Combinatorics of characters and continuation of Li.

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Collaboration at various stages of the work
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Evolution Equations in Combinatorics and Physics :
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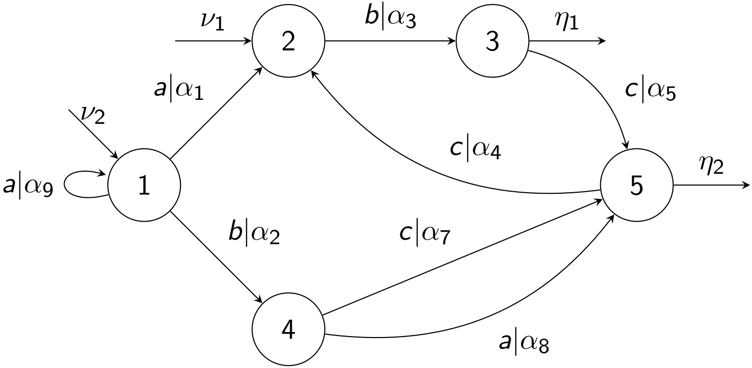
Plan

Multiplicity Automaton (Eilenberg, Schützenberger)



1 S. Eilenberg, *Automata, Languages, and Machines (Vol. A)* Acad. Press, New York, 1974

2 M.P. Schützenberger, *On the definition of a family of automata, Inf. and Contr., 4 (1961)*, 245-270.

Multiplicity automaton (linear representation) & behaviour

Linear representation

$$\nu = \begin{pmatrix}\nu_2 & \nu_1 & 0 & 0 & 0\end{pmatrix}, \qquad \eta = \begin{pmatrix}0 & 0 & \eta_1 & 0 & \eta_2\end{pmatrix}^T$$

$$\mu(a) = \begin{pmatrix} \alpha_9 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 0 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mu(c) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_5 \\ 0 & 0 & 0 & 0 & \alpha_7 \\ 0 & \alpha_4 & 0 & 0 & 0 \end{pmatrix}$$

Behaviour

$$\mathcal{A}(w) = \nu\,\mu(w)\,\eta = \sum_{\substack{i,j \\ \text{states}}} \nu(i) \underbrace{\Big(\sum weight(p)\Big)}_{\substack{\text{weight of all paths } (i) \to (j) \\ \text{with label } w}} \eta(j)$$

Operations and definitions on series

Addition, Scaling: as for functions because $R\langle\langle X\rangle\rangle = R^{X^*}$
Concatenation: $f.g(w) = \sum_{w=uv} f(u)g(v)$
Polynomials: Series s.t. $supp(f) = \{w\}_{f(w)\neq 0}$ is finite.
The set of polynomials will be denoted $R\langle X\rangle$.
Pairing: $\langle S \mid P\rangle = \sum_{w\in X^*} S(w)P(w)$ (S series, P polynomial)
Summation: $\sum_{i\in I} S_i$ summable iff f or all $w \in X^*$, $i \mapsto \langle S_i \mid w\rangle$ is finitely supported. This corresponds to the product topology (with R discrete). In particular, we have

$$\sum_{i\in I} S_i := \sum_{w\in X^*} \Big(\sum_{i\in I} \langle S_i \mid w\rangle\Big)\, w$$

Star: For all series S s.t. $\langle S \mid 1_{X^*}\rangle = 0$, the family $(S^n)_{n\geq 0}$ is summable and we set $S^* := \sum_{n\geq 0} S^n = 1 + S + S^2 + \cdots \ (= (1-S)^{-1})$.
Shifts: $\langle u^{-1}S \mid w\rangle = \langle S \mid uw\rangle$, $\langle Su^{-1} \mid w\rangle = \langle S \mid wu\rangle$

Rational series (Sweedler & Schützenberger)

Theorem A

Let $S \in k\langle\langle X \rangle\rangle$ TFAE
i) The family $(Su^{-1})_{u \in X^*}$ is of finite rank.
ii) The family $(u^{-1}S)_{u \in X^*}$ is of finite rank.
iii) The family $(u^{-1}Sv^{-1})_{u,v \in X^*}$ is of finite rank.
iv) It exists $n \in \mathbb{N}$, $\lambda \in k^{1\times n}$, $\mu : X^* \to k^{n\times n}$ (a multiplicative morphism) and $\gamma \in k^{n\times 1}$ such that, for all $w \in X^*$

$$(S, w) = \lambda\mu(w)\gamma \tag{1}$$

v) The series S is in the closure of $k\langle X \rangle$ for $(+, conc, ^*)$ within $k\langle\langle X \rangle\rangle$.

Definition

A series which fulfill one of the conditions of Theorem A will be called *rational*. The set of these series will be denoted by $k^{rat}\langle\langle X \rangle\rangle$.

Sweedler's duals

Remarks

1. (i $\leftrightarrow$ iii) needs k to be a field.
2. (iv) needs X to be finite, (iv $\leftrightarrow$ v) is known as the theorem of Kleene-Schützenberger (M.P. Schützenberger, *On the definition of a family of automata, Inf. and Contr., 4 (1961)*, 245-270.)
3. For the sake of Combinatorial Physics (where the alphabets can be infinite), **(iv)** has been extended to infinite alphabets and replaced by **iv')** The series S is in the rational closure of k^X (linear series) within $k\langle\langle X\rangle\rangle$.
4. This theorem is linked to the following: Representative functions on X^* (see Eichii Abe, Chari & Pressley), Sweedler's duals &c.
5. In the vein of (v) expressions like ab^* or identities like $(ab^*)^*a^* = (a+b)^*$ (Lazard's elimination) will be called rational.

From theory to practice: Schützenberger's calculus

From series to automata

Starting from a series S, one has a way to construct an automaton (finite-stated iff the series is rational) providing that we know how to compute on shifts and one-letter-shifts will be sufficient due to the formula $u^{-1}v^{-1}S = (vu)^{-1}S$.

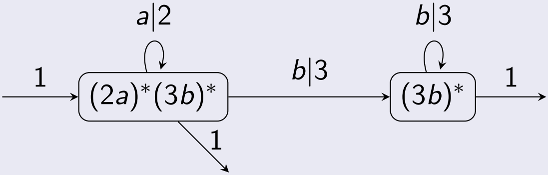
Calculus on rational expressions

In the following, x is a letter, E, F are rational expressions (i.e. expressions built from letters by scalings, concatenations and stars)

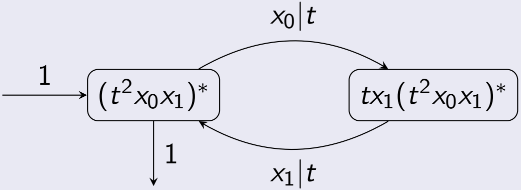
1. x^{-1} is (left and right) linear
2. $x^{-1}(E.F) = x^{-1}(E).F + \langle E \mid 1_{X^*}\rangle x^{-1}(F)$
3. $x^{-1}(E^*) = x^{-1}(E).E^*$

Examples

With $(2a)^*(3b)^*$; $X = \{a, b\}$



With $(t^2x_0x_1)^*$; $X = \{x_0, x_1\}$



From theory to practice: construction starting from S.

- **States** $\boxed{u^{-1}S}$ (constructed step by step)
- **Edges** We shift every state by letters (length) level by level (knowing that $x^{-1}(u^{-1}S) = (ux)^{-1}S$). Two cases:
  Returning state: The state is a linear combination of the already created ones i.e. $x^{-1}(u^{-1}S) = \sum_{v \in F} \alpha(ux, v) v^{-1}S$ (with F finite), then we set the edges

$$\boxed{u^{-1}S} \xrightarrow{x|\alpha_v} \boxed{v^{-1}S}$$

  The created state is new: Then

$$\boxed{u^{-1}S} \xrightarrow{x|1} \boxed{x^{-1}(u^{-1}S)}$$

- **Input** $\boxed{S}$ with the weight 1
- **Outputs** All states $\boxed{T}$ with weight $\langle T \mid 1_{X^*} \rangle$

Link with conc-bialgebras (CAP 17)

We call here conc-bialgebras, structures such that $\mathcal{B} = (k\langle X\rangle, conc, 1_{X^*}, \Delta, \epsilon)$ is a bialgebra and $\Delta(X) \subset (k.X \oplus k.1_{X^*})^{\otimes 2}$. For this, as $k\langle X\rangle$ is a free algebra, it suffices to define Δ and check the axioms on letters. Below, some examples

Shuffle: X is arbitrary $\Delta(x) = x \otimes 1 + 1 \otimes x$ and

$$\Delta(w) = \sum_{I+J=[1\cdots|w|]} w[I] \otimes w[J]$$

Stuffle: $Y = \{y_i\}_{i\geq 1}$, $\Delta(y_k) = y_k \otimes 1 + 1 \otimes y_k + \sum_{i+j=k} y_i \otimes y_j$

q-**infiltration**: X is arbitrary, $\Delta(x) = x \otimes 1 + 1 \otimes x + q\, x \otimes x$ and

$$\Delta(w) = \sum_{I\cup J=[1\cdots|w|]} q^{|I\cap J|} w[I] \otimes w[J]$$

Link with conc-bialgebras/2

In case $\epsilon(P) = \langle P \mid 1_{X^*}\rangle$[a], the restricted (graded) dual is $\mathcal{B}^\vee = (k\langle X\rangle, *, 1_{X^*}, \Delta_{conc}, \epsilon)$ and we can write, for $x \in X$

$$\Delta(x) = x \otimes 1_{X^*} + 1_{X^*} \otimes x + \Delta_+(x) \tag{2}$$

then, the dual law $*$ $(={}^t\,\Delta)$ can be defined by recursion

$$\begin{array}{rcl} w * 1_{X^*} & = & 1_{X^*} * w = w \\ au * bv & = & a(u * bv) + b(au * v) + \varphi(a,b)(u * v) \end{array} \tag{3}$$

where $\varphi ={}^t\,\Delta_+ : k.X \otimes k.X \to k.X$ is an associative law.

[a] which covers all usual combinatorial cases, save Hadamard

Some dual laws

Name	Formula (recursion)	φ	Type
Shuffle [21]	$au \shuffle bv = a(u \shuffle bv) + b(au \shuffle v)$	$\varphi \equiv 0$	I
Stuffle [19]	$x_i u \stuffle x_j v = x_i(u \stuffle x_j v) + x_j(x_i u \stuffle v) + x_{i+j}(u \stuffle v)$	$\varphi(x_i, x_j) = x_{i+j}$	I
Min-stuffle [7]	$x_i u \minstuffle x_j v = x_i(u \minstuffle x_j v) + x_j(x_i u \minstuffle v) - x_{i+j}(u \minstuffle v)$	$\varphi(x_i, x_j) = -x_{i+j}$	III
Muffle [14]	$x_i u \muffle x_j v = x_i(u \muffle x_j v) + x_j(x_i u \muffle v) + x_{i\times j}(u \muffle v)$	$\varphi(x_i, x_j) = x_{i\times j}$	I
q-shuffle [3]	$x_i u \stuffle_q x_j v = x_i(u \stuffle_q x_j v) + x_j(x_i u \stuffle_q v) + q x_{i+j}(u \stuffle_q v)$	$\varphi(x_i, x_j) = q x_{i+j}$	III
q-shuffle$_2$	$x_i u \stuffle_q x_j v = x_i(u \stuffle_q x_j v) + x_j(x_i u \stuffle_q v) + q^{i.j} x_{i+j}(u \stuffle_q v)$	$\varphi(x_i, x_j) = q^{i.j} x_{i+j}$	II
$\texttt{LDIAG}(1, q_s)$ [10] (non-crossed, non-shifted)	$au \shuffle bv = a(u \shuffle bv) + b(au \shuffle v) + q_s^{\lvert a\rvert\lvert b\rvert} a.b(u \shuffle v)$	$\varphi(a,b) = q_s^{\lvert a\rvert\lvert b\rvert}(a.b)$	II
q-Infiltration [12]	$au \uparrow bv = a(u \uparrow bv) + b(au \uparrow v) + q\delta_{a,b} a(u \uparrow v)$	$\varphi(a,b) = q\delta_{a,b} a$	III
AC-stuffle	$au \shuffle_\varphi bv = a(u \shuffle_\varphi bv) + b(au \shuffle_\varphi v) + \varphi(a,b)(u \shuffle_\varphi v)$	$\varphi(a,b) = \varphi(b,a)$ $\varphi(\varphi(a,b),c) = \varphi(a,\varphi(b,c))$	IV
Semigroup-stuffle	$x_t u \shuffle_\perp x_s v = x_t(u \shuffle_\perp x_s v) + x_s(x_t u \shuffle_\perp v) + x_{t\perp s}(u \shuffle_\perp v)$	$\varphi(x_t, x_s) = x_{t\perp s}$	I
φ-shuffle	$au \shuffle_\varphi bv = a(u \shuffle_\varphi bv) + b(au \shuffle_\varphi v) + \varphi(a,b)(u \shuffle_\varphi v)$	$\varphi(a,b)$ law of AAU	V

Of course, the q-shuffle is equal to the (classical) shuffle when $q = 0$. As for the q-infiltration, when $q = 1$, one recovers the infiltration product defined in [6]

A useful property

Proposition B

Let $\mathcal{B} = (k\langle X\rangle, conc, 1_{X^*}, \Delta, \epsilon)$ be a conc-bialgebra, then

1. The space $k^{rat}\langle X\rangle$ is closed by the convolution product $\diamond$ (here ${}^t\Delta$) given by
$$\langle S \diamond T \mid w\rangle = \langle S \otimes T \mid \Delta(w)\rangle \tag{4}$$
2. If k is a $\mathbb{Q}$-algebra and $\Delta_+ \; : \; k.X \to k.X \otimes k.X$ cocommutative, $\mathcal{B}$ is an enveloping algebra iff Δ_+ is moderate[a].
3. If, moreover k is without zero divisors, the characters $(x^*)_{x\in X}$ are algebraically independant over $(k\langle X\rangle, \diamond, 1_{X^*})$ within $(k\langle\langle X\rangle\rangle, \diamond, 1_{X^*})$.

[a] See CAP 2017

A useful property/2

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Independence of characters with respect to polynomials

5

1

I came across the following property :

Let $\mathfrak{g}$ be a Lie algebra over a ring k without zero divisors, $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ be its enveloping algebra. As such, $\mathcal{U}$ is a Hopf algebra and ϵ, its counit, is the only character of $\mathcal{U} \to k$ which vanishes on $\mathfrak{g}$.

Set $\mathcal{U}_+ = ker(\epsilon)$. We build the following filtrations $(N \geq 1)$

$$\mathcal{U}_N = \mathcal{U}_+^N = \underbrace{\mathcal{U}_+ \ldots . \mathcal{U}_+}_{N \text{ times}} \qquad (1)$$

and

$$\mathcal{U}_N^* = \mathcal{U}_{N+1}^{\perp} = \{f \in \mathcal{U}^* | (\forall u \in \mathcal{U}_{N+1})(f(u) = 0)\} \qquad (2)$$

the first one is decreasing and the second one increasing. One shows easily that (with $\diamond$ as the convolution product)

$$\mathcal{U}_p^* \diamond \mathcal{U}_q^* \subset \mathcal{U}_{p+q}^*$$

so that $\mathcal{U}_\infty^* = \cup_{n \geq 1} \mathcal{U}_n^*$ is a convolution subalgebra of $\mathcal{U}^*$.

Now, we can state the

Theorem : The set of characters of $(\mathcal{U}, ., 1_{\mathcal{U}})$ is linearly free w.r.t. $\mathcal{U}_\infty^*$.

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A useful property/3

Remark

Property (3) is no longer true if Δ is not moderate. For example with the Hadamard coproduct and $x \neq y$, one has $y \odot (x)^* = 0$.

Examples

Shuffle: $(\alpha x)^* ⧢ (\beta y)^* = (\alpha x + \beta y)^*$

Stuffle: $(\alpha y_i)^* \uplus (\beta y_j)^* = (\alpha y_i + \beta y_j + \alpha\beta y_{i+j})^*$

***q*-infiltration**: $(\alpha x)^* \uparrow_q (\beta y)^* = (\alpha x + \beta y + \alpha\beta\delta_{x,y} x)^*$

Hadamard: $(\alpha a)^* \odot (\beta b)^* = 1_{X^*}$ if $a \neq b$ and $(\alpha a)^* \odot (\beta a)^* = (\alpha\beta a)^*$

Starting the ladder

$$\begin{array}{ccc} (\mathbb{C}\langle X\rangle, ш, 1_{X^*}) & \xrightarrow{\mathrm{Li}_\bullet} & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\ \downarrow & & \downarrow \\ (\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] & \xrightarrow{\mathrm{Li}_\bullet^{(1)}} & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*} \end{array}$$

Domain of Li (definition)

In order to extend Li to series, we define $Dom(Li;\Omega)$ (or $Dom(Li)$) if the context is clear) as the set of series $S = \sum_{n\geq 0} S_n$ (decomposition by homogeneous components) such that $\sum_{n\geq 0} Li_{S_n}(z)$ converges for the compact convergence in Ω. One sets

$$Li_S(z) := \sum_{n\geq 0} Li_{S_n}(z) \tag{5}$$

Examples

$$Li_{x_0^*}(z) = z,\ \ Li_{x_1^*}(z) = (1-z)^{-1}\ ;\ \ Li_{\alpha x_0^* + \beta x_1^*}(z) = z^{\alpha}(1-z)^{-\beta}$$

Properties of the extended Li

Proposition

With this definition, we have

1. $Dom(Li)$ is a shuffle subalgebra of $\mathbb{C}\langle\langle X\rangle\rangle$ and then so is $Dom^{rat}(Li) := Dom(Li) \cap \mathbb{C}^{rat}\langle\langle X\rangle\rangle$
2. For $S, T \in Dom(Li)$, we have

$$\mathrm{Li}_{S \sqcup\!\sqcup T} = \mathrm{Li}_S \,.\, \mathrm{Li}_T$$

Examples and counterexamples

For $|t| < 1$, one has $(tx_0)^* x_1 \in Dom(Li, D)$ (D is the open unit slit disc), whereas $x_0^* x_1 \notin Dom(Li, D)$.
Indeed, we have to examine the convergence of $\sum_{n\geq 0} \mathrm{Li}_{x_0^n x_1}(z)$, but, for $z \in]0, 1[$, one has $0 < z < \mathrm{Li}_{x_0^n x_1}(z) \in \mathbb{R}$ and therefore, for these values $\sum_{n\geq 0} \mathrm{Li}_{x_0^n x_1}(z) = +\infty$.

Coefficients in the Ladder

$$\begin{array}{ccc}
(\mathbb{C}\langle X\rangle, ш, 1_{X^*}) & \xrightarrow{\mathrm{Li}_\bullet} & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\
\big\downarrow & & \big\downarrow \\
(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] & \xrightarrow{\mathrm{Li}_\bullet^{(1)}} & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\big\downarrow & & \big\downarrow \\
\mathbb{C}\langle X\rangle \, ш \, \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \, ш \, \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle & \xrightarrow{\mathrm{Li}_\bullet^{(2)}} & \mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*}
\end{array}$$

Were, for every additive subgroup $(H,+) \subset (\mathbb{C},+)$, $\mathcal{C}_H$ has been set to the following subring of $\mathbb{C}$

$$\mathcal{C}_H := \mathbb{C}\{z^\alpha(1-z)^{-\beta}\}_{\alpha,\beta\in H} \ . \tag{6}$$

Examples

$$Li_{x_0^*}(z) = z, \ \ Li_{x_1^*}(z) = (1-z)^{-1} \ ; \ \ Li_{\alpha x_0^* + \beta x_1^*}(z) = z^\alpha (1-z)^{-\beta}$$

The arrow $\mathrm{Li}_\bullet^{(1)}$

Proposition

i. The family $\{x_0^*, x_1^*\}$ is algebraically independent over $(\mathbb{C}\langle X\rangle, ⧢, 1_{X^*})$ within $(\mathbb{C}\langle\langle X\rangle\rangle^{\mathrm{rat}}, ⧢, 1_{X^*})$.

ii. $(\mathbb{C}\langle X\rangle, ⧢, 1_{X^*})[x_0^*, x_1^*, (-x_0)^*]$ is a free module over $\mathbb{C}\langle X\rangle$, the family $\{(x_0^*)^{⧢\, k} ⧢ (x_1^*)^{⧢\, l}\}_{(k,l)\in\mathbb{Z}\times\mathbb{N}}$ is a $\mathbb{C}\langle X\rangle$-basis of it.

iii. As a consequence, $\{w ⧢ (x_0^*)^{⧢\, k} ⧢ (x_1^*)^{⧢\, l}\}_{\substack{w\in X^* \\ (k,l)\in\mathbb{Z}\times\mathbb{N}}}$ is a $\mathbb{C}$-basis of it.

iv. $\mathrm{Li}_\bullet^{(1)}$ is the unique morphism from $(\mathbb{C}\langle X\rangle, ⧢, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*]$ to $\mathcal{H}(\Omega)$ such that

$$x_0^* \to z,\ (-x_0)^* \to z^{-1} \text{ and } x_1^* \to (1-z)^{-1}$$

v. $\mathrm{Im}(\mathrm{Li}_\bullet^{(1)}) = \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*}$.

vi. $\ker(\mathrm{Li}_\bullet^{(1)})$ is the (shuffle) ideal generated by $x_0^* ⧢ x_1^* - x_1^* + 1_{X^*}$.

Sketch of the proof for vi.

Let $\mathcal{J}$ be the ideal generated by $x_0^* \sqcup\!\sqcup\, x_1^* - x_1^* + 1_{X^*}$. It is easily checked, from the following formulas[a], for $k \geq 1$,

$$\begin{aligned} w \sqcup\!\sqcup\, x_0^* \sqcup\!\sqcup\, (x_1^*)^{\sqcup\!\sqcup\, k} &\equiv w \sqcup\!\sqcup\, (x_1^*)^{\sqcup\!\sqcup\, k} - w \sqcup\!\sqcup\, (x_1^*)^{\sqcup\!\sqcup\, k-1} \ [\mathcal{J}], \\ w \sqcup\!\sqcup\, (-x_0)^* \sqcup\!\sqcup\, (x_1^*)^{\sqcup\!\sqcup\, k} &\equiv w \sqcup\!\sqcup\, (-x_0)^* \sqcup\!\sqcup\, (x_1^*)^{\sqcup\!\sqcup\, k-1} + w \sqcup\!\sqcup\, (x_1^*)^{\sqcup\!\sqcup\, k} [\mathcal{J}], \end{aligned}$$

that one can rewrite $[\bmod \mathcal{J}]$ any monomial $w \sqcup\!\sqcup\, (x_0^*)^{\sqcup\!\sqcup\, l} \sqcup\!\sqcup\, (x_1^*)^{\sqcup\!\sqcup\, k}$ as a linear combination of such monomials with $kl = 0$. Observing that the image, through $\mathrm{Li}_\bullet^{(1)}$, of the following family is free in $\mathcal{H}(\Omega)$

$$\{w \sqcup\!\sqcup\, (x_1^*)^{\sqcup\!\sqcup\, l} \sqcup\!\sqcup\, (x_0^*)^{\sqcup\!\sqcup\, k}\}_{(w,l,k) \in (X^* \times \mathbb{N} \times \{0\}) \sqcup (X^* \times \{0\} \times \mathbb{Z})} \tag{7}$$

we get the result.

[a] In the Figure below, (w, l, k) codes the element $w \sqcup\!\sqcup\, (x_0^*)^{\sqcup\!\sqcup\, l} \sqcup\!\sqcup\, (x_1^*)^{\sqcup\!\sqcup\, k}$.

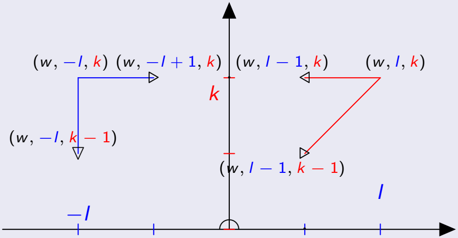


Figure: Rewriting $\mod \mathcal{J}$ of $\{w ш (x_0^*)^{ш\, l} ш (x_1^*)^{ш\, k}\}_{k\in\mathbb{N}, l\in\mathbb{Z}, w\in X^*}$.

End of the ladder: pushing coefficients to $\mathcal{C}_{\mathbb{C}}$

$$\begin{array}{ccc}
(\mathbb{C}\langle X\rangle, ш, 1_{X^*}) & \xrightarrow{\mathrm{Li}_\bullet} & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] & \xrightarrow{\mathrm{Li}_\bullet^{(1)}} & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
\mathbb{C}\langle X\rangle \, ш \, \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \, ш \, \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle & \xrightarrow{\mathrm{Li}_\bullet^{(2)}} & \mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*}
\end{array}$$

Exchangeable (rational) series

The power series S belongs to $\mathbb{C}_{\mathrm{exc}}\langle X\rangle$, iff

$$(\forall u, v \in X^*)((\forall x \in X)(|u|_x = |v|_x) \ \Rightarrow \ \langle S|u\rangle = \langle S|v\rangle). \qquad (8)$$

We will note $\mathbb{C}_{exc}^{rat}\langle X\rangle$, the set of exchangeable rational series i.e.

$$\mathbb{C}_{exc}^{rat}\langle X\rangle := \mathbb{C}_{exc}\langle X\rangle \cap \mathbb{C}^{rat}\langle X\rangle \qquad (9)$$

Lemma (D., HNM, Ngô, 2016)

1. $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle := \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle \bigcap \mathbb{C}_{\mathrm{exc}}\langle\langle X\rangle\rangle = \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$.
2. *For any $x \in X$, from a theorem by Kronecker, one has* $\mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle = \mathrm{span}_{\mathbb{C}}\{(ax)^* ⧢ \mathbb{C}\langle x\rangle | a \in \mathbb{C}\}$ *and*
$$\{(ax)^* ⧢ x^n\}_{(a,n)\in\mathbb{C}\times\mathbb{N}} \tag{10}$$
is a basis of it. When restricted to $(\mathbb{C}^* \times \mathbb{N}) \cup \{(0,0)\}$ this family spans $\mathbb{C}^{rat}_{const}\langle\langle x\rangle\rangle$ (fractions being constant at infinity)
3. $\mathbb{C}\langle X\rangle ⧢ \mathbb{C}^{rat}_{exc}\langle\langle X\rangle\rangle \simeq \mathbb{C}\langle X\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}_{const}\langle\langle x_0\rangle\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}_{const}\langle\langle x_1\rangle\rangle$
4. $\mathrm{Im}(\mathrm{Li}^{(2)}_{\bullet}) = \mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*}$.
5. $\ker(\mathrm{Li}^{(2)}_{\bullet})$ *is the (shuffle) ideal generated by* $x_0^* ⧢ x_1^* - x_1^* + 1_{X^*}$ *(prospective).*

Concluding remarks/1

1. We have coded classical (and extended) polylogarithms with words obtaining a Noncommutative generating series which is a shuffle character
2. This character can be extended by continuity to certain series forming a shuffle subalgebra of Noncommutative formal power series.
3. We have found some remarkable subalgebras of $Dom^{rat}(Li)$, given their bases and described the kernel of the so extended $\mathrm{Li}_\bullet$.

4. Definition of $Dom(Li)$ and $Dom^{rat}(Li)$ have to be refined and their exploration pushed further.
5. Combinatorics of discrete Dyson integrals for various sets of differential forms has to be implemented

Concluding remarks/2

6. Drinfeld-Kohno Lie algebras i.e. algebras presented by

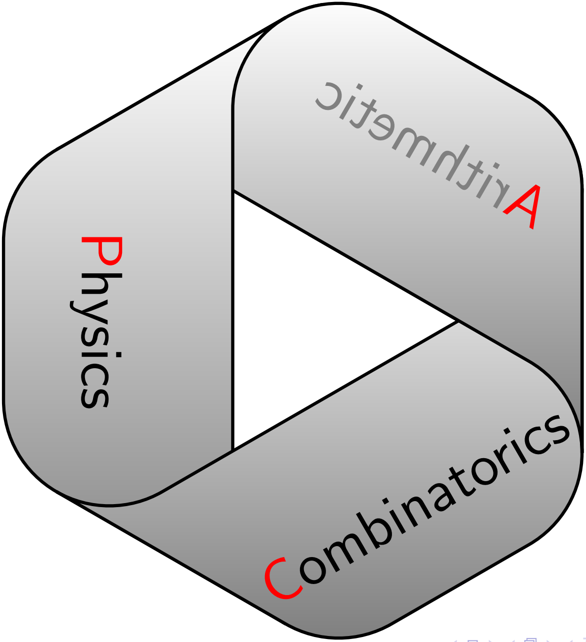
$$DK(A;k) = \langle A \times A \; ; \; \mathbf{R_A} \, \rangle_{k-\text{Lie algebras}} \tag{11}$$

with $\mathbf{R_A}$, the relator

$$\mathbf{R_A} = \begin{cases} (a,a) &= 0 \text{ for } a \in A \\ (a,b) &= (b,a) \text{ for } a,b \in A \\ [(a,c),(a,b)+(b,c)] &= 0 \text{ for } |\{a,b,c\}| = 3, \\ [(a,b),(c,d)] &= 0 \text{ for } |\{a,b,c,d\}| = 4 \end{cases} \tag{12}$$

can be decomposed in several ways as a direct sum of Free Lie algebras giving rise to product of MRS factorisations

$$\chi = \prod_{l \in \mathcal{L}yn(X)}^{\searrow} e^{\chi(S_l)\, P_l} \tag{13}$$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THANK YOU FOR YOUR ATTENTION !