CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems
to usual applications.

G.H.E. Duchamp
Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
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CIP seminar,

Friday conversations:

For this seminar, please have a look at Slide CCRT[n] & ff.

Goal of this series of talks

The goal of this talk is threefold

A bit of category theory: How to construct free objects w.r.t. a functor and two routes to reach the free algebra.

Representation theory: Categories of modules, semi-simplicity, isomorphism classes i.e. the framework of Kronecker coefficients

MRS factorisation: A local system of coordinates for Hausdorff groups

Wikipedia says

Representation theory is a branch of mathematics that studies abstract algebraic structures by representing their elements as linear transformations of vector spaces .../...
The success of representation theory has led to numerous generalizations. One of the most general is in category theory.

As our track is based on Combinatorial Physics and Experimental/Computational Mathematics, we will have a practical approach of the three main points of view

- Algebraic
- Geometric
- Combinatorial
- Categorical

Matters

1. Representation theory or theories
    1. Geometric point of view
    2. Combinatorial point of view (Ram and Barcelo manifesto)
    3. Categorical point of view
2. From groups to algebras
   Here is a bit of rep. theory of the symmetric group, deformations, Wedderburn-Artin and idempotents
3. Irreducible and indecomposable modules
4. Characters, central functions and shifts
5. Reductibility and invariant inner products
   Here stands Joseph's result
6. Commutative characters
   Here are time-ordered exponentials, iterated integrals, evolution equations and Minh's results
7. Lie groups Cartan theorem
   Here is BTT

1. Let $\mathcal{C}_{left}$, $\mathcal{C}_{right}$ be two categories and $F: \mathcal{C}_{right} \to \mathcal{C}_{left}$ a (covariant) functor between them

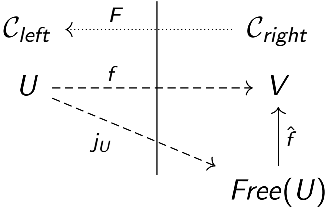


Figure: A solution of the universal problem w.r.t. the functor F is the datum, for each $U \in \mathcal{C}_{left}$, of a pair $(j_U, Free(U))$ (with $j_U \in Hom(U, F[Free(U)])$, $Free(U) \in \mathcal{C}_{right}$) such that, for all $f \in Hom(U, F[V])$ it exists a unique $\hat{f} \in Hom(Free(U), V)$ with $F[\hat{f}] \circ j_U = f$. Elements in $Hom(U, F[V])$ are called heteromorphisms their set is noted $Het_F(U, V)$.

$$\left(\forall f \in Hom(U, F[V])\right)\left(\exists!\, \hat{f} \in Hom(Free(U), V)\right)\left(F(\hat{f}) \circ j_U = f\right)$$

The pair $U \to Free(U)$ is, in fact, a functor.

Which, in turn, will prove to be left-adjoint to F

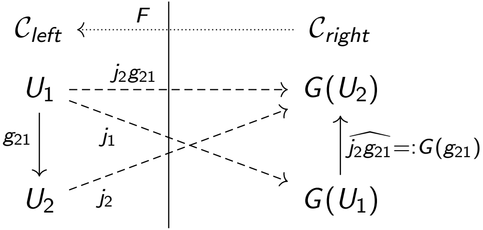


Figure: Making a free functor G ($= Free$) from F: for any morphism $g_{21} \in Hom(U_1, U_2)$, $G(g_{21})$ is the unique morphism in $Hom(U_1, U_2)$ such that

$$F[G(g_{21})] \circ j_1 = j_2 g_{21} \qquad (**)$$

We now prove that G is a functor.

- If $U_1 = U_2$ and $g_{21} = Id_{U_1}$, then $j_1 = j_2 = j_2 g_{21}$ and $F[Id_{G(U_1)}] \circ j_1 = j_1 = j_2 g_{21}$ hence $G[Id_{U_1}] = Id_{G(U_1)}$
- **A remark**: $Het(?, ?)$ is intended to give a symmetric middle term/step to the adjunction chain $Hom(U, F[V]) =: Het_F(U, V) \simeq Het^G(U, V) := Hom(G(U), V)$ $\simeq$ being constructed by a set of bijections.

Functor G from $Free/2$

- Let now $U_1 \xhookrightarrow{g_{21}} U_2 \xhookrightarrow{g_{32}} U_3$ be a chain of C_{left}-morphisms.
  We have

$$F[G(g_{21})] \circ j_1 = j_2 \circ g_{21} \text{ and } F[G(g_{32})] \circ j_2 = j_3 \circ g_{32}$$

then

$$F[G(g_{32}) \circ G(g_{21})] \circ j_1 \stackrel{(1)}{=} F[G(g_{32})] \circ F[G(g_{21})] \circ j_1 \stackrel{(2)}{=}$$
$$F[G(g_{32})] \circ j_2 \circ g_{21} \stackrel{(3)}{=} j_3 \circ g_{32} \circ g_{21}$$

(1) because F is a functor, (2) is Eq. (**) applied to indices 21,
(3) is Eq. (**) applied to indices 32.
Now, we know that $g \in Hom(U, U')$ being given, the solution
$X \in Hom(G(U), G(U'))$ of

$$F[X] \circ j_1 = j_2 \circ g$$

is unique. Then $G(g_{32}) \circ G(g_{21}) = G(g_{32} \circ g_{21})$ □

Composition of functors F and G

Piling free structures.

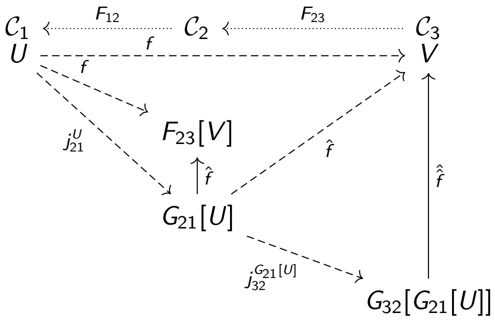
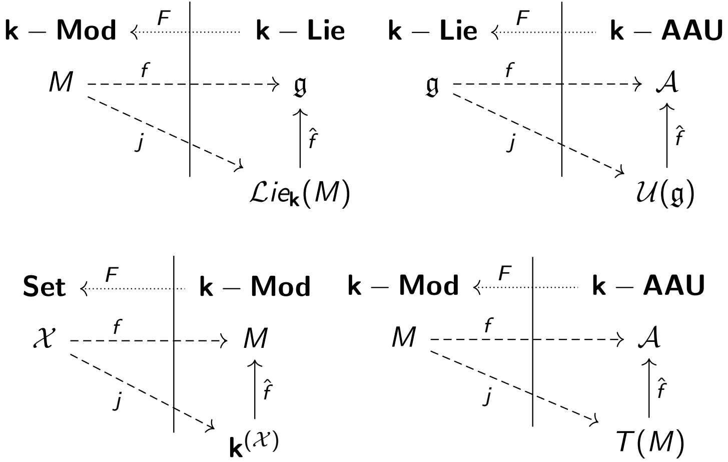


Figure: $\left[F_{12}[j_{32}^{G_{21}[U]}], G_{32}[G_{21}[U]]\right]$ is a solution of the universal problem for $F_{12}F_{23}$.

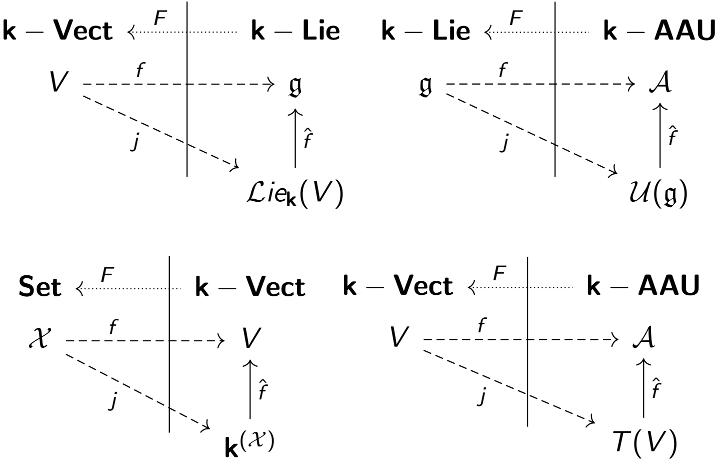
Proof: In fact, $Het_{F_{12}F_{23}}(U, V) = Hom(U, F_{12}F_{23}[V]) = Het_{F_{12}}(U, F_{23}[V])$, hence existence of $\hat{f} \in Hom(G_{21}[U], F_{23}[V]) \ = Het_{F_{23}}(G_{21}[U], V)$, hence again existence of $\hat{\hat{f}}$. Uniqueness of $\hat{\hat{f}}$ is left to the reader.

CCRT[2] First example: $T = UL$.



$$T(M) = \mathcal{U}(\mathcal{L}ie_{\mathbf{k}}(M)) \qquad \mathbf{k}\langle\mathcal{X}\rangle = T(\mathbf{k}^{(\mathcal{X})})$$

First example: $T = UL$, $\mathbf{k}$ field.



$$T(V) = \mathcal{U}(\mathcal{L}ie_{\mathbf{k}}(V)) \qquad \mathbf{k}\langle\mathcal{X}\rangle = T(\mathbf{k}^{(\mathcal{X})})$$

An immediate (and although rich) example/1

Piling free structures/2

1. First, $\mathcal{C}_1 = \mathbf{Set}$ (sets and maps) and $\mathcal{C}_2 = \mathbf{Mon}$ (monoids and morphisms) gives you the triple $(\mathcal{X}, j_{21}, \mathcal{X}^*)$

Usually $\mathcal{X}$, a set, is seen as an *alphabet* that is to say a *set of non commuting variables.* Let us introduce the ring $\mathbf{k}$ of coefficients

2. With $\mathcal{C}_2 = \mathbf{Mon}$ (monoids and morphisms) and $\mathcal{C}_3 = \mathbf{k} - \mathbf{AAU}$ ($\mathbf{k}$-associative algebras with unit), one gets $\mathbf{k}[M]$ the algebra of a monoid M, we get the triple $(M, j_{32}, \mathbf{k}[M])$ and,
3. by transitivity of free objects with $\mathcal{C}_1 = \mathbf{Set}$ (sets and maps) and $\mathcal{C}_3$ as above, we get the triple $(\mathcal{X}, j_{31}, \mathbf{k}\langle\mathcal{X}\rangle)$, $\mathbf{k}\langle\mathcal{X}\rangle = \mathbf{k}[\mathcal{X}^*]$ being the algebra of noncommutative polynomials.
4. we immediately obtain that $\mathbf{k}\langle\mathcal{X}\rangle = \mathbf{k}[\mathcal{X}^*]$ is free with $\{w\}_{w\in\mathcal{X}^*}$ (this will be useful for the principal pairing)

An immediate (and although rich) example/2

5. let us observe here that $\mathbf{k}\langle\mathcal{X}\rangle$ can be reached, instead of

$$[\mathbf{Set}] \longrightarrow [\mathbf{Mon}] \longrightarrow [\mathbf{k} - \mathbf{AAU}]$$

by another path, and this will provide a host of other very interesting (combinatorial) bases.

6. the preceding route amounts to the formula $\mathbf{k}\langle\mathcal{X}\rangle = \mathbf{k}[\mathcal{X}^*]$, but it can be also proved that $\mathbf{k}\langle\mathcal{X}\rangle = \mathcal{U}(\mathcal{L}ie_{\mathbf{k}}\langle X\rangle)$

$$[\mathbf{Set}] \longrightarrow [\mathbf{k} - \mathbf{Lie}] \longrightarrow [\mathbf{k} - \mathbf{AAU}]$$

An immediate (and although rich) example/3

Piling free structures and dual bases

7. from the first (obvious) way (sets to monoids to **k**-AAU) we got the basis $\{w\}_{w\in\mathcal{X}^*}$ which provides the fine grading of $\mathbf{k}\langle\mathcal{X}\rangle$. indeed to each word $w \in \mathcal{X}^*$, we can associate the family

$$\beta(w) = (|w|_x)_{x\in\mathcal{X}} \in \mathbb{N}^{(\mathcal{X})}$$

8. therefore, due to this partitioning of the basis (of words), we get

$$\mathbf{k}\langle\mathcal{X}\rangle = \bigoplus_{\alpha\in\mathbb{N}^{(\mathcal{X})}} \mathbf{k}_\alpha\langle\mathcal{X}\rangle \quad (1)$$

where $\mathbf{k}_\alpha\langle\mathcal{X}\rangle := span_{\mathbf{k}}\{w|\beta(w) = \alpha\}$.

An immediate (and although rich) example/4

Graded bases through free Lie algebra

9. each $\mathbf{k}_\alpha\langle\mathcal{X}\rangle$ is free of dimension $\dfrac{|\alpha|\,!}{\alpha\,!}$; for example with two letters a, b, we have $\mathbf{k}\langle\mathcal{X}\rangle = \bigoplus_{(p,q)\in\mathbb{N}^2}\mathbf{k}_{(p,q)}\langle\mathcal{X}\rangle$ and $\dim(\mathbf{k}_{(p,q)}\langle\mathcal{X}\rangle) = \dfrac{(p+q)\,!}{p\,!q\,!} = \dbinom{p}{q}$.
10. this fine grading is a grading of algebra as

$$\mathbf{k}_\alpha\langle\mathcal{X}\rangle\mathbf{k}_\beta\langle\mathcal{X}\rangle \subset \mathbf{k}_{\alpha+\beta}\langle\mathcal{X}\rangle \ ; \ 1_{\mathcal{X}^*} \in \mathbf{k}_0\langle\mathcal{X}\rangle \tag{2}$$

11. now through the second route (sets-Lie-AAU), we can construct many finely homogeneous bases of $\mathbf{k}\langle\mathcal{X}\rangle$ using the following scheme
    - pick any finely homogeneous basis of $\mathcal{L}ie_{\mathbf{k}}\langle X\rangle$, $(P_i)_{i\in I}$ (we will construct at least one)
    - (Totally) order I and form the PBW basis (of $\mathbf{k}\langle\mathcal{X}\rangle$). it is finely homogeneous (due to eq. 2).
    - use this for MRS factorisation (unfolded below after semi-simplicity)

CCRT[4] Semi-simple categories of modules/1

Next steps

1. Semi-simple categories of modules
2. Link with non-degenerate bilinear forms + examples

About pronunciation

Here are examples of pronunciation

`https://www.linguee.fr/anglais-francais/traduction/semi.html`

including: "semi-detached house", "semi-public", "semiconductor". On this ground, I think that, until there is evidence to the contrary, the "i" of "semi-simple" should sound as in "fish" and not as in "file".

Semisimple categories of modules/2

- Semi-simple categories of modules, see in general
  `https://ncatlab.org/nlab/show/semisimple+category`

Definition

Let R be a ring. We note R-**Mod** the category of R-modules (whatever the size) the arrows being that of R-linear mappings between objects.

Remarks

1. This is a category with direct sums (coproducts) and products.
2. Subcategory of finite length modules (ex. finite dim when R is a **k** algebra) admit (finite) decompositions (Krull) in indecomposables. Another example will be subcategory of semi-simple modules (see below).
3. In the preceding case (finite dim when R is a **k** algebra) it is a subcategory
4. Link with non-degenerate bilinear forms + examples

Semisimple categories of modules/3

Definition: Simple and semi-simple modules

1. A module $M \in R$-**Mod** is said simple if it is not (0) and if its set of submodules is $\{(0), M\}$
2. A module $M \in R$-**Mod** is said semi-simple iff $M = \oplus_{i \in I} M_i$ where M_i are simple submodules of M.

Proposition [A]

Let $M \in R$-**Mod**

1. If M is such that $M = \sum_{i \in I} M_i$ where M_i are simple submodules of M and $N \subset_{submod} M$, then it exists $J \subset I$ such that $M = (\oplus_{i \in J} M_i) \oplus N$.
2. In particular a submodule or a quotient of a semi-simple module is semi-simple.

Proof

- A.1) Let $\mathfrak{S} \subset 2^I$ defined on 2^I

$$\mathfrak{S} = \{J \subset I | (\oplus_{i \in J} M_i) \oplus N \text{ is well defined}\} \tag{3}$$

  The set of $\mathfrak{S}$ is non-empty and of finite character. Then, by Tukey-Teichmüller theorem it admits at least a maximal element foor inclusion. Let J_0 be such an element. If $J_0 = I$ we are done, otherwise let $i \in I \setminus J_0$ and set $T = \big((\oplus_{i \in J_0} M_i) \oplus N\big)$. We cannot have $M_i \cap T = (0)$ otherwise we would get $J_0 \cup \{i\} \in \mathfrak{S}$ and $i \in J_0$, a contradiction. Remains $M_i \subset T$ because M_i is simple. Hence $(\forall i \in I \setminus J_0)(M_i \subset T)$ and this entails $M = T$.
  Remark that, setting N to (0), one obtains that if a module is a sum (direct or not) of simple submodules, then it is semi-simple.

- A.2) We suppose $M = \oplus_{i \in I} M_i$ to be semi-simple. Let $f: \ M \twoheadrightarrow Q$, . Setting $N = ker(f)$ in the preceding situation, we get a subfamily $(M_i)_{i \in J}$ such that $M = (\oplus_{i \in J} M_i) \oplus N$. Then, by f, $(\oplus_{i \in J} M_i) \simeq Q$ and we are done. Now, if N is any submodule of M, by (A.1), it is direct summand and we can write $M = N \oplus N_1$ with projectors p_N, p_{N_1}. From $p_N: \ M \twoheadrightarrow N$ we are done.

Case when R_s itself is semi-simple

Any ring R can be considered as a $R - R$ bimodule by the left and right actions (for $a, b \in R$), $\lambda_a(m) = a.m,\ \rho_b(m) = m.b$. these two actions commute. By definition R_s is the left-module defined by the action $\lambda_a(m)$. We have the following

Proposition [B]

If R_s is semi-simple, all R-module is so.

Proof.

We suppose that R_s is semi-simple. Let M be a R-module, then for all $x \in M$ the (principal) R-submodule $R.x$ is a semi-simple image (that of $t \to t.x$), hence semisimple. The result is then a consequence of Proposition [A].1 in view of the fact that $M = \sum_{x \in M} R.x$. □

A sufficient condition for R_s to be semi-simple

Proposition [C]

Under the preceding conditions

1. If R_s is semi-simple then every left ideal is direct summand of R_s within the lattice of left ideals.
2. The converse is true in the case when this lattice[a] satisfies ACC+DCC chain conditions.

   https://en.wikipedia.org/wiki/Ascending_chain_condition

[a] The lattice of left ideals.

For hilbertian traces, see Dieudonné XV.6 [4].
In the category of modules, ACC is Noetherian, DCC is Artinian.
Next steps: Frobenius characteristics, characters, case of finite groups, the symmetric group, Kronecker, Littlewood-Richardson and Clebsch–Gordan coefficients.

Proof of Proposition [C]

1) In fact this is true of every semi-simple module by Proposition [A].1.
2) As in (1), this converse is true for every module satisfying the same conditions (i.e. every submodule is direct summand + ACC + DCC). Let M be such a module, we build the following double sequence

1. (Init.) $C_0 = ((0), M)$
2. (Running) $C_n = (\oplus_{i=1}^n N_i, Q_n)$ with N_i simple submodules of M and $\oplus_{i=1}^n N_i \oplus Q_n = M$
3. (Halt) $Q_n = (0)$ (then we are done)
4. (Next Step) Suppose $C_n = (\oplus_{i=1}^n N_i, Q_n)$ with $Q_n \neq (0)$ (non-halting step) then we choose a minimal submodule Q_{min} of M among those such that $(0) \subsetneq Q \subset Q_n$ (it is possible because M satisfies DCC). We set $N_{n+1} = Q_{min}$ and remark that the family $(N_i)_{1\leq i\leq n+1}$ is in direct sum and, by hypothesis, it exists Q_{n+1} such that $\oplus_{i=1}^{n+1} N_i \oplus Q_{n+1} = M$ then set $C_{n+1} = (\oplus_{i=1}^{n+1} N_i, Q_{n+1})$

Proof of Proposition [C]/2 and first applications

- **Proof that this algorithm halts**) unless $M = (0)$ there is at least one step. Let $n+1$ be any valid rank of a step. By construction $\oplus_{i=1}^{n} N_i \subsetneq \oplus_{i=1}^{n+1} N_i$, a strictly increasing sequence of submodules. By ACC this sequence must be finite.
- **Semi-simplicity**) Let m is the last index of the sequence C_n. We have $Q_m = (0)$ and then $M = \oplus_{i=1}^{m} N_i$. CQFD

CCRT[5] Finite groups, states and semi-simplicity

1. **Applies to** Every finite dimensional $*$-algebra which admits a faithful state (FS) then is semi-simple. See below.
2. **and in particular to** $\mathcal{A} = \mathbb{C}[G]$ where G is a finite group. With

$$\Big(\sum_{g\in G} \alpha(g)\, g\Big)^* := \Big(\sum_{g\in G} \overline{\alpha(g)}\, g^{-1}\Big)$$

and $\varphi(Q) = \langle 1_G | Q \rangle$

CCRT[6] States, isometries, orbits and orthogonality in star-algebras.

- First of all, note that spectral theory fails dramatically in the absence of a complete norm [a].
- Let $\mathcal{A}$ be an $*$-algebra ($x \to x^*$ is semi-linear, involutive and an anti-automorphism)
- $\mathcal{C}_+(\mathcal{A})$, generated by elements of the form $\sum_{i\in F} x_i x_i^*$ (with F finite) is an hermitian (self-dual) convex cone
- $State(\mathcal{A})$ is the set of linear forms $f \in \mathcal{A}^*$ such that $z \in \mathcal{C}_+(\mathcal{A}) \implies f(z) \geq 0$ and $f(1) = 1$
- A faithful state (FS) is such that

$$z \in \mathcal{C}_+(\mathcal{A}) \text{ and } f(z) = 0 \implies z = 0$$

[a] See discussion in https://math.stackexchange.com/questions/1520974

CCRT[6] cont'd/1

Proposition [D]

A finite dimensional star-algebra with a non-degenerate state is semi-simple with all its commutants $\simeq \mathbb{C}$.

Unfolding [D]

1. Let φ be one of these faithful states and set

$$\langle x|y\rangle = \varphi(x^* y) \tag{4}$$

it is a non-degenerate hermitian form such that, identically $\langle x|a.y\rangle = \langle a^*.x|y\rangle$. Then, if $\mathcal{J}$ is a left-ideal of the algebra $\mathcal{A}$, then it is easy to prove that $\mathcal{J}^\perp$ is a left ideal

2. In particular with the preceding setting ($\mathcal{A} = \mathbb{C}[G]$ where G is a finite group, star-structure and state) we have the result.

3. We decompose $\mathcal{A}$ into minimal left ideals with algorithm of slide 21 and get $\mathcal{A} = \oplus_{j\in F}\mathcal{J}_j$. We can then write $1_{\mathcal{A}} = \sum_{i\in F} p_i$.

Construction of the matrix units.

4. One can prove that $\mathcal{J}_i = \mathcal{A}\, p_j$ and $p_j p_i = p_i p_j = \delta_{ij} p_i$ (complete orthogonal family of minimal idempotents)
5. The lemma $Hom_{\mathcal{A}}(\mathcal{A}.e, \mathcal{A}.f) \simeq e.\mathcal{A}.f$ (sandwich) gives Wedderburn's decomposition.
6. For e, f idempotents then $eaf \to (x \to xeaf)$ is an iso of **k**-spaces between $e.\mathcal{A}.f$ and $Hom_{\mathcal{A}}(\mathcal{A}.e, \mathcal{A}.f)$ the inverse being $f \to f(e)$ (note that $f(e) \in e.\mathcal{A}.f$).
7. Return to $1_{\mathcal{A}} = \sum_{i \in F} p_i$ (each p_i is minimal) and set $i \sim j \Longleftrightarrow e_i.\mathcal{A}.e_j \neq (0)$ (block equivalence)
8. Take a block C, order $C = \{i_1 < i_2 < \cdots < i_m\}$ totally.
9. For $1 \leq j < m$ choose $a_{i_j} \in e_{i_j}.\mathcal{A}.e_{i_{j+1}} \setminus (0)$
10. **Remark** that 4 to 9 applies to every finite dimensional F-semi-simple algebras, in particular, with $F = \mathbb{R}$ you can get **Real**, **Complex** and **Quaternion** blocks.

Classification of finite-dim. semi-simple F- and $\mathbb{R}$-algebras

Wedderburn-Artin Theorem [1], p116 and [18].

Let F be a field and $\mathcal{A}$ a finite-dimensional F-algebra, then $\mathcal{A}$ is semi-simple iff it exists numbers $(n_i)_{1\leq i\leq r}$ and $(D_i)_{1\leq i\leq r}$ finite-dimensional F-division algebras such that

$$\mathcal{A} \simeq D_1^{n_1\times n_1} \times \cdots \times D_r^{n_r\times n_r} \tag{5}$$

Classification of finite-dim. semi-simple $\mathbb{R}$-algebras

As there are only three types of finite-dim. $\mathbb{R}$-division algebras i.e. $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, we have a complete description of the finite-dim. semi-simple $\mathbb{R}$-algebras: you have **Real**, **Complex** and **Quaternion** blocks.

Classification of finite-dim. semi-simple $\mathbb{C}$-algebras

In this case, as $\mathbb{C}$ is central, you have only **Complex** blocks (as $\mathbb{C} \subset \mathbb{H}$, but is not central).

Case of an *-algebra ($\mathbf{k} = \mathbb{C}$) with a (FS).

11. Let now $\mathcal{A}$ be a finite-dimensional *-algebra (over $\mathbb{C}$) with a (FS) φ.
12. From (4), we get that $\mathcal{A}$ is semi-simple.
13. On the other hand, one checks that the left-regular representation constructed by $\gamma_a(x) = a.x$ and $\rho(a) = \gamma_a$ is an *-faithful *-representation of $\mathcal{A}$
14. $\mathcal{A}$ can be given, through this faithful representation the structure of C*-algebra (not in general with the norm $||x||_1 = \sqrt{\langle x|x\rangle}$ though[a])
15. If $\mathcal{A}$ has the structure of C*-algebra for the norm $||x||_1$ then $\mathcal{A} \simeq \mathbb{C}$.
16. Sketch of the proof: Using the decomposition (3) through orthogonal ideal process, we get that the p_i are self-adjoint, then if $|F| > 1$ we can set $F = F_1 \sqcup F_2$ with $|F_i| > 0$, then with $e_i = \sum_{j\in F_i} p_j$, we get two self-adjoint idempotents and

$$||e_i|| = ||e_1 + e_2|| = ||e_1 - e_2|| = 1$$

makes the parallelogram rule fail.

A remark

Remark[a]. – The fact that A be a star-algebra of finite dimension, sum of matrix algebras is by no means sufficient to imply that the projectors on the blocks are *-invariant nor $A \simeq \mathbb{C}$ as shows the following counterexample. Take $B = \mathbb{C}^{n \times n}$ (algebra of complex square matrices of dimension $n > 0$) and $A = B \oplus B$ with the anti-automorphism $(X, Y)^\star = (Y^*, X^*)$. Then $(A, \star)$ is easily checked to be a star algebra (i.e. involutive algebra). It is of finite dimension, sum of matrix algebras but $dim_{\mathbb{C}} = 2n^2 \neq 1$. Indeed, the existence of a faithful state is crucial as there is none over A.

[a] For a counterexample with words, see `https://math.stackexchange.com/questions/829182`.

Universal problem without functor: Coproducts

All here is stated within the same category $\mathcal{C}$.

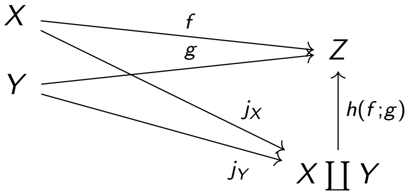


Figure: Coproduct $(j_X, j_Y; X \coprod Y)$.

$$
\begin{aligned}
&\big(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)\big) \\
&\big(\exists!\, h(f;g) \in Hom(X \coprod Y, Z)\big) \\
&\big(h(f;g) \circ j_X = f \text{ and } h(f;g) \circ j_Y = g\big)
\end{aligned} \tag{6}
$$

Coproducts: Sets

All here is stated within the category **Set**.

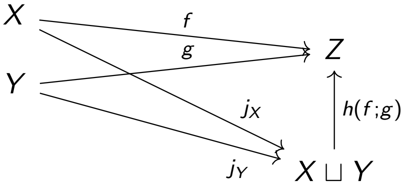


Figure: Coproduct $(j_X, j_Y; X \sqcup Y)$.

$$\big(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)\big)$$
$$\big(\exists!\, h(f;g) \in Hom(X \sqcup Y, Z)\big)$$
$$\big(h(f;g) \circ j_X = f \text{ and } h(f;g) \circ j_Y = g\big) \tag{7}$$

Coproducts: Vector Spaces

All here is stated within the same category $\mathbf{k} - \mathbf{Vect}$.

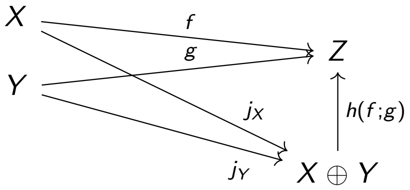


Figure: Coproduct $(j_X, j_Y; X \oplus Y)$ here $h(f;g) = f \oplus g$.

$$\begin{aligned}&\big(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)\big)\\&\big(\exists!\, h(f;g) \in Hom(X \oplus Y, Z)\big)\\&\big(h(f;g) \circ j_X = f \text{ and } h(f;g) \circ j_Y = g\big)\end{aligned} \tag{8}$$

Coproducts: $\mathbf{k} - \mathbf{CAAU}$

All here is stated within the same category $\mathbf{k} - \mathbf{CAAU}$.

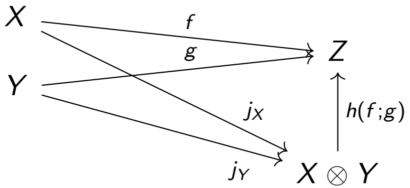


Figure: Coproduct $(j_X, j_Y; X \otimes Y)$ here $h(f;g) = f \otimes g$.

$$\begin{aligned}&\big(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)\big)\\&\big(\exists!\, h(f;g) \in Hom(X \otimes Y, Z)\big)\\&\big(h(f;g) \circ j_X = f \text{ and } h(f;g) \circ j_Y = g\big)\end{aligned} \tag{9}$$

All here is stated within the same category *Augmented* $\mathbf{k}-\mathbf{AAU}$.

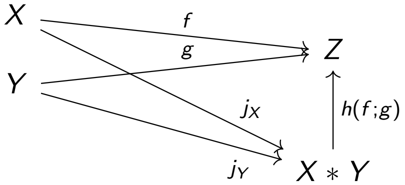


Figure: Coproduct $(j_X, j_Y; X*Y)$ here $h(f;g) = f*g$.

$$\begin{aligned}&\big(\forall (f,g) \in Hom(X,Z) \times Hom(Y,Z)\big)\\&\big(\exists!\, h(f;g) \in Hom(X*Y,Z)\big)\\&\big(h(f;g)\circ j_X = f \text{ and } h(f;g)\circ j_Y = g\big)\end{aligned} \tag{10}$$

Continuation: An immediate (and although rich) example/5

Words and Lyndon words, details.

Algebraic structure

- Concatenation: This law is noted *conc*
- With the empty word as neutral, the set of words is the free monoid $(X^*, conc, 1_{X^*})$
- The pairing between series and polynomials is defined by

$$\langle S|P\rangle = \sum_{w\in\mathcal{X}^*} \langle S|w\rangle\langle P|w\rangle$$

Coding by words gives access to a welter of structures, studies, relations and results (algebra, geometry, topology, probability, combinatorics on words, on polynomials and series). We will use in particular their complete factorisation by Lyndon words.

An immediate (and although rich) example/6

Words and classes

Example with $\mathcal{X} = \{a, b\},\ a < b$, in red Lyndon words ($= \mathcal{L}yn\mathcal{X}$).

Length	*words*
0	1_{X^*}
1	$a,\ b$
2	$aa,\ ab,\ ba,\ bb$
3	$aaa,\ aab,\ aba,\ abb,\ baa,\ bab,\ bba,\ bbb$
4	$a^4,\ a^3b,\ a^2ba,\ a^2b^2,\ aba^2,\ abab,\ ab^2a,\ ab^3$
	$ba^3,\ ba^2b,\ baba,\ babb,\ b^2a^2,\ b^2ab,\ b^3a,\ b^4$

Two properties of Lyndon words

1. All $\ell \in \mathcal{L}yn\mathcal{X} \setminus \mathcal{X}$ factorises (not uniquely in general) as $\ell = \ell_1\ell_2,\ \ell_1 \prec \ell_2,\ \ell_i \in \mathcal{L}yn\mathcal{X}$ (ex. $a^3ba^2bab = a^3b|a^2bab = a^3ba^2b|ab$), the one with the longest right factor will be called standard $\sigma(\ell) = (\ell_1, \ell_2)$.
2. Every word $w \in \mathcal{X}^*$ factorises uniquely $w = \ell_1^{i_1} \ldots \ell_k^{i_k}$ with $\ell_1 \succ \ldots \succ \ell_k, (\ell_i \in \mathcal{L}ynX)$

An immediate (and although rich) example/7

Shuffle product(s)

Non deformed case

Coming from the route where $\mathbf{k}\langle\mathcal{X}\rangle = \mathcal{U}(\mathcal{L}ie_{\mathbf{k}}\langle X\rangle)$, we have a structure of coalgebra on $\mathbf{k}\langle\mathcal{X}\rangle$ its comultiplication is given by its value on letters

$$\Delta_{ш}(x) = x \otimes 1_{\mathcal{X}^*} + 1_{\mathcal{X}^*} \otimes x \tag{11}$$

Then shuffle product is defined as a dual law, for each $w \in \mathcal{X}^*$ by

$$\langle P ш Q | w\rangle = \langle P \otimes Q | \Delta_{ш}(w)\rangle \tag{12}$$

We get the following recursion for shuffle products

$$w ш 1_{\mathcal{X}^*} = 1_{\mathcal{X}^*} ш w = w \qquad \text{for any word } w \in \mathcal{X}^*; \tag{13}$$

$$au ш bv = a(u ш bv) + b(au ш v) \tag{14}$$

An immediate (and although rich) example/8

Two bases in duality/1: Combinatorial constructions

Lyndon basis

$$\begin{array}{rcl} P_x = & x & \text{for } x \in X, \\ P_\ell = & [P_s, P_r] & \text{for } \ell \in \mathcal{L}yn\mathcal{X} \setminus \mathcal{X} \text{ and } \sigma(\ell) = (s, r), \\ P_w = & P_{\ell_1}^{i_1} \ldots P_{\ell_k}^{i_k} & \text{for } w = \ell_1^{i_1} \ldots \ell_k^{i_k}, \ell_1 \succ \ldots \succ \ell_k,\ (\ell_i \in \mathcal{L}yn\mathcal{X}). \end{array}$$

where $\succ$ stands for the lexicographic (strict) ordering defined from $x_0 \prec x_1$.

Triangular property

Indeed $\{P_w\}_{w \in X^*}$ is lower unitriangular w.r.t. words (this property, joined with the fact that this family is finely homogeneous, implies that $\{P_w\}_{w \in X^*}$ is a basis of $\mathbf{k}\langle \mathcal{X} \rangle$)

$$P_w = w + \sum_{v \succ w, \beta(v) = \beta(w)} c_v v \text{ with } c_v \in \mathbb{Z} \tag{15}$$

Dual bases

Construction of $(S_w)_{w\in\mathcal{X}^\alpha}$

For each multidegree α, let $\mathcal{X}^\alpha$ be the (finite) set of words with multidegree α and T_α be the lower unitriangular matrix of $\{P_w\}_{\beta(w)=\alpha}$ w.r.t. words of $\mathcal{X}^\alpha$ then, the matrix *transpose*(T^{-1}) defines a family $(S_w)_{w\in\mathcal{X}^\alpha}$ such that

1. $S_w = w + \sum_{v\prec w,\beta(v)=\beta(w)} d_v v$ with $d_v \in \mathbb{Z}$
2. For all $u, v \in \mathcal{X}^\alpha$, $\langle S_u | P_v\rangle = \delta_{u,v}$.
3. The quantification of the preceding property can be extended to all $u, v \in \mathcal{X}^*$ due to the fact that the decomposition (1) is, in fact, orthogonal.

Schützenberger's basis (**k** is a $\mathbb{Q}$-algebra)

M. -P. Schützenberger proved that, when **k** is a $\mathbb{Q}$-algebra, the basis $(S_w)_{w\in\mathcal{X}^*}$ can be computed recursively as follows

$$\begin{aligned}
S_x &= x && \text{for } x \in \mathcal{X},\\
S_l &= xS_u, && \text{for } l = xu \in \mathcal{L}yn\mathcal{X} \setminus \mathcal{X},\\
S_w &= \frac{S_{l_1}^{\shuffle\, i_1} \shuffle \ldots \shuffle S_{l_k}^{\shuffle\, i_k}}{i_1! \ldots i_k!} && \text{for } w = l_1^{i_1} \ldots l_k^{i_k}, l_1 \succ \ldots \succ l_k.
\end{aligned}$$

Triangular properties (recall)

$$P_w = w + \sum_{v \succ w, \beta(v)=\beta(w)} c_v v \quad \text{and} \quad S_w = w + \sum_{v \prec w, \beta(v)=\beta(w)} d_v v. \tag{16}$$

We recall that the bases $\{S_w\}_{w\in X^*}$ and $\{P_w\}_{w\in X^*}$ are lower and upper triangular respectively and that they are (finely) graded (all the monomials have the same partial degrees).

Table of these bases

Example (First values)

Let $X = \{x_0, x_1\}$ with $x_0 \prec x_1$.

l	P_l	S_l
x_0	x_0	x_0
x_1	x_1	x_1
x_0x_1	$[x_0, x_1]$	x_0x_1
$x_0^2x_1$	$[x_0, [x_0, x_1]]$	$x_0^2x_1$
$x_0x_1^2$	$[[x_0, x_1], x_1]$	$x_0x_1^2$
$x_0^3x_1$	$[x_0, [x_0, [x_0, x_1]]]$	$x_0^3x_1$
$x_0^2x_1^2$	$[x_0, [[x_0, x_1], x_1]]$	$x_0^2x_1^2$
$x_0x_1^3$	$[[[x_0, x_1], x_1], x_1]$	$x_0x_1^3$
$x_0^4x_1$	$[x_0, [x_0, [x_0, [x_0, x_1]]]]$	$x_0^4x_1$
$x_0^3x_1^2$	$[x_0, [x_0, [[x_0, x_1], x_1]]]$	$x_0^3x_1^2$
$x_0^2x_1x_0x_1$	$[[x_0, [x_0, x_1]], [x_0, x_1]]$	$2x_0^3x_1^2 + x_0^2x_1x_0x_1$
$x_0^2x_1^3$	$[x_0, [[[x_0, x_1], x_1], x_1]]$	$x_0^2x_1^3$
$x_0x_1x_0x_1^2$	$[[x_0, x_1], [[x_0, x_1], x_1]]$	$3x_0^2x_1^3 + x_0x_1x_0x_1^2$
$x_0x_1^4$	$[[[[x_0, x_1], x_1], x_1], x_1]$	$x_0x_1^4$

Factorisation of the diagonal as a resolution of identity.

Resolution of identity as an infinite product

Now we are in the position of writing the principal factorisation of the diagonal series. In here, series multiply by shuffle on the left and concatenation on the right.

$$\mathcal{D}_X := \sum_{w\in X^*} w\otimes w = \sum_{w\in X^*} S_w \otimes P_w = \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(S_l\otimes P_l) \qquad (17)$$

Application to factorisation of characters

If we have a shuffle-character $\chi : (\mathbf{k}\langle\mathcal{X}\rangle, ш, 1_{\mathcal{X}^*}) \to \mathcal{A}$, we act on the left

$$\chi = \sum_{w\in X^*} \chi(w)\otimes w = \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(\chi(S_l)\otimes P_l) \qquad (18)$$

But with a conc-character $\chi : (\mathbf{k}\langle\mathcal{X}\rangle, conc, 1_{\mathcal{X}^*}) \to \mathcal{A}$, we act on the right

$$\chi = \sum_{w\in X^*} w\otimes \chi(w) = \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(S_l\otimes \chi(P_l)) \qquad (19)$$

Conclusion

1. The values of iterated integrals (standard of regularized) are shuffle-characters, then we have factorisations and they constitute multiplicative regularizations.
2. The values of matrix representations of the free monoid (as the transitions of rational series for instance) are conc-characters and we get useful factorizations of them.
3. In the next talk (friday morning ?), we will see the deformed case through CQMM and applications to harmonic sums.

Thank you for your attention.

Exercises/1

Ex1: States and pre-states

Let $\mathcal{A}$ be a complex finite-dimensional $*$-algebra. A FS (Faithful State) is a linear form $\varphi \in \mathcal{A}^*$ such that

$$(\forall x \in \mathcal{A} \setminus \{0\})(\varphi(x^* x) > 0) \tag{20}$$

1. Prove that the bilinear form $\langle x|y\rangle := \frac{1}{2}\big(\varphi(x^* y) + \overline{\varphi(y^* x)}\big)$ is a non-degenerate hermitian scalar product[a] such that, identically
$$\langle x|a.y\rangle = \langle a^*.x|y\rangle$$
2. Prove that a complex finite-dimensional $*$-algebra admitting a FS is semi-simple.

[a] I take the convention of semi-linearity on the left (see the link "Hilbert modules" below).

Exercises/2

Ex1: States and pre-states/2

Let G be a finite group, set $\mathcal{A} = \mathbb{C}[G]$ and, for $a = \sum_{g \in G} \alpha(g)\, g$, set $a^* = \sum_{g \in G} \overline{\alpha(g)}\, g^{-1}$

3. Prove that $(\mathcal{A}, *)$ is an $*$-algebra
4. Prove that $\varphi \in \mathcal{A}^*$ defined by $\varphi(a) = \alpha(1)$ is a FS on $\mathcal{A}$.

Links

1. Categorical framework(s)

   https://ncatlab.org/nlab/show/category
   https://en.wikipedia.org/wiki/Category_(mathematics)

2. Universal problems

   https://ncatlab.org/nlab/show/universal+construction
   https://en.wikipedia.org/wiki/Universal_property

3. Paolo Perrone, *Notes on Category Theory with examples from basic mathematics*, 181p (2020)
   arXiv:1912.10642 [math.CT]

   https://en.wikipedia.org/wiki/Abstract_nonsense

4. Heteromorphism

   https://ncatlab.org/nlab/show/heteromorphism

5. D. Ellerman, *MacLane, Bourbaki, and Adjoints: A Heteromorphic Retrospective*, David EllermanPhilosophy Department, University of California at Riverside

Links/2

6. https://en.wikipedia.org/wiki/Category_of_modules
7. https://ncatlab.org/nlab/show/Grothendieck+group
8. Traces and hilbertian operators

   https://hal.archives-ouvertes.fr/hal-01015295/document
9. State on a star-algebra

   https://ncatlab.org/nlab/show/state+on+a+star-algebra
10. Hilbert module

    https://ncatlab.org/nlab/show/Hilbert+module

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