MRS[1] factorizations and applications

Séminaire Combinatoire, Informatique et Physique,

3, 17 et 24 Novembre 2020, Villetaneuse

V. Hoang Ngoc Minh
Université de Lille,
1, Place Déliot,
59024 Lille, France.

[1] Mélançon-Reutenauer-Schützenberger.

Outline

INTRODUCTION

Zeta functions in several variables and polyzetas

Let $\mathcal{H}_r = \{(s_1,\ldots,s_r) \in \mathbb{C}^r | \forall m = 1,\ldots,r, \Re(s_1)+\ldots+\Re(s_m) > m\}$, for $r \in \mathbb{N}_+$, the following zeta function converges for $(s_1,\ldots,s_r) \in \mathcal{H}_r$

$$\zeta(s_1,\ldots,s_r) = \sum_{n_1>\ldots>n_r>0} n_1^{-s_1}\ldots n_r^{-s_r}.$$

By a Abel's theorem, for $n \in \mathbb{N}, z \in \mathbb{C}, |z| < 1$, this value can be obtained as

$$\zeta(s_1,\ldots,s_r) = \lim_{n\to+\infty} \mathrm{H}_{s_1,\ldots,s_r}(n) = \lim_{z\to 1} \mathrm{Li}_{s_1,\ldots,s_r}(z),$$

where the following functions are well defined for $(s_1,\ldots,s_r) \in \mathbb{C}^r$

$$\mathrm{Li}_{s_1,\ldots,s_r}(z) = \sum_{n_1>\ldots>n_r>0} \frac{z^{n_1}}{n_1^{s_1}\ldots n_r^{s_r}} \quad \text{and} \quad \frac{\mathrm{Li}_{s_1,\ldots,s_r}(z)}{1-z} = \sum_{n\geq 0} \mathrm{H}_{s_1,\ldots,s_r}(n) z^n.$$

$$\mathcal{Z} := \mathrm{span}_{\mathbb{Q}}\{\mathrm{Li}_{s_1,\ldots,s_r}(1)\}_{\substack{s_1,\ldots,s_r\in\mathbb{N}_+\\ s_1>1}} = \mathrm{span}_{\mathbb{Q}}\{\mathrm{H}_{s_1,\ldots,s_r}(+\infty)\}_{\substack{s_1,\ldots,s_r\in\mathbb{N}_+\\ s_1>1}}.$$

We use the correspondence between words over $X = \{x_0, x_1\}$, $Y = \{y_k\}_{k\geq 1}$:

$$(s_1,\ldots,s_r) \in \mathbb{N}_+^r \leftrightarrow y_{s_1}\ldots y_{s_r} \in Y^* \underset{\pi_Y}{\overset{\pi_X}{\rightleftharpoons}} x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1 \in X^*x_1.$$

$\mathrm{Li}_{s_1,\ldots,s_r}(z) = \alpha_{z_0}^{z}(x_0^{s_1-1}x_1\ldots x_0^{s_k-1}x_1)$, where $z_0 \rightsquigarrow z$ is a path in a simply connected domain, in $\mathbb{C}$, with a subdivision $(z_0, z_1, \ldots, z_k, z)$ and

$$\alpha_{z_0}^{z}(x_{i_1}\ldots x_{i_k}) = \int_{z_0}^{z} \omega_{i_1}(z_1)\ldots\int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k) \quad \text{with} \quad \begin{cases} \omega_0(z) = dz/z, \\ \omega_1(z) = dz/(1-z). \end{cases}$$

Words and noncommutative formal series

- Let $(X^*, 1_{X^*})$ (resp. $(Y^*, 1_{Y^*})$) be the free monoid generated by X (resp. Y) equipped with the order $x_1 \succ x_0$ (resp. $y_1 \succ y_2 \succ \ldots$).
- $A\langle\mathcal{X}\rangle$ (resp. $A\langle\langle\mathcal{X}\rangle\rangle$): set of polynomials (resp. formal series) with coef. in the commutative ring $A \supset \mathbb{Q}$ over $\mathcal{X}$ which denotes X or Y.
- On $(A\langle\mathcal{X}\rangle, \mathrm{conc}, 1_{\mathcal{X}^*}, \Delta_{\sqcup\!\sqcup}, \varepsilon)$ (resp. $(A\langle Y\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{\uplus}, \varepsilon)$), for $x, y \in \mathcal{X}, y_i, y_j \in Y$ and $u, v \in \mathcal{X}^*$ (resp. Y^*), one defines
  - $\Delta_{\sqcup\!\sqcup} x = x \otimes 1_{\mathcal{X}^*} + 1_{\mathcal{X}^*} \otimes x$, or equivalently $u \sqcup\!\sqcup 1_{\mathcal{X}^*} = 1_{\mathcal{X}^*} \sqcup\!\sqcup u = u$ and $xu \sqcup\!\sqcup yv = x(u \sqcup\!\sqcup yv) + y(xu \sqcup\!\sqcup v)$,
  - $\Delta_{\uplus} y_i = y_i \otimes 1_{Y^*} + 1_{Y^*} \otimes y_i + \sum_{k+l=i} y_k \otimes y_l$, or equivalently $u \uplus 1_{Y^*} = 1_{Y^*} \uplus u = u$ and $x_i u \uplus y_j v = y_i(u \uplus y_j v) + y_j(y_i u \uplus v) + y_{i+j}(u \uplus v)$.
- Considering A as the differential ring of holomorphic functions on a simply connected domain Ω, denoted by $(\mathcal{H}(\Omega), \partial)$ and equipped with 1_Ω as neutral element, the differential ring $(\mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle, \mathrm{d})$ is defined, for any $S \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle$, as follows

$$\mathrm{d}S = \sum_{w \in \mathcal{X}^*} (\partial\langle S|w\rangle) w \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle.$$

$\mathrm{Const}(\mathcal{H}(\Omega)) = \mathbb{C}.1_\Omega$ and $\mathrm{Const}(\mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle) = \mathbb{C}.1_\Omega\langle\langle\mathcal{X}\rangle\rangle$.

Comb. of noncommutative co-commutative bialgebras

Concatenation-shuffle bialgebra[2], $(A\langle\mathcal{X}\rangle, \mathrm{conc}, \Delta_{\sqcup\!\sqcup}, 1_{\mathcal{X}^*}, \varepsilon)$:

$$\mathcal{D}_{\mathcal{X}} := \sum_{w\in\mathcal{X}^*} w\otimes w = \sum_{w\in\mathcal{X}^*} S_w \otimes P_w = \prod_{l\in\mathcal{L}yn\mathcal{X}}^{\searrow} e^{S_l\otimes P_l} \quad \text{(MRS-factorization)},$$

where $\{P_l\}_{l\in\mathcal{L}yn\mathcal{X}}$ is a basis of $\mathcal{L}ie_A\langle\mathcal{X}\rangle$, defined by

$$P_l = l \text{ if } l\in\mathcal{X} \text{ and } P_l = [P_u, P_v] \text{ if } l\in\mathcal{L}yn\mathcal{X}, \mathrm{st}(l) = (u, v),$$

$\{P_w\}_{w\in\mathcal{X}^*}$ is the PBW basis of $\mathcal{U}(\mathcal{L}ie_A\langle\mathcal{X}\rangle)$ and $\{S_w\}_{w\in\mathcal{X}^*}$ is its dual basis, containing the pure transcendence basis, $\{S_l\}_{l\in\mathcal{L}yn\mathcal{X}}$, of $(A\langle\mathcal{X}\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*})$.

Concatenation-stuffle bialgebra[3], $(A\langle Y\rangle, \mathrm{conc}, \Delta_{\uplus}, 1_{Y^*}, \varepsilon)$:

$$\mathcal{D}_{Y} := \sum_{w\in Y^*} w\otimes w = \sum_{w\in Y^*} \Sigma_w \otimes \Pi_w = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\Sigma_l\otimes \Pi_l} \quad (\uplus - \text{modified MRS-factorization}),$$

where $\{\Pi_l\}_{l\in\mathcal{L}ynY}$ is a basis of $\mathrm{Prim}(Y)$, defined by[4]

$$\Pi_l = \pi_1(l) \text{ if } l\in Y \text{ and } \Pi_l = [\Pi_u, \Pi_v] \text{ if } l\in\mathcal{L}ynY, \mathrm{st}(l) = (u, v),$$

$\{\Pi_w\}_{w\in X^*}$ is the PBW basis of $\mathcal{U}(\mathrm{Prim}(Y))$ and $\{\Sigma_w\}_{w\in Y^*}$ is its dual basis, containing the pure transcendence basis, $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$, of $(A\langle Y\rangle, \uplus, 1_{Y^*})$.

---

[2] Letters are primitive, for $\Delta_{\sqcup\!\sqcup}$.

[3] Only letter y_1 is primitive, for $\Delta_{\uplus}$.

[4] π_1 denotes a $\uplus$-modified eulerian projector.

FIRST STRUCTURES AND ABEL LIKE THEOREMS

First structures of polylogarithms and harmonic sums

$z \in \Omega = \mathbb{C} \setminus \widetilde{\{0,1\}}$. For any $k \geq 0$ let $\mathrm{Li}_{x_0^k}(z) := \alpha_1^z(x_0^k) = \log^k(z)/k$.

1. $\{\mathrm{Li}_w\}_{w \in X^*}$ is $\mathbb{C}$-linearly independent. Hence, the following morphism of algebras is injective
$$\begin{aligned} \mathrm{Li}_\bullet : (\mathbb{C}\langle X \rangle, ш, 1_{X^*}) &\to (\mathbb{C}\{\mathrm{Li}_w\}_{w \in X^*}, ., 1), \\ x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 &\mapsto \mathrm{Li}_{x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1} \ (\textit{i.e.}\ \mathrm{Li}_{s_1,\ldots,s_r}). \end{aligned}$$
Thus, $\{\mathrm{Li}_l\}_{l \in \mathcal{L}ynX}$ (resp. $\{\mathrm{Li}_{S_l}\}_{l \in \mathcal{L}ynX}$) is algebraically independent.

2. The following morphism of algebras is injective
$$\begin{aligned} \mathrm{P}_\bullet : (\mathbb{C}\langle Y \rangle, \uplus, 1_{Y^*}) &\to (\mathbb{C}\{\mathrm{P}_w\}_{w \in Y^*}, \odot, 1), \\ w &\mapsto \mathrm{P}_w(z) := \frac{\mathrm{Li}_{\pi_X w}(z)}{1-z} = \sum_{n \geq 0} \mathrm{H}_w(n) z^n. \end{aligned}$$
Hence, $\{\mathrm{P}_w\}_{w \in Y^*}$ is $\mathbb{C}$-linearly independent. It follows that, $\{\mathrm{P}_l\}_{l \in \mathcal{L}ynY}$ (resp. $\{\mathrm{P}_{\Sigma_l}\}_{l \in \mathcal{L}ynY}$) is algebraically independent, for[5] $\odot$.

3. The following morphism of algebras is injective
$$\begin{aligned} \mathrm{H}_\bullet : (\mathbb{C}\langle Y \rangle, \uplus, 1_{Y^*}) &\to (\mathbb{C}\{\mathrm{H}_w\}_{w \in Y^*}, ., 1), \\ y_{s_1} \ldots y_{s_r} &\mapsto \mathrm{H}_{y_{s_1} \ldots y_{s_r}} \ (\textit{i.e.}\ \mathrm{H}_{s_1,\ldots,s_r}). \end{aligned}$$
Hence, $\{\mathrm{H}_w\}_{w \in Y^*}$ is $\mathbb{C}$-linearly independent. It follows that, $\{\mathrm{H}_l\}_{l \in \mathcal{L}ynY}$ (resp. $\{\mathrm{H}_{\Sigma_l}\}_{l \in \mathcal{L}ynY}$) is algebraically independent.

[5] For any $u, v \in Y, \mathrm{P}_u \odot \mathrm{P}_v = \mathrm{P}_{u \uplus v}$.

Towards more about structure of $\mathcal{Z}$

4. The following polymorphism of algebras is surjective

$$\zeta : \begin{array}{c}(\mathbb{C}[\mathcal{L}ynX - X], ⧢, 1_{X^*})\\ (\mathbb{C}[\mathcal{L}ynY - \{y_1\}], ⧢\!\!\!\!\pm, 1_{Y^*})\end{array} \twoheadrightarrow (\mathcal{Z}, ., 1),$$

$$\begin{array}{c}x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1\\ y_{s_1}\ldots y_{s_r}\end{array} \mapsto \zeta(s_1, \ldots, s_r).$$

$\forall l_1, l_2 \in \mathcal{L}ynX - X, \zeta(l_1 ⧢ l_2) = \zeta((\pi_Y l_1) ⧢\!\!\!\!\pm (\pi_Y l_2)) = \zeta(l_1)\zeta(l_2).$

ζ can be extended as characters:

$\zeta_{⧢} : (\mathbb{C}\langle X\rangle, ⧢, 1_{X^*}) \to (\mathcal{Z}, ., 1), \quad \zeta_{⧢\!\!\!\!\pm} : (\mathbb{C}\langle Y\rangle, ⧢\!\!\!\!\pm, 1_{Y^*}) \to (\mathcal{Z}, ., 1),$

s.t. $\zeta_{⧢}(x_0) = 0 = \log(1),$

$\zeta_{⧢}(x_1) = 0 = \mathrm{f.p.}_{z\to 1}\log(1-z), \quad \{(1-z)^a \log^b(1-z)\}_{a\in\mathbb{Z}, b\in\mathbb{N}},$

$\zeta_{⧢\!\!\!\!\pm}(y_1) = 0 = \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_1(n), \quad \{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z}, b\in\mathbb{N}}.$

$\forall k \geq 0, \mathcal{A}_k := \mathrm{span}_{\mathbb{Z}}\{\zeta(s_1, \ldots, s_r), s_1 + \ldots + s_r = k\}_{s_1+\ldots+s_r \in \mathcal{H}_r\cap\mathbb{N}^r, r\geq 0}$, and[6]
$\mathcal{Z}_k := \mathrm{span}_{\mathbb{Q}}\{\zeta(w), |w| = k\}_{w\in x_0X^*x_1} = \mathrm{span}_{\mathbb{Q}}\{\zeta(w), (w) = k\}_{w\in(Y-\{y_1\})Y^*}.$

Conjecture 1 (Zagier's dimension conjecture)

$\forall k \geq 0, d_k := \dim \mathcal{A}_k$. *Then* $d_0 = 1, d_1 = 0, d_2 = 1, d_k = d_{k-2} + d_{k-3}$ $(k \geq 3)$.

$$\bigoplus_{k\geq 0}\mathcal{A}_k \longrightarrow \mathcal{Z} \text{ is injective ? } \mathcal{Z} = \bigoplus_{k\geq 0}\mathcal{Z}_k ?$$

---

[6] For $w = x_{s_1}\ldots x_{s_r} \in \mathcal{X}^*, |w| = r$. If $\mathcal{X} = Y$ then $(w) = s_1 + \ldots + s_r$ being the weight of $(s_r, \ldots, s_r)$. Thus, $\mathcal{Z}_k = \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{A}_k$.

Noncommutative generating series

$$\mathrm{L}(z) := \sum_{w\in X^*} \mathrm{Li}_w(z)w = (\mathrm{Li}_\bullet \otimes \mathtt{Id})\mathcal{D}_X = e^{-\log(1-z)x_1}\mathrm{L}_{\mathtt{reg}}(z)e^{\log(z)x_0},$$

$$\mathrm{H}(n) := \sum_{w\in Y^*} \mathrm{H}_w(n)w = (\mathrm{H}_\bullet \otimes \mathtt{Id})\mathcal{D}_Y = e^{\mathrm{H}_{y_1}(n)y_1}\mathrm{H}_{\mathtt{reg}}(n),$$

where $\mathrm{L}_{\mathtt{reg}} := \prod^{\searrow}_{\substack{l\in\mathcal{L}ynX\\ l\neq x_0,x_1}} e^{\mathrm{Li}_{S_l} P_l}$ and $\mathrm{H}_{\mathtt{reg}} := \prod^{\searrow}_{\substack{l\in\mathcal{L}ynY\\ l\neq y_1}} e^{\mathrm{H}_{\Sigma_l}\Pi_l}$. We put also[7]

$$Z_{\sqcup\!\sqcup} := \mathrm{L}_{\mathtt{reg}}(1) = \prod^{\searrow}_{\substack{l\in\mathcal{L}ynX\\ l\neq x_0,x_1}} e^{\zeta(S_l)P_l} \quad \text{and} \quad Z_{\boxplus} := \mathrm{H}_{\mathtt{reg}}(+\infty) = \prod^{\searrow}_{\substack{l\in\mathcal{L}ynY\\ l\neq y_1}} e^{\zeta(\Sigma_l)\Pi_l}.$$

Let $\gamma_{1_{Y^*}} := 1$ and, for any $w \in Y^+$, let

$$\gamma_w := \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_w(n), \quad \{n^a \log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}.$$

Then $\gamma_\bullet$ realizes a character on $(\mathbb{C}\langle Y\rangle, \boxplus, 1_{Y^*})$, *i.e.*

$$\gamma_{y_1} = \gamma \quad \text{and} \quad \forall l \in \mathcal{L}ynY - \{y_1\}, \quad \gamma_{\Sigma_l} = \zeta(\Sigma_l).$$

$$Z_\gamma := \sum_{w\in Y^*} \gamma_w w = \prod^{\searrow}_{l\in\mathcal{L}ynY} e^{\gamma_{\Sigma_l}\Pi_l} = e^{\gamma y_1} Z_{\boxplus}.$$

L and $Z_{\sqcup\!\sqcup}$ (resp. H and $Z_\gamma, Z_{\boxplus}$) are group-like, for $\Delta_{\sqcup\!\sqcup}$ (resp. $\Delta_{\boxplus}$).

[7] $Z_{\sqcup\!\sqcup}$ corresponds to the Drindfel'd associator, Φ_{KZ}.

More about generating series

Let us consider

$$\texttt{Mono}(z) := \sum_{n\geq 0} \mathrm{P}_{y_1^n} y_1^n \in \mathcal{H}(\Omega)\langle\langle y_1\rangle\rangle \quad \text{and} \quad \texttt{Const} := \sum_{k\geq 0} \mathrm{H}_{y_1^k} y_1^k.$$

Since, for any $k \geq 1$, by definition, $\mathrm{P}_{y_1^k}(z) = (1-z)^{-1}\,\mathrm{Li}_{x_1^k}(z)$ with $\mathrm{Li}_{x_1^k}(z) = (-\log(1-z))^k/k!$ then

$$\texttt{Mono}(z) = (1-z)^{-1} e^{-\log(1-z)y_1} \quad \text{and} \quad \texttt{Const} = \exp\left(-\sum_{k\geq 0} \mathrm{H}_{y_k} \frac{(-y_1)^k}{k}\right).$$

Let us also consider[8]

$$B(y_1) := \exp\left(\gamma y_1 - \sum_{k\geq 2} \zeta(k) \frac{(-y_1)^k}{k}\right) \text{ and } B'(y_1) := \exp\left(\sum_{k\geq 2} \zeta(k) \frac{(-y_1)^k}{k}\right).$$

Note that,

$$\texttt{Const}^{-1} = \sum_{n\geq 0} \mathrm{H}_{y_1^n}(-y_1)^n = \exp\left(\sum_{k\geq 0} \mathrm{H}_{y_k} \frac{(-y_1)^k}{k}\right).$$

[8] $\mathbb{C}\langle\langle y_1\rangle\rangle \ni B(y_1) = \Gamma^{-1}(1+y_1)$. $B'(y_1)$ corresponds to the Ecalle's mould $\texttt{Mono}$.

Chen generating series of ω_0 and ω_1 along a path $z_0 \rightsquigarrow z$

$$C_{z_0\rightsquigarrow z} := \sum_{w\in X^*} \alpha_{z_0}^{z}(w)w \quad \text{with} \quad \begin{cases} \omega_0(z) = z^{-1}dz, \\ \omega_1(z) = (1-z)^{-1}dz. \end{cases}$$

By a Ree's theorem, $C_{z_0\rightsquigarrow z}$ is group-like, for $\Delta_{\sqcup\!\sqcup}$, and is solution[9] of[10]

$$(DE) \quad \mathrm{d}S = ((1-z)^{-1}x_0 + (1-z)^{-1}x_1)S.$$

Let g be the transformation $z \mapsto 1-z$. Then $g^*\omega_0 = -\omega_1$ and $g^*\omega_1 = -\omega_0$. Hence,

$$C_{g(z_0)\rightsquigarrow g(z)} = \sum_{w\in X^*} \alpha_{g(z_0)}^{g(z)}(w)w = \sum_{w\in X^*} \alpha_{z_0}^{z}(w)\sigma(w) = \sigma(C_{z_0\rightsquigarrow z}),$$

where σ is the morphism defined by $\sigma(x_0) = -x_1$ and $\sigma(x_1) = -x_0$.

On the other hand, L is also solution of (DE) and one has

$$\mathrm{L}(z) = C_{z_0\rightsquigarrow z}\mathrm{L}(z_0) \quad \text{and} \quad \mathrm{L}(g(z)) = C_{g(z_0)\rightsquigarrow g(z)}\mathrm{L}(g(z_0)).$$

Since $\mathrm{L}(z) \sim_0 e^{x_0 \log(z)}$ then

$$C_{g(z_0)\rightsquigarrow g(z)} = \sigma(\mathrm{L}(z)\mathrm{L}^{-1}(z_0)) \sim_{z_0\to 0} \sigma(\mathrm{L}(z))e^{x_1\log(z_0)}.$$

Proposition 1

Let σ be the letter morphism s.t. $\sigma(x_0) = -x_1$ and $\sigma(x_1) = -x_0$. Then

$$\mathrm{L}(1-z) = \sigma(\mathrm{L}(z))Z_{\sqcup\!\sqcup}.$$

[9] It can be obtained by a Picard iteration, initialized at $\langle C_{z_0\rightsquigarrow z}|1_{\mathcal{X}^*}\rangle = 1_\Omega 1_{\mathcal{X}^*}$.

[10] For $x_0 = A/2\mathrm{i}\pi$ and $x_1 = -B/2\mathrm{i}\pi$, (DE) is nothing else (KZ_3).

Abel like results and bridge equations

Since[11] $\mathrm{L}(z) = \sigma(\mathrm{L}(1-z))Z_{\sqcup\!\sqcup} = e^{x_0 \log(z)}\sigma(\mathrm{L}_{\mathrm{reg}}(1-z))e^{-x_1 \log(1-z)}Z_{\sqcup\!\sqcup}$ then $\mathrm{L}(z) \sim_1 e^{-x_1 \log(1-z)} Z_{\sqcup\!\sqcup}$ and then $\mathrm{H}(n) \sim_{+\infty} \mathtt{Const(n)}\pi_Y Z_{\sqcup\!\sqcup}$.

Theorem 1 (first Abel like theorem)

$$\lim_{z\to 1} e^{y_1 \log(1-z)}\pi_Y \mathrm{L}(z) = \pi_Y Z_{\sqcup\!\sqcup} = \lim_{n\to\infty} \mathtt{Const(n)}^{-1}\mathrm{H}(n).$$

Corollary 2 (bridge equations)

$$Z_\gamma = B(y_1)\pi_Y Z_{\sqcup\!\sqcup} \iff Z_{\uplus} = B'(y_1)\pi_Y Z_{\sqcup\!\sqcup}.$$

Remarque 1

On the one hand, by identification coefficients, for $w \in X^* x_1$,

$\zeta_{\sqcup\!\sqcup}(w) = \langle Z_{\sqcup\!\sqcup} | w\rangle = \mathrm{f.p.}_{z\to 1} \mathrm{Li}_w(z)$, $\{(1-z)^a \log^b(1-z)\}_{a\in\mathbb{Z}, b\in\mathbb{N}}$.

On the other hand, by an $\uplus$-modified Radford theorem, for $w \in Y^*$,

$\zeta_{\uplus}(w) = \langle Z_{\uplus} | w\rangle = \mathrm{f.p.}_{n\to+\infty} \mathrm{H}_w(n)$, $\{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z}, b\in\mathbb{N}}$.

In particular[12], $\zeta_{\sqcup\!\sqcup}(x_1) = \zeta_{\uplus}(y_1) = 0$.

---

[11] By a Hoffman's duality, *i.e.* $\zeta(\rho(\tilde{w})) = \zeta(w)$ (where ρ is the morphism defined by $\rho(x_0) = x_1, \rho(x_1) = x_0$ and $\tilde{w}$ is mirror of w), we get $\sigma(Z_{\sqcup\!\sqcup}^{-1}) = Z_{\sqcup\!\sqcup}$.

[12] These coefficients of singular and asymptotic expansions can be changed if we use other comparison scales.

Cloned Abel like results and cloned bridge equations

Let $e^C \in \mathrm{Gal}_{\mathbb{C}}(DE) = \{e^C\}_{C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X \rangle\rangle}$ and $\overline{\mathrm{L}} := \mathrm{L}e^C, \overline{Z}_{\sqcup\!\sqcup} := Z_{\sqcup\!\sqcup} e^C$.
Hence, $\overline{\mathrm{L}}(z) \sim_1 e^{-x_1 \log(1-z)} \overline{Z}_{\sqcup\!\sqcup}$ and then $\overline{\mathrm{H}}(n) \sim_{+\infty} \mathtt{Const(n)} \pi_Y \overline{Z}_{\sqcup\!\sqcup}$.

Theorem 3 (cloned first Abel like theorem)

$$\lim_{z \to 1} e^{y_1 \log(1-z)} \pi_Y \overline{\mathrm{L}}(z) = \pi_Y \overline{Z}_{\sqcup\!\sqcup} = \lim_{n \to \infty} \mathtt{Const(n)}^{-1} \overline{\mathrm{H}}(n).$$

If[13] $\overline{Z}_{\sqcup\!\sqcup} \in dm(A) := \{Z_{\sqcup\!\sqcup} e^C \mid C \in \mathcal{L}ie_A\langle\langle X \rangle\rangle, \langle e^C | x_0 \rangle = \langle e^C | x_1 \rangle = 0\}$
then[14] $\overline{Z}_\gamma = e^{\gamma y_1} \overline{Z}_{\uplus}$ and it follows that

Corollary 4 (cloned bridge equations)

If $\overline{Z}_{\sqcup\!\sqcup} \in dm(A)$ *then* $\overline{Z}_\gamma = B(y_1) \pi_Y \overline{Z}_{\sqcup\!\sqcup} \iff \overline{Z}_{\uplus} = B'(y_1) \pi_Y \overline{Z}_{\sqcup\!\sqcup}$.

Remarque 2

Since $\{S_l\}_{l \in \mathcal{L}ynX}$ *and* $\{\Sigma_l\}_{l \in \mathcal{L}ynY}$ *are homogenous in weight then the local coordinates of* $\overline{Z}_{\uplus}$ *and* $\overline{Z}_{\sqcup\!\sqcup}$ *are homogenous polynomial on convergent polyzetas, with coefficients in* A. *Hence, if* $\gamma \notin A$ *then* γ *is transcendent over the* A*-algebra generated by convergent polyzetas.*

[13] $dm(A)$ contains $DM(A)$ introduced by P. Cartier and G. Racinet and it is a strict normal subgroup of $\mathrm{Gal}_A(DE)$ (recall that $\mathbb{Q} \subset A \subset \mathbb{C}$).

[14] For $w \in Y^*$, one has $\langle \overline{Z}_{\uplus} | w \rangle = \mathrm{f.p.}_{n \to +\infty} \overline{\mathrm{H}}_w(n)$, $\{n^a \mathrm{H}_1^b(n)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}$ and $\langle \overline{Z}_\gamma | w \rangle = \mathrm{f.p.}_{n \to +\infty} \overline{\mathrm{H}}_w(n)$, $\{n^a \log^b(n)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}$.

STRUCTURE OF POLYZETAS

Generalized Euler's gamma constante

Identifying the coefficients of $y_1^k w, w \in X^*, k \in \mathbb{N}$ in $Z_\gamma = B(y_1)\pi_Y Z_{\sqcup\!\sqcup}$, one has

1. $$\gamma_{y_1^k} = \sum_{\substack{s_1,\ldots,s_k>0\\ s_1+\ldots+ks_k=k}} \frac{(-1)^k}{s_1!\ldots s_k!}(-\gamma)^{s_1}\left(-\frac{\zeta(2)}{2}\right)^{s_2}\ldots\left(-\frac{\zeta(k)}{k}\right)^{s_k}.$$

Example 5

$$\gamma_{1,1} = \frac{1}{2}(\gamma^2 - \zeta(2)), \qquad \gamma_{1,1,1} = \frac{1}{6}(\gamma^3 - 3\zeta(2)\gamma + 2\zeta(3)).$$

2. $$\gamma_{y_1^k w} = \sum_{i=0}^{k} \frac{\zeta(x_0[(-x_1)^{k-i} \sqcup\!\sqcup \pi_X w])}{i!}\left(\sum_{j=1}^{i} b_{i,j}(\gamma, -\zeta(2), 2\zeta(3), \ldots)\right),$$

   where $k \in \mathbb{N}_+, w \in Y^+$ and $b_{n,k}(t_1,\ldots,t_k)$ are Bell polynomials.

Example 6

$$\begin{aligned}
\gamma_{1,7} &= \zeta(7)\gamma + \zeta(3)\zeta(5) - \tfrac{54}{175}\zeta(2)^4,\\
\gamma_{1,1,6} &= \frac{4}{35}\zeta(2)^3\gamma^2 + (\zeta(2)\zeta(5) + \frac{2}{5}\zeta(3)\zeta(2)^2 - 4\zeta(7))\gamma\\
&\quad +\zeta(6,2) + \frac{19}{35}\zeta(2)^4 + \frac{1}{2}\zeta(2)\zeta(3)^2 - 4\zeta(3)\zeta(5).
\end{aligned}$$

Homogenous polynomials relations[15] on local coordinates

Identifying the local coordinates in $Z_\gamma = B(y_1)\pi_Y Z_{⧢}$, one has

	Polynomial relations on $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Polynomial relations on $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2}) = \zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4}) = \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1}) = \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2}) = \frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2}) = \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2}) = 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1}) = -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1}) = -\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2}) = \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2}) = \frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3}) = \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4}) = \zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6}) = \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2}) = \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1}) = \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1}) = \frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2}) = -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3}) = \frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1}) = 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2}) = \frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2}) = \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1}) = -\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2}) = \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3}) = \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3}) = \frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4}) = \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$

[15] These polynomials relations are independent from γ and similarly for the case where the ring of their coefficients is the ring A.

Homogenous polynomials generating ker ζ

	$\{Q_l\}_{l\in\mathcal{L}ynY-\{y_1\}}$	$\{Q_l\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}-\frac{3}{2}\Sigma_{y_3})=0$	$\zeta(S_{x_0x_1^2}-S_{x_0^2x_1})=0$
4	$\zeta(\Sigma_{y_4}-\frac{2}{5}\Sigma_{y_2}^{\uplus 2})=0$	$\zeta(S_{x_0^3x_1}-\frac{2}{5}S_{x_0x_1}^{\text{ш} 2})=0$
	$\zeta(\Sigma_{y_3y_1}-\frac{3}{10}\Sigma_{y_2}^{\uplus 2})=0$	$\zeta(S_{x_0^2x_1^2}-\frac{1}{10}S_{x_0x_1}^{\text{ш} 2})=0$
	$\zeta(\Sigma_{y_2y_1^2}-\frac{2}{3}\Sigma_{y_2}^{\uplus 2})=0$	$\zeta(S_{x_0x_1^3}-\frac{2}{5}S_{x_0x_1}^{\text{ш} 2})=0$
5	$\zeta(\Sigma_{y_3y_2}-3\Sigma_{y_3}\uplus\Sigma_{y_2}-5\Sigma_{y_5})=0$	$\zeta(S_{x_0^3x_1^2}-S_{x_0^2x_1}\,\text{ш}\,S_{x_0x_1}+2S_{x_0^4x_1})=0$
	$\zeta(\Sigma_{y_4y_1}-\Sigma_{y_3}\uplus\Sigma_{y_2})+\frac{5}{2}\Sigma_{y_5})=0$	$\zeta(S_{x_0^2x_1x_0x_1}-\frac{3}{2}S_{x_0^4x_1}+S_{x_0^2x_1}\,\text{ш}\,S_{x_0x_1})=0$
	$\zeta(\Sigma_{y_2^2y_1}-\frac{3}{2}\Sigma_{y_3}\uplus\Sigma_{y_2}-\frac{25}{12}\Sigma_{y_5})=0$	$\zeta(S_{x_0^2x_1^3}-S_{x_0^2x_1}\,\text{ш}\,S_{x_0x_1}+2S_{x_0^4x_1})=0$
	$\zeta(\Sigma_{y_3y_1^2}-\frac{5}{12}\Sigma_{y_5})=0$	$\zeta(S_{x_0x_1x_0x_1^2}-\frac{1}{2}S_{x_0^4x_1})=0$
	$\zeta(\Sigma_{y_2y_1^3}-\frac{1}{4}\Sigma_{y_3}\uplus\Sigma_{y_2})+\frac{5}{4}\Sigma_{y_5})=0$	$\zeta(S_{x_0x_1^4}-S_{x_0^4x_1})=0$
6	$\zeta(\Sigma_{y_6}-\frac{8}{35}\Sigma_{y_2}^{\uplus 3})=0$	$\zeta(S_{x_0^5x_1}-\frac{8}{35}S_{x_0x_1}^{\text{ш} 3})=0$
	$\zeta(\Sigma_{y_4y_2}-\Sigma_{y_3}^{\uplus 2}-\frac{4}{21}\Sigma_{y_2}^{\uplus 3})=0$	$\zeta(S_{x_0^4x_1^2}-\frac{6}{35}S_{x_0x_1}^{\text{ш} 3}-\frac{1}{2}S_{x_0^2x_1}^{\text{ш} 2})=0$
	$\zeta(\Sigma_{y_5y_1}-\frac{2}{7}\Sigma_{y_2}^{\uplus 3}-\frac{1}{2}\Sigma_{y_3}^{\uplus 2})=0$	$\zeta(S_{x_0^3x_1x_0x_1}-\frac{4}{105}S_{x_0x_1}^{\text{ш} 3})=0$
	$\zeta(\Sigma_{y_3y_1y_2}-\frac{17}{30}\Sigma_{y_2}^{\uplus 3}+\frac{9}{4}\Sigma_{y_3}^{\uplus 2})=0$	$\zeta(S_{x_0^3x_1^3}-\frac{23}{70}S_{x_0x_1}^{\text{ш} 3}-S_{x_0^2x_1}^{\text{ш} 2})=0$
	$\zeta(\Sigma_{y_3y_2y_1}-3\Sigma_{y_3}^{\uplus 2}-\frac{9}{10}\Sigma_{y_2}^{\uplus 3})=0$	$\zeta(S_{x_0^2x_1x_0x_1^2}-\frac{2}{105}S_{x_0x_1}^{\text{ш} 3})=0$
	$\zeta(\Sigma_{y_4y_1^2}-\frac{3}{10}\Sigma_{y_2}^{\uplus 2}-\frac{3}{4}\Sigma_{y_3}^{\uplus 2})=0$	$\zeta(S_{x_0^2x_1^2x_0x_1}-\frac{89}{210}S_{x_0x_1}^{\text{ш} 3}+\frac{3}{2}S_{x_0^2x_1}^{\text{ш} 2})=0$
	$\zeta(\Sigma_{y_2^2y_1^2}-\frac{11}{63}\Sigma_{y_2}^{\uplus 2}-\frac{1}{4}\Sigma_{y_3}^{\uplus 2})=0$	$\zeta(S_{x_0^2x_1^4}-\frac{6}{35}S_{x_0x_1}^{\text{ш} 3}-\frac{1}{2}S_{x_0^2x_1}^{\text{ш} 2})=0$
	$\zeta(\Sigma_{y_3y_1^3}-\frac{1}{21}\Sigma_{y_2}^{\uplus 3})=0$	$\zeta(S_{x_0x_1x_0x_1^3}-\frac{8}{21}S_{x_0x_1}^{\text{ш} 3}-S_{x_0^2x_1}^{\text{ш} 2})=0$
	$\zeta(\Sigma_{y_2y_1^4}-\frac{17}{50}\Sigma_{y_2}^{\uplus 3}+\frac{3}{16}\Sigma_{y_3}^{\uplus 2})=0$	$\zeta(S_{x_0x_1^5}-\frac{8}{35}S_{x_0x_1}^{\text{ш} 3})=0$

One has $\mathcal{R}_{\mathcal{X}}\subseteq\ker\zeta$, where $\left\{\begin{array}{l}\mathcal{R}_Y:=(\mathbb{Q}\{Q_l\}_{l\in\mathcal{L}ynY-\{y_1\}},\uplus,1_{Y^*})\\ \mathcal{R}_X:=(\mathbb{Q}\{Q_l\}_{l\in\mathcal{L}ynX-X},\text{ш},1_{X^*})\end{array}\right\}$.

Noetherian rewriting system & irreducible coordinates[16]

	Rewriting among $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Rewriting among $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}) \to \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2}) \to \zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4}) \to \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1}) \to \frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1}) \to \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2}) \to \frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2}) \to \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3}) \to \frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2}) \to 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2}) \to -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1}) \to -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1}) \to -\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1}) \to \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3}) \to -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2}) \to \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2}) \to \frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3}) \to \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4}) \to \zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6}) \to \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1}) \to \frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2}) \to \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2}) \to \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1}) \to \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1}) \to \frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2}) \to -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3}) \to \frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1}) \to 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2}) \to \frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2}) \to \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1}) \to -\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2}) \to \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4}) \to \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3}) \to \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3}) \to \frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4}) \to \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5}) \to \frac{8}{35}\zeta(S_{x_0x_1})^3$

$$\mathcal{Z}_{irr}^{\leq 2}(\mathcal{X}) \subset \cdots \mathcal{Z}_{irr}^{\leq p}(\mathcal{X}) \subset \cdots \subset \mathcal{Z}_{irr}^{\infty}(\mathcal{X}) = \cup_{p\geq 2}\mathcal{Z}_{irr}^{\leq p}(\mathcal{X}).$$

[16] The set of irreducible coordinates forms algebraic generator system for $\mathcal{Z}$.

Noetherian rewriting system & totally ordered $\mathcal{L}_{irr}^{\infty}(\mathcal{X})$

	Rewriting among $\{\Sigma_l\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Rewriting among $\{S_l\}_{\mathcal{L}ynX-X}$
3	$\Sigma_{y_2y_1} \rightarrow \frac{3}{2}\Sigma_{y_3}$	$S_{x_0x_1^2} \rightarrow S_{x_0^2x_1}$
4	$\Sigma_{y_4} \rightarrow \frac{2}{5}\Sigma_{y_2}^2$	$S_{x_0^3x_1} \rightarrow \frac{2}{5}S_{x_0x_1}^2$
	$\Sigma_{y_3y_1} \rightarrow \frac{3}{10}\Sigma_{y_2}^2$	$S_{x_0^2x_1^2} \rightarrow \frac{1}{10}S_{x_0x_1}^2$
	$\Sigma_{y_2y_1^2} \rightarrow \frac{2}{3}\Sigma_{y_2}^2$	$S_{x_0x_1^3} \rightarrow \frac{2}{5}S_{x_0x_1}^2$
5	$\Sigma_{y_3y_2} \rightarrow 3\Sigma_{y_3}\Sigma_{y_2} - 5\Sigma_{y_5}$	$S_{x_0^3x_1^2} \rightarrow -S_{x_0^2x_1}S_{x_0x_1} + 2S_{x_0^4x_1}$
	$\Sigma_{y_4y_1} \rightarrow -\Sigma_{y_3}\Sigma_{y_2} + \frac{5}{2}\Sigma_{y_5}$	$S_{x_0^2x_1x_0x_1} \rightarrow -\frac{3}{2}S_{x_0^4x_1} + S_{x_0^2x_1}S_{x_0x_1}$
	$\Sigma_{y_2^2y_1} \rightarrow \frac{3}{2}\Sigma_{y_3}\Sigma_{y_2} - \frac{25}{12}\Sigma_{y_5}$	$S_{x_0^2x_1^3} \rightarrow -S_{x_0^2x_1}S_{x_0x_1} + 2S_{x_0^4x_1}$
	$\Sigma_{y_3y_1^2} \rightarrow \frac{5}{12}\Sigma_{y_5}$	$S_{x_0x_1x_0x_1^2} \rightarrow \frac{1}{2}S_{x_0^4x_1}$
	$\Sigma_{y_2y_1^3} \rightarrow \frac{1}{4}\Sigma_{y_3}\Sigma_{y_2} + \frac{5}{4}\Sigma_{y_5}$	$S_{x_0x_1^4} \rightarrow S_{x_0^4x_1}$
6	$\Sigma_{y_6} \rightarrow \frac{8}{35}\Sigma_{y_2}^3$	$S_{x_0^5x_1} \rightarrow \frac{8}{35}S_{x_0x_1}^3$
	$\Sigma_{y_4y_2} \rightarrow \Sigma_{y_3}^2 - \frac{4}{21}\Sigma_{y_2}^3$	$S_{x_0^4x_1^2} \rightarrow \frac{6}{35}S_{x_0x_1}^3 - \frac{1}{2}S_{x_0^2x_1}^2$
	$\Sigma_{y_5y_1} \rightarrow \frac{2}{7}\Sigma_{y_2}^3 - \frac{1}{2}\Sigma_{y_3}^2$	$S_{x_0^3x_1x_0x_1} \rightarrow \frac{4}{105}S_{x_0x_1}^3$
	$\Sigma_{y_3y_1y_2} \rightarrow -\frac{17}{30}\Sigma_{y_2}^3 + \frac{9}{4}\Sigma_{y_3}^2$	$S_{x_0^3x_1^3} \rightarrow \frac{23}{70}S_{x_0x_1}^3 - S_{x_0^2x_1}^2$
	$\Sigma_{y_3y_2y_1} \rightarrow 3\Sigma_{y_3}^2 - \frac{9}{10}\Sigma_{y_2}^3$	$S_{x_0^2x_1x_0x_1^2} \rightarrow \frac{2}{105}S_{x_0x_1}^3$
	$\Sigma_{y_4y_1^2} \rightarrow \frac{3}{10}\Sigma_{y_2}^3 - \frac{3}{4}\Sigma_{y_3}^2$	$S_{x_0^2x_1^2x_0x_1} \rightarrow -\frac{89}{210}S_{x_0x_1}^3 + \frac{3}{2}S_{x_0^2x_1}^2$
	$\Sigma_{y_2^2y_1^2} \rightarrow \frac{11}{63}\Sigma_{y_2}^3 - \frac{1}{4}\Sigma_{y_3}^2$	$S_{x_0^2x_1^4} \rightarrow \frac{6}{35}S_{x_0x_1}^3 - \frac{1}{2}S_{x_0^2x_1}^2$
	$\Sigma_{y_3y_1^3} \rightarrow \frac{1}{21}\Sigma_{y_2}^3$	$S_{x_0x_1x_0x_1^3} \rightarrow \frac{8}{21}S_{x_0x_1}^3 - S_{x_0^2x_1}^2$
	$\Sigma_{y_2y_1^4} \rightarrow \frac{17}{50}\Sigma_{y_2}^3 + \frac{3}{16}\Sigma_{y_3}^2$	$S_{x_0x_1^5} \rightarrow \frac{8}{35}S_{x_0x_1}^3$

$$\mathcal{L}_{irr}^{\le 2}(\mathcal{X}) \subset \cdots \mathcal{L}_{irr}^{\le p}(\mathcal{X}) \subset \cdots \subset \mathcal{L}_{irr}^{\infty}(\mathcal{X}) = \cup_{p\ge 2}\mathcal{L}_{irr}^{\le p}(\mathcal{X}).$$

$$\left\{\begin{array}{c} \forall l \in \mathcal{L}ynY - \{y_1\}, \Sigma_l \in \mathcal{L}_{irr}^{\infty}(Y) \\ \forall l \in \mathcal{L}ynX - X, S_l \in \mathcal{L}_{irr}^{\infty}(X) \end{array}\right\} \iff Q_l = 0.$$

Im and ker of $\zeta : \begin{matrix}(\mathbb{Q}[\{S_l\}_{l\in\mathcal{L}ynX-X}], ш ,1_{X^*}) \\ (\mathbb{Q}[\{\Sigma_l\}_{l\in\mathcal{L}ynY-\{y_1\}}], \uplus ,1_{Y^*})\end{matrix} \twoheadrightarrow (\mathcal{Z},.,1)$

Identification of local coordinates yields a family of polynomials $\{Q_l\}_{l\in\mathcal{L}yn\mathcal{X}}$, homogenous in weight, and a family of algebraic generators $\mathcal{Z}_{irr}^{\infty}(\mathcal{X})$ s.t. the restriction $\zeta : \mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})] \to \mathcal{Z} = \mathbb{Q}[\mathcal{Z}_{irr}^{\infty}(\mathcal{X})] = \mathbb{Q}[\{\zeta(p)\}_{p\in\mathcal{L}_{irr}^{\infty}(\mathcal{X})}]$ is bijective.
$\mathrm{Im}\,\zeta = \mathcal{Z} = \mathbb{Q}[\{\zeta(p)\}_{p\in\mathcal{L}_{irr}^{\infty}(\mathcal{X})}]$ and

$$(\mathbb{Q}[\{S_l\}_{l\in\mathcal{L}ynX-X}], ш ,1_{X^*}) = \mathcal{R}_X \oplus (\mathbb{Q}[\mathcal{L}_{irr}^{\infty}(X)], ш ,1_{X^*}),$$
$$(\mathbb{Q}[\{\Sigma_l\}_{l\in\mathcal{L}ynY-\{y_1\}}], \uplus ,1_{Y^*}) = \mathcal{R}_Y \oplus (\mathbb{Q}[\mathcal{L}_{irr}^{\infty}(Y)], \uplus ,1_{Y^*}).$$

$\ker\zeta = \mathcal{R}_{\mathcal{X}}$ because, for $Q \in \ker\zeta$, two cases can occur:

1. if $Q \notin \mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})]$ then $Q \equiv_{\mathcal{R}_{\mathcal{X}}} Q'$ s.t. $Q' \in \mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})]$ and then
2. if $Q \in \mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})]$ then $Q \equiv_{\mathcal{R}_{\mathcal{X}}} 0$.

$\ker\zeta$ is then an ideal generated by homogenous polynomials $\{Q_l\}_{l\in\mathcal{L}yn\mathcal{X}}$.
If the decompositions and equalities above with the properties of the announced complements hold, it would prove that $\ker\zeta$ is graded and then $\mathcal{Z}$ is also graded because $\mathrm{Im}\,\zeta \cong \mathbb{Q}1_{Y^*} \oplus (Y - \{y_1\})\mathbb{Q}\langle Y\rangle/\ker\zeta$ (resp. $\mathrm{Im}\,\zeta \cong \mathbb{Q}1_{X^*} \oplus x_0\mathbb{Q}\langle X\rangle x_1/\ker\zeta$).
Now, let $\xi := \zeta(P)$, where $\mathbb{Q}\langle\mathcal{X}\rangle \ni P \notin \ker\zeta$, homogenous of weight n.
Each monomial $\xi^n, n \geq 1$, is of different weight (because $\mathcal{Z}_p\mathcal{Z}_n \subset \mathcal{Z}_{p+n}$).
Thus ξ could not satisfy $\xi^n + a_{n-1}\xi^{n-1} + \ldots = 0$ and then is transcendent.
Hence, $\mathbb{Q}[\{\zeta(p)\}_{p\in\mathcal{L}_{irr}^{\infty}(\mathcal{X})}]$ is graded and $\{\zeta(p)\}_{p\in\mathcal{L}_{irr}^{\infty}(\mathcal{X})}$ are transcendent.

Concluding remarks

For any $l \in \mathcal{L}yn\mathcal{X} - \mathcal{X}$, one has $l \succeq y_n$ (resp. $l \succeq x_0^{n-1}x_1$). In particular, $\Sigma_{y_n} = y_n \in \mathcal{L}ynY$ and $S_{x_0^{n-1}x_1} = x_0^{n-1}x_1 \in \mathcal{L}ynX$. Next,

1. $\zeta(2) = \zeta(\Sigma_{y_2}) = \zeta(S_{x_0x_1})$ is then irreducible and, by the Euler's identity about the ratio $\zeta(2k)/\pi^{2k}$, one deduces then, for $k > 1$, $\Sigma_{y_{2k}} = y_{2k} \notin \mathcal{L}_{irr}^{\infty}(Y)$ and $S_{x_0^{2k-1}x_1} = x_0^{2k-1}x_1 \notin \mathcal{L}_{irr}^{\infty}(X)$,
2. $\Sigma_{y_{2n+1}} = y_{2n+1} \in \mathcal{L}_{irr}^{\infty}(Y)$ and $S_{x_0^{2n}x_1} = x_0^{2n}x_1 \in \mathcal{L}_{irr}^{\infty}(X)$.

Up to weight 12, the Zagier's dimension conjecture holds meaning that $\mathcal{Z}_{irr}^{\leq 12}(\mathcal{X})$ is algebraically independent:

$$\mathcal{Z}_{irr}^{\leq 12}(X) = \{\zeta(S_{x_0x_1}), \zeta(S_{x_0^2x_1}), \zeta(S_{x_0^4x_1}), \zeta(S_{x_0^6x_1}), \zeta(S_{x_0x_1^2x_0x_1^4}), \zeta(S_{x_0^8x_1}), \zeta(S_{x_0x_1^2x_0x_1^6}), \zeta(S_{x_0^{10}x_1}), \zeta(S_{x_0x_1^3x_0x_1^7}), \zeta(S_{x_0x_1^2x_0x_1^8}), \zeta(S_{x_0x_1^4x_0x_1^6})\}.$$

$$\mathcal{L}_{irr}^{\leq 12}(X) = \{S_{x_0x_1}, S_{x_0^2x_1}, S_{x_0^4x_1}, S_{x_0^6x_1}, S_{x_0x_1^2x_0x_1^4}, S_{x_0^8x_1}, S_{x_0x_1^2x_0x_1^6}, S_{x_0^{10}x_1}, S_{x_0x_1^3x_0x_1^7}, S_{x_0x_1^2x_0x_1^8}, S_{x_0x_1^4x_0x_1^6}\}.$$

$$\mathcal{Z}_{irr}^{\leq 12}(Y) = \{\zeta(\Sigma_{y_2}), \zeta(\Sigma_{y_3}), \zeta(\Sigma_{y_5}), \zeta(\Sigma_{y_7}), \zeta(\Sigma_{y_3y_1^5}), \zeta(\Sigma_{y_9}), \zeta(\Sigma_{y_3y_1^7}), \zeta(\Sigma_{y_{11}}), \zeta(\Sigma_{y_2y_1^9}), \zeta(\Sigma_{y_3y_1^9}), \zeta(\Sigma_{y_2^2y_1^8})\}.$$

$$\mathcal{L}_{irr}^{\leq 12}(Y) = \{\Sigma_{y_2}, \Sigma_{y_3}, \Sigma_{y_5}, \Sigma_{y_7}, \Sigma_{y_3y_1^5}, \Sigma_{y_9}, \Sigma_{y_3y_1^7}, \Sigma_{y_{11}}, \Sigma_{y_2y_1^9}, \Sigma_{y_3y_1^9}, \Sigma_{y_2^2y_1^8}\}.$$

THANK YOU FOR YOUR ATTENTION