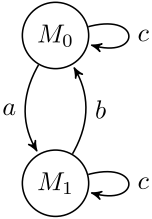

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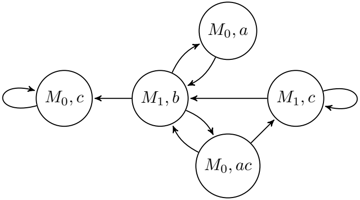
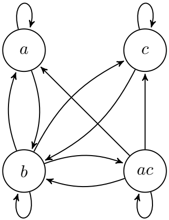


traces in an automaton

Framework- random sampling from a Markov chain

- Take account of "concurrency" and "states"
- The executions of 1-safe Petri nets are understood up to traces.



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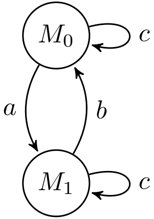

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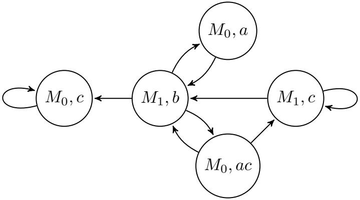

traces in an automaton

Uniform measure on trace monoids

Trace monoid : $\mathcal{M} = \langle a, b, c \mid a \cdot c = c \cdot a \rangle$

- set of cliques $\mathscr{C}$: ε, a, b, c, ac
- Möbius polynomial $\mu(x) = \sum_{c \in \mathscr{C}} (-1)^{|c|} x^{|c|} = 1 - 3x + x^2$
- Möbius inversion formula : $G(x) = \sum_{u \in \mathcal{M}} x^{|u|} = \frac{1}{\mu(x)}$

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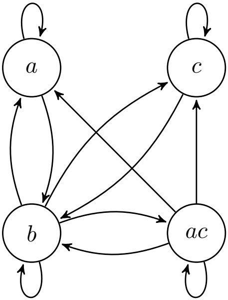
Theorem (Abbes, Mairesse 2015)

There exists a unique uniform measure ν on $\partial\mathcal{M}$, satisfying:
$\forall u \in \mathcal{M}, \quad \nu(\uparrow u) = p_0^{|u|}$
p_0 *: the root of smallest modulus of $\mu(x)$.*

Uniform measure on trace monoids

Theorem (Abbes, Mairesse 2015)

Let ν be the uniform measure on $\partial\mathcal{M}$. Then the canonical normal decomposition of a trace is a realization of the Markov chain with initial probability measure h which is the Möbius transform of ν.



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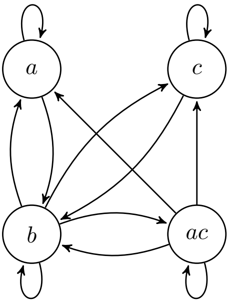
- # paths with length k in the automaton = # traces with height k in a trace monoid

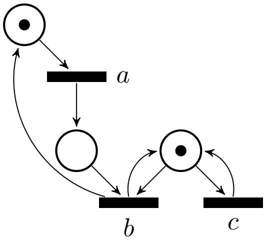


Uniform measure for actions on trace monoids

Theorem (Abbes 2015)

Let $X \times \mathcal{M} \to X$ be an irreducible partial action. Then there exists a uniform Markov measure, satisfying :

$$\forall \alpha \in X \quad \forall x \in \mathcal{M}_\alpha, \quad \nu_\alpha(\uparrow x) = p_0^{|x|}\Gamma(\alpha, \alpha \cdot x).$$



- $G_\alpha(x) = \sum_{u \in \mathcal{M}_\alpha} x^{|u|}$
- $\Gamma(\alpha, \beta) = \lim_{x \to p_0} \frac{G_\beta(x)}{G_\alpha(x)}$
- $\mu_{\alpha,\beta}(x) = \sum_{\gamma \in \mathscr{C}_{\alpha,\beta}} (-1)^{|\gamma|} x^{|\gamma|}$
- Möbius matrix:

$$\mu(x) = (\mu_{\alpha,\beta})(x)$$

Properties of Γ function

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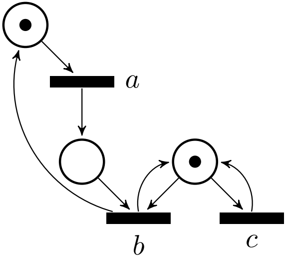
- cocycle relation: $\Gamma(\alpha, \gamma) = \Gamma(\alpha, \beta)\Gamma(\beta, \gamma)$
$$\Gamma(\alpha, \alpha) = 1$$
- Fix a state α_0,

$$(\Gamma(\alpha_0, \beta))_\beta \in \ker \mu(p_0)$$

Calculation of Γ function



$$M_0 \to M_0 : \varepsilon, c, \quad M_0 \to M_1 : a, ac,$$
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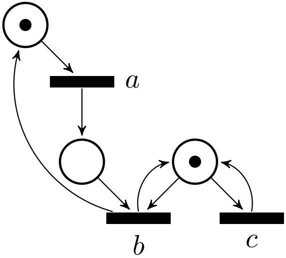

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$$\mu(x) = \begin{matrix} M_0 \\ M_1 \end{matrix} \begin{pmatrix} 1-x & -x+x^2 \\ -x & 1-x \end{pmatrix}.$$

$$p_0 = \frac{\sqrt{5}+1}{2}$$

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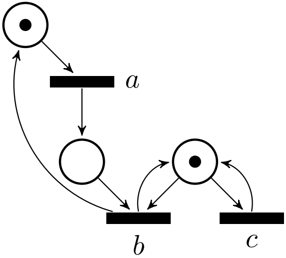

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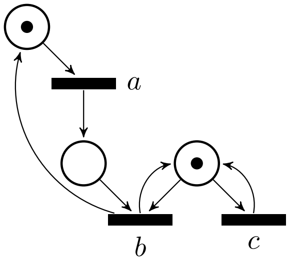

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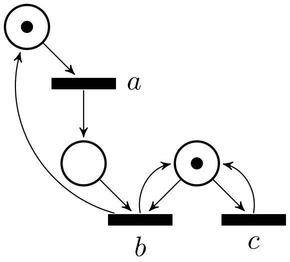

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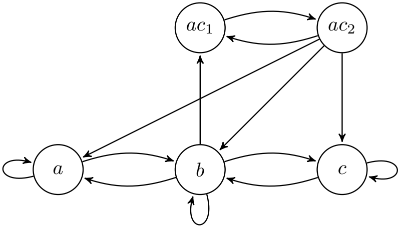
$$-p_0 + (1-p_0)\cdot\lambda = 0 \Longrightarrow \quad \lambda = \frac{p_0}{1-p_0} = \frac{1}{p_0} = \frac{\sqrt{5}-1}{2}.$$

Uniqueness of uniform measure on trace monoids

New proof from the linear algebra point of view

- Construct the expanded automaton of cliques
  $\mathcal{M} = \langle a, b, c \mid a \cdot c = c \cdot a \rangle$



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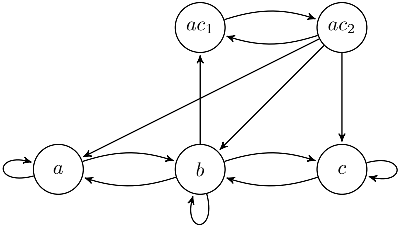

$$
\begin{array}{c}
(a,1)\\
(b,1)\\
(c,1)\\
(ac,1)\\
(ac,2)
\end{array}
\quad
\begin{pmatrix}
1 & 1 & 0 & 0 & 0\\
1 & 1 & 1 & 1 & 0\\
0 & 1 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1\\
1 & 1 & 1 & 1 & 0
\end{pmatrix}
$$

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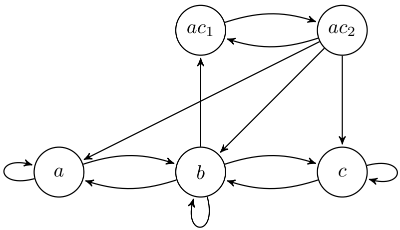

$$
\begin{array}{c}
(a,1) \\ (b,1) \\ (c,1) \\ (ac,1) \\ (ac,2)
\end{array}
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
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$$

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$$v_{(c,i)} = \frac{1}{p^{i-1}} h(c).$$

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Uniqueness of uniform measure on trace monoids

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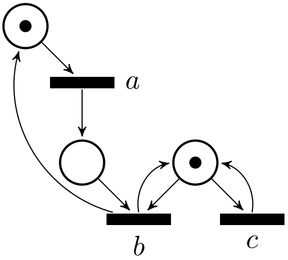
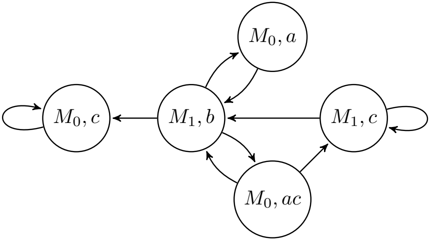
- Get the uniqueness of the uniform measure

Uniqueness of uniform measure on actions

Example for 1-safe petri net





$$\nu_{M_0}(C_1 = c) = \nu_{M_0}(\uparrow c) - \nu_{M_0}(\uparrow (ac))$$

Uniqueness of uniform measure on actions

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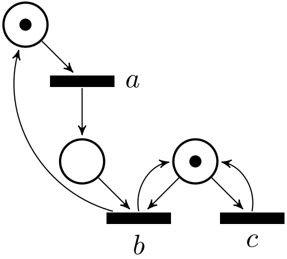

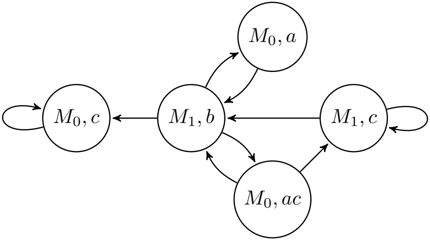

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Uniqueness of uniform measure on actions

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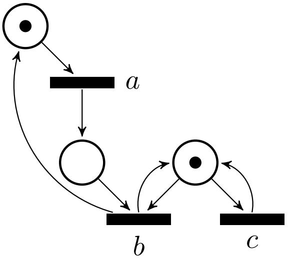
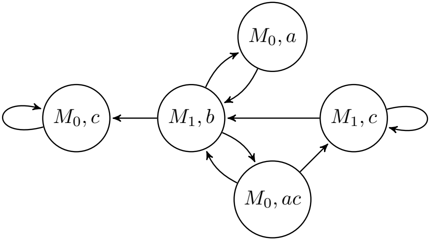
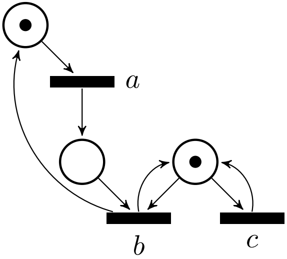


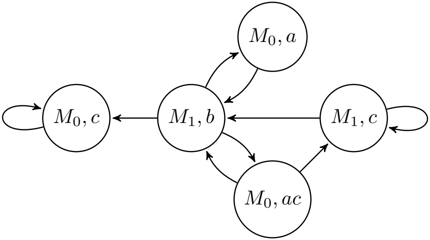


$$\begin{aligned}\nu_{M_0}(C_1 = c) &= \nu_{M_0}(\uparrow c) - \nu_{M_0}(\uparrow (ac)) \\ &= p_0 \cdot \Gamma(M_0, M_0) - p_0^2 \cdot \Gamma(M_0, M_1) \\ &= p_0 \cdot 1 - p_0^2 \cdot \frac{1}{p_0} = 0\end{aligned}$$

Uniqueness of uniform measure on actions

Example for 1-safe petri net





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  $\rightarrow$ can not apply Perron-Frobenius Theorem

Uniqueness of uniform measure on actions

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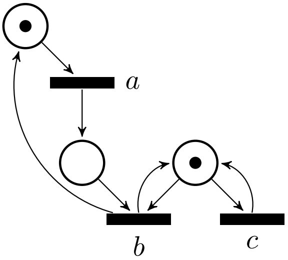

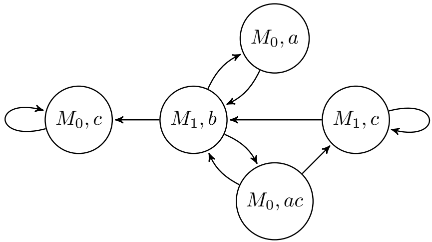

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  $\rightarrow$ can not apply Perron-Frobenius Theorem
- Some state never go through under ν_{M_0}

Ongoing directions

- A systematic way to calculate Γ function

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- Complete the proof of the uniqueness of uniform measure on actions in a trace monoid
- Random generation for actions on a trace monoid
- For the purpose of model checking (application of this theory)

Thank you!