Lazard's elimination and applications

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Outline

INTRODUCTION

$$A = B \coprod Z \quad A^* = B^*(ZB^*)^* = (B^*Z)^*B^* \quad A^* = \prod_{l \in \mathcal{L}ynA}^{\searrow} l^*$$

Zeta functions in several variables and polyzetas

Let $\mathcal{H}_r = \{(s_1, \ldots, s_r) \in \mathbb{C}^r | \forall m = 1, \ldots, r, \Re(s_1) + \ldots + \Re(s_m) > m\}$, for $r \in \mathbb{N}_+$, the following zeta function converges for $(s_1, \ldots, s_r) \in \mathcal{H}_r$

$$\zeta(s_1, \ldots, s_r) = \sum_{n_1 > \ldots > n_r > 0} n_1^{-s_1} \ldots n_r^{-s_r}.$$

By a Abel's theorem, for $n \in \mathbb{N}, z \in \mathbb{C}, |z| < 1$, this value can be obtained as

$$\zeta(s_1, \ldots, s_r) = \lim_{n \to +\infty} \mathrm{H}_{s_1, \ldots, s_r}(n) = \lim_{z \to 1} \mathrm{Li}_{s_1, \ldots, s_r}(z),$$

where the following functions are well defined for $(s_1, \ldots, s_r) \in \mathbb{C}^r, |z| < 1$

$$\mathrm{Li}_{s_1, \ldots, s_r}(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}} \quad \text{and} \quad \frac{\mathrm{Li}_{s_1, \ldots, s_r}(z)}{1 - z} = \sum_{n \geq 0} \mathrm{H}_{s_1, \ldots, s_r}(n) z^n.$$

$$\mathcal{Z} := \mathrm{span}_{\mathbb{Q}}\{\mathrm{Li}_{s_1, \ldots, s_r}(1)\}_{\substack{s_1, \ldots, s_r \in \mathbb{N}_+ \\ s_1 > 1, r > 0}} = \mathrm{span}_{\mathbb{Q}}\{\mathrm{H}_{s_1, \ldots, s_r}(+\infty)\}_{\substack{s_1, \ldots, s_r \in \mathbb{N}_+ \\ s_1 > 1, r > 0}}.$$

We use the correspondence between words over $X = \{x_0, x_1\}$, $Y = \{y_k\}_{k \geq 1}$:

$$(s_1, \ldots, s_r) \in \mathbb{N}_+^r \leftrightarrow y_{s_1} \ldots y_{s_r} \in Y^* \underset{\pi_Y}{\overset{\pi_X}{\rightleftharpoons}} x_0^{s_1 - 1} x_1 \ldots x_0^{s_r - 1} x_1 \in X^* x_1.$$

$\mathrm{Li}_{s_1, \ldots, s_r}(z) = \alpha_{z_0}^z(x_0^{s_1 - 1} x_1 \ldots x_0^{s_k - 1} x_1)$, where $z_0 \rightsquigarrow z$ is a path in a simply connected domain, $\widetilde{\mathbb{C} \setminus \{0, 1\}}$, with a subdivision $(z_0, z_1, \ldots, z_k, z)$ and

$$\alpha_{z_0}^z(x_{i_1} \ldots x_{i_k}) = \int_{z_0}^{z} \omega_{i_1}(z_1) \ldots \int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k) \quad \text{with} \quad \begin{cases} \omega_0(z) = dz/z, \\ \omega_1(z) = dz/(1 - z). \end{cases}$$

Words and noncommutative formal series

- Let $(\mathcal{X}^*, 1_{\mathcal{X}^*})$ be the free monoid generated by $\mathcal{X}$, denoting the alphabet $X := \{x_0, x_1\}$ or $Y := \{y_k\}_{k\geq 1}$ (equipped with the order $x_1 \succ x_0$ and $y_1 \succ y_2 \succ \ldots$, respectively).
- Let $\mathcal{L}yn\mathcal{X}$ denote set of Lyndon words over $\mathcal{X}$.
- Let $A\langle\mathcal{X}\rangle$ (resp. $A\langle\langle\mathcal{X}\rangle\rangle$) be the set of polynomials (resp. series) and $\mathcal{L}ie_A\langle\mathcal{X}\rangle$ (resp. $\mathcal{L}ie_A\langle\langle\mathcal{X}\rangle\rangle$ be the set of Lie polynomials (resp. series) over $\mathcal{X}$ with coefficients in A.
- On[1] $(A\langle\mathcal{X}\rangle, \mathrm{conc}, 1_{\mathcal{X}^*}, \Delta_{\sqcup\!\sqcup}, \varepsilon)$ & $(A\langle Y\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{\stackrel{-}{\sqcup\!\sqcup}}, \varepsilon)$, we get

$$\mathcal{D}_{\mathcal{X}} := \sum_{w\in\mathcal{X}^*} w\otimes w = \sum_{w\in\mathcal{X}^*} S_w \otimes P_w = \prod_{l\in\mathcal{L}yn\mathcal{X}}^{\searrow} e^{S_l\otimes P_l},$$

$$\mathcal{D}_{Y} := \sum_{w\in Y^*} w\otimes w = \sum_{w\in Y^*} \Sigma_w \otimes \Pi_w = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\Sigma_l\otimes \Pi_l},$$

where $\{P_l\}_{l\in\mathcal{L}yn\mathcal{X}}$ (resp. $\{\Pi_l\}_{l\in\mathcal{L}ynY}$) is a basis of Lie algebra of primitive elements and $\{S_l\}_{l\in\mathcal{L}yn\mathcal{X}}$ (resp. $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$) is a pure transcendence basis of $(A\langle\mathcal{X}\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*})$ (resp. $(A\langle Y\rangle, \stackrel{-}{\sqcup\!\sqcup}, 1_{Y^*})$).

[1] $\forall x \in \mathcal{X}, \Delta_{\sqcup\!\sqcup}(x) = 1_{\mathcal{X}^*}\otimes x + x\otimes 1_{\mathcal{X}^*}$ and
$\forall y_k \in Y, \Delta_{\sqcup\!\sqcup}(y_k) = 1_{Y^*}\otimes y_k + y_k\otimes 1_{Y^*} + \sum_{i+j=k} y_i \otimes y_j$.

First structures of polylogarithms and harmonic sums

1. The following morphisms of algebras is injective

$$
\begin{array}{rcl}
\mathrm{Li}_\bullet : (\mathbb{C}\langle X\rangle, \shuffle, 1_{X^*}) & \to & (\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}, ., 1),\\
x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1 & \mapsto & \mathrm{Li}_{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1} \ (\textit{i.e.}\ \mathrm{Li}_{s_1,\ldots,s_r}),\\
x_0 & \mapsto & \log(z),\\
\mathrm{H}_\bullet : (\mathbb{C}\langle Y\rangle, \stuffle, 1_{Y^*}) & \to & (\mathbb{C}\{\mathrm{H}_w\}_{w\in Y^*}, ., 1),\\
y_{s_1}\ldots y_{s_r} & \mapsto & \mathrm{H}_{y_{s_1}\ldots y_{s_r}} \ (\textit{i.e.}\ \mathrm{H}_{s_1,\ldots,s_r}).
\end{array}
$$

Hence, $\{\mathrm{Li}_w\}_{w\in X^*}$ and $\{\mathrm{H}_w\}_{w\in Y^*}$ are $\mathbb{C}$-linearly independent. It follows that, $\{\mathrm{H}_l\}_{l\in\mathcal{L}ynY}$ and $\{\mathrm{Li}_l\}_{l\in\mathcal{L}ynX}$ (resp. $\{\mathrm{H}_{\Sigma_l}\}_{l\in\mathcal{L}ynY}$ and $\{\mathrm{Li}_{S_l}\}_{l\in\mathcal{L}ynX}$) are algebraically independent.

2. The following polymorphism of algebras is surjective

$$
\begin{array}{rcl}
\zeta : \begin{array}{c}(\mathbb{C}1_{X^*} \oplus x_0\mathbb{C}\langle X\rangle x_1, \shuffle, 1_{X^*})\\ (\mathbb{C}1_{Y^*} \oplus (Y-\{y_1\})\mathbb{C}\langle Y\rangle, \stuffle, 1_{Y^*})\end{array} & \twoheadrightarrow & (\mathcal{Z}, ., 1),\\
\begin{array}{c}x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1\\ y_{s_1}\ldots y_{s_r}\end{array} & \mapsto & \zeta(s_1,\ldots,s_r).
\end{array}
$$

$$\forall l_1, l_2 \in \mathcal{L}ynX - X, \zeta(l_1 \shuffle l_2) = \zeta((\pi_Y l_1) \stuffle (\pi_Y l_2)) = \zeta(l_1)\zeta(l_2).$$

ζ can be extended as characters, for $\shuffle$ and $\stuffle$, respectively :

$$\zeta_{\shuffle} : (\mathbb{C}\langle X\rangle, \shuffle, 1_{X^*}) \to (\mathcal{Z}, ., 1), \quad \zeta_{\stuffle} : (\mathbb{C}\langle Y\rangle, \stuffle, 1_{Y^*}) \to (\mathcal{Z}, ., 1),$$

s.t.
$$
\begin{array}{ll}
\zeta_{\shuffle}(x_0) = 0 = \log(1), & \\
\zeta_{\shuffle}(x_1) = 0 = \mathrm{f.p.}_{z\to 1}\log(1-z), & \{(1-z)^a\log^b(1-z)\}_{a\in\mathbb{Z}_{\le -1}, b\in\mathbb{N}}\\
\zeta_{\stuffle}(y_1) = 0 = \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_1(n), & \{n^a\mathrm{H}_1^b(n)\}_{a\in\mathbb{Z}_{\le -1}, b\in\mathbb{N}}
\end{array}
$$

NONCOMMUTATIVE GENERATING SERIES

Noncommutative generating series

$$\mathrm{L}(z) := \sum_{w\in X^*} \mathrm{Li}_w(z)w = (\mathrm{Li}_\bullet \otimes \mathtt{Id})\mathcal{D}_X = \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\mathrm{Li}_{S_l} P_l},$$

$$\mathrm{H}(n) := \sum_{w\in Y^*} \mathrm{H}_w(n)w = (\mathrm{H}_\bullet \otimes \mathtt{Id})\mathcal{D}_Y = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\mathrm{H}_{\Sigma_l}\Pi_l},$$

$$Z_{\sqcup\!\sqcup} := \sum_{w\in X^*} \zeta_{\sqcup\!\sqcup}(w)w = (\zeta_{\sqcup\!\sqcup} \otimes \mathtt{Id})\mathcal{D}_X = \prod_{l\in\mathcal{L}ynX\setminus X}^{\searrow} e^{\zeta(S_l)P_l},$$

$$Z_{\uplus} := \sum_{w\in Y^*} \zeta_{\uplus}(w)w = (\zeta_{\uplus} \otimes \mathtt{Id})\mathcal{D}_Y = \prod_{l\in\mathcal{L}ynY\setminus\{y_1\}}^{\searrow} e^{\zeta(\Sigma_l)\Pi_l}.$$

Let $\gamma_\bullet$ be the character on $(\mathbb{C}\langle Y\rangle, \uplus, 1_{Y^*})$ defined by $\gamma_{1_{Y^*}} = 1$ and[2]

$$\forall l \in \mathcal{L}ynY, \quad \gamma_{\Sigma_l} := \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_{\Sigma_l}(n) = \zeta(\Sigma_l), \quad \{n^a \log^b(n)\}_{a\in\mathbb{Z}_{\leq -1}, b\in\mathbb{N}}.$$

$$Z_\gamma := \sum_{w\in Y^*} \gamma_w w = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\gamma_{\Sigma_l}\Pi_l} = e^{\gamma y_1} Z_{\uplus}.$$

---

[2] $\gamma_{\Sigma_{y_1}} = \gamma_{y_1} = \gamma$ and, for any $l \in \mathcal{L}ynY \setminus \{y_1\}$ (resp. $\mathcal{L}ynX \setminus X$), $\zeta(\Sigma_l)$ (resp. $\zeta(S_l)$) is convergent because the polynomial, homogenous of weight (l), Σ_l (resp. S_l) belongs to $(Y \setminus \{y_1\})\mathbb{Q}\langle Y\rangle$ (resp. $x_0\mathbb{Q}\langle X\rangle x_1$) (so is Π_l (resp. P_l)).

Gradation of L and $Z_{ш}$

Let the operation $\circ$ be defined, for any $l \in \mathbb{N}$ and $P \in \mathbb{C}\langle X\rangle$, by $x_1x_0^l \circ P = x_1(x_0^l ш P)$. Then[3]

$$\begin{aligned}
\mathrm{L}(z) &= \sum_{k\geq 0}\sum_{w\in x_0^* ш x_1^k} \mathrm{Li}_w(z)w \\
&= e^{x_0\log(z)}\Big(1_{X^*} + \sum_{k\geq 1}\sum_{l_1,\cdots,l_k\geq 0} \mathrm{Li}_{x_1x_0^{l_1}\circ\cdots\circ x_1x_0^{l_k}}(z)\prod_{i=1}^{k}\mathrm{ad}_{-x_0}^{l_i}x_1\Big) \\
&= \sum_{k\geq 0}\int_0^z \omega_1(t_k)\cdots\int_0^{t_{k-1}}\omega_1(t_1)\kappa_k(z,t_1,\cdots,t_k),
\end{aligned}$$

where, for any $k\geq 0, \kappa_k(z,t_1,\cdots,t_k)$ is the formal power series given by

$$\begin{aligned}
\kappa_k(z,t_1,\cdots,t_k) &= e^{x_0[\log(z)-\log(t_1)]}x_1\cdots e^{x_0[\log(t_{k-1})-\log(t_k)]}x_1e^{x_0\log(t_k)} \\
&= e^{x_0\log(z)}e^{\mathrm{ad}_{-x_0\log(t_1)}}x_1\cdots e^{\mathrm{ad}_{-x_0\log(t_k)}}x_1 \\
&= e^{x_0\log(z)}\sum_{l_1,\cdots,l_k\geq 0}\prod_{i=1}^{k}\frac{\log^{l_i}(t_i)}{l_i!}\mathrm{ad}_{-x_0}^{l_i}x_1.
\end{aligned}$$

$\{\mathrm{ad}_{-x_0}^{l_1}x_1\cdots\mathrm{ad}_{-x_0}^{l_k}x_1\}_{k\geq 0}^{l_1,\cdots,l_k\geq 0}$ and $\{x_1x_0^{l_1}\circ\cdots\circ x_1x_0^{l_k}\}_{k\geq 0}^{l_1,\cdots,l_k\geq 0}$ are dual bases of $\mathcal{U}(\mathcal{J})$ and $\mathcal{U}(\mathcal{J})^{\vee}$, respectively, because $\mathrm{Li}_\bullet$ is injective.

$$Z_{ш} = \sum_{k\geq 0}\sum_{l_1,\cdots,l_k\geq 0}\zeta_{ш}(x_1x_0^{l_1}\circ\cdots\circ x_1x_0^{l_k})\prod_{i=0}^{k}\mathrm{ad}_{-x_0}^{l_i}x_1.$$

[3] $\mathrm{supp}(x_1x_0^{l_1}\circ\cdots\circ x_1x_0^{l_k}) = \{w\in x_1X^* \mid |w|_{x_1}=k, |w|_{x_0}=l_1+\cdots+l_k\}$.

Abel like results and bridge equations

Let π_Y denote the projector $(A \oplus A\langle X\rangle x_1, .) \to (A\langle Y\rangle, .)$, mapping $x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1$ to $y_{s_1} \ldots y_{s_r}$.

Theorem 1 (first Abel like theorem)

Let $\mathtt{Const} := \sum_{k\geq 0} \mathrm{H}_{y_1^k} y_1^k = e^{-\sum_{k\geq 0} \mathrm{H}_{y_k}(-y_1)^k/k}$. *Then*

$$\lim_{z\to 1} e^{y_1 \log(1-z)} \pi_Y \mathrm{L}(z) = \pi_Y Z_{⧢} = \lim_{n\to\infty} \mathtt{Const(n)}^{-1} \mathrm{H}(n).$$

Corollary 2 (bridge equations)

Let $B'(y_1) := e^{\sum_{k\geq 2} \zeta(k)(-y_1)^k/k}$ *and* $B(y_1) := e^{\gamma y_1 - \sum_{k\geq 2} \zeta(k)(-y_1)^k/k}$. *Then*

$$Z_\gamma = B(y_1)\pi_Y Z_{⧢} \iff Z_{⧢\!\!\!+} = B'(y_1)\pi_Y Z_{⧢}.$$

Example 3 (generalized Euler's constant)

From the first bridge equation, for $w \in X^*, k \in \mathbb{N}$, identifying the coefficients of $y_1^k w$, one has

$$\begin{aligned}
\gamma_{1,1} &= \tfrac{1}{2}(\gamma^2 - \zeta(2)),\\
\gamma_{1,1,1} &= \tfrac{1}{6}(\gamma^3 - 3\zeta(2)\gamma + 2\zeta(3)),\\
\gamma_{1,7} &= \zeta(7)\gamma + \zeta(3)\zeta(5) - \tfrac{54}{175}\zeta(2)^4,\\
\gamma_{1,1,6} &= \tfrac{4}{35}\zeta(2)^3\gamma^2 + (\zeta(2)\zeta(5) + \tfrac{2}{5}\zeta(3)\zeta(2)^2 - 4\zeta(7))\gamma\\
&\quad + \zeta(6,2) + \tfrac{19}{35}\zeta(2)^4 + \tfrac{1}{2}\zeta(2)\zeta(3)^2 - 4\zeta(3)\zeta(5).
\end{aligned}$$

COMPUTATIONAL EXAMPLES

Homogenous polynomials relations[4] on local coordinates

Identifying the local coordinates in $Z_\gamma = B(y_1)\pi_Y Z_{ш}$, one has

	Polynomial relations on $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Polynomial relations on $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2}) = \zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4}) = \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1}) = \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2}) = \frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2}) = \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2}) = 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1}) = -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1}) = -\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2}) = \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2}) = \frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3}) = \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4}) = \zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6}) = \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2}) = \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1}) = \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1}) = \frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2}) = -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3}) = \frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1}) = 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2}) = \frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2}) = \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1}) = -\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2}) = \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3}) = \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3}) = \frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4}) = \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$

[4] These polynomials relations are independent from γ and similarly for the case where the ring of their coefficients is the ring A.

Homogenous polynomials generating inside ker ζ

	$\{Q_l\}_{l\in\mathcal{L}ynY-\{y_1\}}$	$\{Q_l\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}-\frac{3}{2}\Sigma_{y_3})=0$	$\zeta(S_{x_0x_1^2}-S_{x_0^2x_1})=0$
4	$\zeta(\Sigma_{y_4}-\frac{2}{5}\Sigma_{y_2}^{\stuffle 2})=0$	$\zeta(S_{x_0^3x_1}-\frac{2}{5}S_{x_0x_1}^{\shuffle 2})=0$
	$\zeta(\Sigma_{y_3y_1}-\frac{3}{10}\Sigma_{y_2}^{\stuffle 2})=0$	$\zeta(S_{x_0^2x_1^2}-\frac{1}{10}S_{x_0x_1}^{\shuffle 2})=0$
	$\zeta(\Sigma_{y_2y_1^2}-\frac{2}{3}\Sigma_{y_2}^{\stuffle 2})=0$	$\zeta(S_{x_0x_1^3}-\frac{2}{5}S_{x_0x_1}^{\shuffle 2})=0$
5	$\zeta(\Sigma_{y_3y_2}-3\Sigma_{y_3}\stuffle\Sigma_{y_2}-5\Sigma_{y_5})=0$	$\zeta(S_{x_0^3x_1^2}-S_{x_0^2x_1}\shuffle S_{x_0x_1}+2S_{x_0^4x_1})=0$
	$\zeta(\Sigma_{y_4y_1}-\Sigma_{y_3}\stuffle\Sigma_{y_2})+\frac{5}{2}\Sigma_{y_5})=0$	$\zeta(S_{x_0^2x_1x_0x_1}-\frac{3}{2}S_{x_0^4x_1}+S_{x_0^2x_1}\shuffle S_{x_0x_1})=0$
	$\zeta(\Sigma_{y_2^2y_1}-\frac{3}{2}\Sigma_{y_3}\stuffle\Sigma_{y_2}-\frac{25}{12}\Sigma_{y_5})=0$	$\zeta(S_{x_0^2x_1^3}-S_{x_0^2x_1}\shuffle S_{x_0x_1}+2S_{x_0^4x_1})=0$
	$\zeta(\Sigma_{y_3y_1^2}-\frac{5}{12}\Sigma_{y_5})=0$	$\zeta(S_{x_0x_1x_0x_1^2}-\frac{1}{2}S_{x_0^4x_1})=0$
	$\zeta(\Sigma_{y_2y_1^3}-\frac{1}{4}\Sigma_{y_3}\stuffle\Sigma_{y_2})+\frac{5}{4}\Sigma_{y_5})=0$	$\zeta(S_{x_0x_1^4}-S_{x_0^4x_1})=0$
6	$\zeta(\Sigma_{y_6}-\frac{8}{35}\Sigma_{y_2}^{\stuffle 3})=0$	$\zeta(S_{x_0^5x_1}-\frac{8}{35}S_{x_0x_1}^{\shuffle 3})=0$
	$\zeta(\Sigma_{y_4y_2}-\Sigma_{y_3}^{\stuffle 2}-\frac{4}{21}\Sigma_{y_2}^{\stuffle 3})=0$	$\zeta(S_{x_0^4x_1^2}-\frac{6}{35}S_{x_0x_1}^{\shuffle 3}-\frac{1}{2}S_{x_0^2x_1}^{\shuffle 2})=0$
	$\zeta(\Sigma_{y_5y_1}-\frac{2}{7}\Sigma_{y_2}^{\stuffle 3}-\frac{1}{2}\Sigma_{y_3}^{\stuffle 2})=0$	$\zeta(S_{x_0^3x_1x_0x_1}-\frac{4}{105}S_{x_0x_1}^{\shuffle 3})=0$
	$\zeta(\Sigma_{y_3y_1y_2}-\frac{17}{30}\Sigma_{y_2}^{\stuffle 3}+\frac{9}{4}\Sigma_{y_3}^{\stuffle 2})=0$	$\zeta(S_{x_0^3x_1^3}-\frac{23}{70}S_{x_0x_1}^{\shuffle 3}-S_{x_0^2x_1}^{\shuffle 2})=0$
	$\zeta(\Sigma_{y_3y_2y_1}-3\Sigma_{y_3}^{\stuffle 2}-\frac{9}{10}\Sigma_{y_2}^{\stuffle 3})=0$	$\zeta(S_{x_0^2x_1x_0x_1^2}-\frac{2}{105}S_{x_0x_1}^{\shuffle 3})=0$
	$\zeta(\Sigma_{y_4y_1^2}-\frac{3}{10}\Sigma_{y_2}^{\stuffle 2}-\frac{3}{4}\Sigma_{y_3}^{\stuffle 2})=0$	$\zeta(S_{x_0^2x_1^2x_0x_1}-\frac{89}{210}S_{x_0x_1}^{\shuffle 3}+\frac{3}{2}S_{x_0^2x_1}^{\shuffle 2})=0$
	$\zeta(\Sigma_{y_2^2y_1^2}-\frac{11}{63}\Sigma_{y_2}^{\stuffle 2}-\frac{1}{4}\Sigma_{y_3}^{\stuffle 2})=0$	$\zeta(S_{x_0^2x_1^4}-\frac{6}{35}S_{x_0x_1}^{\shuffle 3}-\frac{1}{2}S_{x_0^2x_1}^{\shuffle 2})=0$
	$\zeta(\Sigma_{y_3y_1^3}-\frac{1}{21}\Sigma_{y_2}^{\stuffle 3})=0$	$\zeta(S_{x_0x_1x_0x_1^3}-\frac{8}{21}S_{x_0x_1}^{\shuffle 3}-S_{x_0^2x_1}^{\shuffle 2})=0$
	$\zeta(\Sigma_{y_2y_1^4}-\frac{17}{50}\Sigma_{y_2}^{\stuffle 3}+\frac{3}{16}\Sigma_{y_3}^{\stuffle 2})=0$	$\zeta(S_{x_0x_1^5}-\frac{8}{35}S_{x_0x_1}^{\shuffle 3})=0$

One has $\mathcal{R}_{\mathcal{X}}\subseteq\ker\zeta$, where $\begin{cases}\mathcal{R}_Y:=(\mathbb{Q}\{Q_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}},\stuffle,1_{Y^*})\\ \mathcal{R}_X:=(\mathbb{Q}\{Q_l\}_{l\in\mathcal{L}ynX\setminus X},\shuffle,1_{X^*})\end{cases}$.

Noetherian rewriting system & irreducible coordinates[5]

	Rewriting among	$\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Rewriting among	$\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1})$	$\rightarrow \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2})$	$\rightarrow \zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4})$	$\rightarrow \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1})$	$\rightarrow \frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1})$	$\rightarrow \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2})$	$\rightarrow \frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2})$	$\rightarrow \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3})$	$\rightarrow \frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2})$	$\rightarrow 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2})$	$\rightarrow -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1})$	$\rightarrow -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1})$	$\rightarrow -\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1})$	$\rightarrow \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3})$	$\rightarrow -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2})$	$\rightarrow \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2})$	$\rightarrow \frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3})$	$\rightarrow \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4})$	$\rightarrow \zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6})$	$\rightarrow \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1})$	$\rightarrow \frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2})$	$\rightarrow \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2})$	$\rightarrow \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1})$	$\rightarrow \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1})$	$\rightarrow \frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2})$	$\rightarrow -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3})$	$\rightarrow \frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1})$	$\rightarrow 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2})$	$\rightarrow \frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2})$	$\rightarrow \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1})$	$\rightarrow -\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2})$	$\rightarrow \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4})$	$\rightarrow \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3})$	$\rightarrow \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3})$	$\rightarrow \frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4})$	$\rightarrow \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5})$	$\rightarrow \frac{8}{35}\zeta(S_{x_0x_1})^3$

$$\mathcal{Z}_{irr}^{\leq 2}(\mathcal{X}) \subset \cdots \mathcal{Z}_{irr}^{\leq p}(\mathcal{X}) \subset \cdots \subset \mathcal{Z}_{irr}^{\infty}(\mathcal{X}) = \cup_{p\geq 2}\mathcal{Z}_{irr}^{\leq p}(\mathcal{X}).$$

[5] The set of irreducible local coordinates forms algebraic generator system for $\mathcal{Z}$.

Noetherian rewriting system & totally ordered $\mathcal{L}_{irr}^{\infty}(\mathcal{X})$

	Rewriting among $\{\Sigma_l\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Rewriting among $\{S_l\}_{\mathcal{L}ynX-X}$
3	$\Sigma_{y_2y_1} \to \frac{3}{2}\Sigma_{y_3}$	$S_{x_0x_1^2} \to S_{x_0^2x_1}$
4	$\Sigma_{y_4} \to \frac{2}{5}\Sigma_{y_2}^2$	$S_{x_0^3x_1} \to \frac{2}{5}S_{x_0x_1}^2$
	$\Sigma_{y_3y_1} \to \frac{3}{10}\Sigma_{y_2}^2$	$S_{x_0^2x_1^2} \to \frac{1}{10}S_{x_0x_1}^2$
	$\Sigma_{y_2y_1^2} \to \frac{2}{3}\Sigma_{y_2}^2$	$S_{x_0x_1^3} \to \frac{2}{5}S_{x_0x_1}^2$
5	$\Sigma_{y_3y_2} \to 3\Sigma_{y_3}\Sigma_{y_2} - 5\Sigma_{y_5}$	$S_{x_0^3x_1^2} \to -S_{x_0^2x_1}S_{x_0x_1} + 2S_{x_0^4x_1}$
	$\Sigma_{y_4y_1} \to -\Sigma_{y_3}\Sigma_{y_2} + \frac{5}{2}\Sigma_{y_5}$	$S_{x_0^2x_1x_0x_1} \to -\frac{3}{2}S_{x_0^4x_1} + S_{x_0^2x_1}S_{x_0x_1}$
	$\Sigma_{y_2^2y_1} \to \frac{3}{2}\Sigma_{y_3}\Sigma_{y_2} - \frac{25}{12}\Sigma_{y_5}$	$S_{x_0^2x_1^3} \to -S_{x_0^2x_1}S_{x_0x_1} + 2S_{x_0^4x_1}$
	$\Sigma_{y_3y_1^2} \to \frac{5}{12}\Sigma_{y_5}$	$S_{x_0x_1x_0x_1^2} \to \frac{1}{2}S_{x_0^4x_1}$
	$\Sigma_{y_2y_1^3} \to \frac{1}{4}\Sigma_{y_3}\Sigma_{y_2} + \frac{5}{4}\Sigma_{y_5}$	$S_{x_0x_1^4} \to S_{x_0^4x_1}$
6	$\Sigma_{y_6} \to \frac{8}{35}\Sigma_{y_2}^3$	$S_{x_0^5x_1} \to \frac{8}{35}S_{x_0x_1}^3$
	$\Sigma_{y_4y_2} \to \Sigma_{y_3}^2 - \frac{4}{21}\Sigma_{y_2}^3$	$S_{x_0^4x_1^2} \to \frac{6}{35}S_{x_0x_1}^3 - \frac{1}{2}S_{x_0^2x_1}^2$
	$\Sigma_{y_5y_1} \to \frac{2}{7}\Sigma_{y_2}^3 - \frac{1}{2}\Sigma_{y_3}^2$	$S_{x_0^3x_1x_0x_1} \to \frac{4}{105}S_{x_0x_1}^3$
	$\Sigma_{y_3y_1y_2} \to -\frac{17}{30}\Sigma_{y_2}^3 + \frac{9}{4}\Sigma_{y_3}^2$	$S_{x_0^3x_1^3} \to \frac{23}{70}S_{x_0x_1}^3 - S_{x_0^2x_1}^2$
	$\Sigma_{y_3y_2y_1} \to 3\Sigma_{y_3}^2 - \frac{9}{10}\Sigma_{y_2}^3$	$S_{x_0^2x_1x_0x_1^2} \to \frac{2}{105}S_{x_0x_1}^3$
	$\Sigma_{y_4y_1^2} \to \frac{3}{10}\Sigma_{y_2}^3 - \frac{3}{4}\Sigma_{y_3}^2$	$S_{x_0^2x_1^2x_0x_1} \to -\frac{89}{210}S_{x_0x_1}^3 + \frac{3}{2}S_{x_0^2x_1}^2$
	$\Sigma_{y_2^2y_1^2} \to \frac{11}{63}\Sigma_{y_2}^3 - \frac{1}{4}\Sigma_{y_3}^2$	$S_{x_0^2x_1^4} \to \frac{6}{35}S_{x_0x_1}^3 - \frac{1}{2}S_{x_0^2x_1}^2$
	$\Sigma_{y_3y_1^3} \to \frac{1}{21}\Sigma_{y_2}^3$	$S_{x_0x_1x_0x_1^3} \to \frac{8}{21}S_{x_0x_1}^3 - S_{x_0^2x_1}^2$
	$\Sigma_{y_2y_1^4} \to \frac{17}{50}\Sigma_{y_2}^3 + \frac{3}{16}\Sigma_{y_3}^2$	$S_{x_0x_1^5} \to \frac{8}{35}S_{x_0x_1}^3$

$$\mathcal{L}_{irr}^{\le 2}(\mathcal{X}) \subset \cdots \mathcal{L}_{irr}^{\le p}(\mathcal{X}) \subset \cdots \subset \mathcal{L}_{irr}^{\infty}(\mathcal{X}) = \cup_{p\ge 2}\mathcal{L}_{irr}^{\le p}(\mathcal{X}).$$

$$\forall l \in \left\{\begin{matrix} \mathcal{L}ynY \setminus \{y_1\} \\ \mathcal{L}ynX \setminus X \end{matrix}\right\}, \quad \left\{\begin{matrix} \Sigma_l \in \mathcal{L}_{irr}^{\infty}(Y) \\ S_l \in \mathcal{L}_{irr}^{\infty}(X) \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} \Sigma_l \to \Sigma_l \\ S_l \to S_l \end{matrix}\right\} \Leftrightarrow Q_l = 0.$$

STRUCTURE OF POLYZETAS

Identification of local coordinates $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX\setminus X}$ $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$

The identification of local coordinates in $Z_\gamma = B(y_1)\pi_Y Z_{\sqcup\!\sqcup}$, leads to

1. A family of algebraic generators $\mathcal{Z}_{irr}^{\infty}(\mathcal{X})$ of $\mathcal{Z}$ constructed as follows
$$\mathcal{Z}_{irr}^{\leq 2}(\mathcal{X}) \subset \cdots \subset \mathcal{Z}_{irr}^{\leq p}(\mathcal{X}) \subset \cdots \subset \mathcal{Z}_{irr}^{\infty}(\mathcal{X}) = \bigcup_{p\geq 2} \mathcal{Z}_{irr}^{\leq p}(\mathcal{X})$$
and their inverse image, by a section of ζ,
$$\mathcal{L}_{irr}^{\leq 2}(\mathcal{X}) \subset \cdots \subset \mathcal{L}_{irr}^{\leq p}(\mathcal{X}) \subset \cdots \subset \mathcal{L}_{irr}^{\infty}(\mathcal{X}) = \bigcup_{p\geq 2} \mathcal{L}_{irr}^{\leq p}(\mathcal{X})$$
such that the following restriction is bijective
$$\zeta : \mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})] \to \mathcal{Z} = \mathbb{Q}[\mathcal{Z}_{irr}^{\infty}(\mathcal{X})] = \mathbb{Q}[\{\zeta(p)\}_{p\in\mathcal{L}_{irr}^{\infty}(\mathcal{X})}].$$
2. A ideal $\mathcal{R}_{\mathcal{X}}$ generated by the polynomials $\{Q_l\}_{\substack{l\in\mathcal{L}yn\mathcal{X}\\ l\neq y_1,x_0,x_1}}$ homogenous in weight $(=(l))$ such that the following assertions are equivalent
   i. $Q_l = 0$,
   ii. $\Sigma_l \to \Sigma_l$ (resp. $S_l \to S_l$),
   iii. $\Sigma_l \in \mathcal{L}_{irr}^{\infty}(Y)$ (resp. $S_l \in \mathcal{L}_{irr}^{\infty}(X)$).

   $0 \neq Q_l$ is led by Σ_l (resp. S_l), being transcendent over $\mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})]$, and $\Sigma_l \to \Upsilon_l$ (resp. $S_l \to U_l$) being homogenous of weight $p = (l)$ and belonging to $\mathbb{Q}[\mathcal{L}_{irr}^{\leq p}(\mathcal{X})]$. In other terms, $\Sigma_l = Q_l + \Upsilon_l$ (resp. $S_l = Q_l + U_l$), *i.e.* $\mathrm{span}_{\mathbb{Q}} \begin{matrix}\{S_l\}_{l\in\mathcal{L}ynX\setminus X}\\ \{\Sigma_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}\end{matrix} = \mathcal{R}_{\mathcal{X}} \oplus \mathrm{span}_{\mathbb{Q}} \mathcal{L}_{irr}^{\infty}(\mathcal{X})$.

Im and ker of $\zeta : \begin{matrix}(\mathbb{Q}[\{S_l\}_{l\in\mathcal{L}ynX\setminus X}], ⧢, 1_{X^*})\\(\mathbb{Q}[\{\Sigma_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}], ⧢, 1_{Y^*})\end{matrix} \twoheadrightarrow (\mathcal{Z}, ., 1)$

Let $Q \in \mathcal{R}_{\mathcal{X}} \cap \mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})]$. Since $\mathcal{R}_{\mathcal{X}} \subseteq \ker\zeta$ then $\zeta(Q) = 0$. Restricted on $\mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})]$, the polymorphism ζ is bijectice, it follows then $Q = 0$.
For any $w \in \begin{matrix}x_0X^*x_1\\(Y\setminus\{y_1\})Y^*\end{matrix}$, by the Radford's Theorem, $\zeta(w) \in \mathbb{Q}[\mathcal{Z}_{irr}^{\infty}(\mathcal{X})]$.
Thus, for $P \in \begin{matrix}\mathbb{Q}[\{S_l\}_{l\in\mathcal{L}ynX\setminus X}]\\\mathbb{Q}[\{\Sigma_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}]\end{matrix}$, $P \notin \ker\zeta$, by linearity, $\zeta(P) \in \mathbb{Q}[\mathcal{Z}_{irr}^{\infty}(\mathcal{X})]$.

Proposition 1

$$\begin{aligned}\mathbb{Q}[\{S_l\}_{l\in\mathcal{L}ynX\setminus X}] &= \mathcal{R}_X \oplus \mathbb{Q}[\mathcal{L}_{irr}^{\infty}(X)],\\ \mathbb{Q}[\{\Sigma_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}] &= \mathcal{R}_Y \oplus \mathbb{Q}[\mathcal{L}_{irr}^{\infty}(Y)],\end{aligned}$$

(as v.s. associated to ⧢ or ⧢-subalgebras). By duality & CQMM,

$$\begin{aligned}\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X\rangle \setminus X) &= \mathcal{J}_X \oplus \mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle\{P_l\}_{l\in\mathcal{L}ynX:S_l\in\mathcal{L}_{irr}^{\infty}(X)}\rangle),\\ \mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle Y\rangle \setminus \{y_1\}) &= \mathcal{J}_Y \oplus \mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle\{\Pi_l\}_{l\in\mathcal{L}ynX:\Sigma_l\in\mathcal{L}_{irr}^{\infty}(Y)}\rangle),\end{aligned}$$

where $\mathcal{J}_X$ (resp. $\mathcal{J}_Y$) is a Lie ideal generated by $\{P_l\}_{l\in\mathcal{L}ynX:S_l\notin\mathcal{L}_{irr}^{\infty}(X)}$ (resp. $\{\Pi_l\}_{l\in\mathcal{L}ynY:\Sigma_l\notin\mathcal{L}_{irr}^{\infty}(Y)}$).

Now, let $Q \in \ker\zeta$, $\langle Q|1_{\mathcal{X}^*}\rangle = 0$. Then $Q = Q_1 + Q_2$ with $Q_1 \in \mathcal{R}_{\mathcal{X}}$ and $Q_2 \in \mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})]$. Thus, $Q \equiv_{\mathcal{R}_{\mathcal{X}}} Q_1 \in \mathcal{R}_{\mathcal{X}}$.

Corollary 4

$\mathbb{Q}[\{\zeta(p)\}_{p\in\mathcal{L}_{irr}^{\infty}(\mathcal{X})}] = \mathcal{Z} = \mathrm{Im}\,\zeta$ *and* $\mathcal{R}_{\mathcal{X}} = \ker\zeta$.

Structure of polyzetas

As an ideal generated by homogenous polynomials, $\ker\zeta$ is graded. Since $\mathrm{Im}\,\zeta \cong \mathbb{Q}1_{Y^*} \oplus (Y - \{y_1\})\mathbb{Q}\langle Y\rangle/\ker\zeta \cong \mathbb{Q}1_{X^*} \oplus x_0\mathbb{Q}\langle X\rangle x_1/\ker\zeta$ then

Corollary 5

$\mathcal{Z}$ is graded.

Now, let $\xi := \zeta(P)$, where $\mathbb{Q}\langle\mathcal{X}\rangle \ni P \notin \ker\zeta$, homogenous in weight. Each monomial $\xi^n, n \geq 1$, is of different weight (because $\mathcal{Z}_p\mathcal{Z}_n \subset \mathcal{Z}_{p+n}$). Thus ξ could not satisfy $\xi^n + a_{n-1}\xi^{n-1} + \ldots = 0$, with $a_{n-1}, \ldots \in \mathbb{Q}$.

Any $s \in \mathcal{L}_{irr}^{\infty}(\mathcal{X})$ is homogenous in weight then $\zeta(s)$ is transcendent over $\mathbb{Q}$.

For any $l \in \mathcal{L}yn\mathcal{X}, l \neq y_1, x_0, x_1$, one has $l \succeq y_n$ (resp. $l \succeq x_0^{n-1}x_1$). In particular, $\Sigma_{y_n} = y_n \in \mathcal{L}ynY$ and $S_{x_0^{n-1}x_1} = x_0^{n-1}x_1 \in \mathcal{L}ynX$. Next,

1. $\zeta(2) = \zeta(\Sigma_{y_2}) = \zeta(S_{x_0x_1})$ is then irreducible and, by the Euler's identity about the ratio $\zeta(2k)/\pi^{2k}$, one deduces then, for $k > 1$, $\Sigma_{y_{2k}} = y_{2k} \notin \mathcal{L}_{irr}^{\infty}(Y)$ and $S_{x_0^{2k-1}x_1} = x_0^{2k-1}x_1 \notin \mathcal{L}_{irr}^{\infty}(X)$,
2. $\Sigma_{y_{2n+1}} = y_{2n+1} \in \mathcal{L}_{irr}^{\infty}(Y)$ and $S_{x_0^{2n}x_1} = x_0^{2n}x_1 \in \mathcal{L}_{irr}^{\infty}(X)$.

THANK YOU FOR YOUR ATTENTION