The Combinatorics of Harmonic Sums and Polylogarithms at negative integer multi- indices

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20-10-2015

Abstract

Let $Y_0 = \{y_s\}_{s\geq 0}$ be an infinite alphabet. We define Y_0^* to be the (free) monoid of words on the alphabet Y_0. Then each of elements $w \in Y_0^*$ can be writen in the form $w = y_{s_1} \dots y_{s_r}$ for any r-uplet $(s_1, \dots, s_r) \in \mathbb{N}^r$. Let $r \in \mathbb{N}$, and $z \in \mathbb{C}$ such that $|z| < 1$, then the following *Polylogarithm* is well defined

$$\mathrm{Li}^-_{s_1,\dots,s_r}(z) := \sum_{n_1 > \dots > n_r > 0} z^{n_1} n_1^{s_1} \dots n_r^{s_r}. \tag{1}$$

The Taylor expansion of the function $(1-z)^{-1}\,\mathrm{Li}_{s_1,\dots,s_r}(z)$ is given by $\frac{\mathrm{Li}^-_{s_1,\dots,s_r}(z)}{1-z} = \sum_{N\geq 0} \mathrm{H}^-_{s_1,\dots,s_r}(N)\, z^N$, where the coefficient $\mathrm{H}^-_{s_1,\dots,s_r} : \mathbb{N} \longrightarrow \mathbb{Q}$ is an arithmetic function, also called *Harmonic sum*, which can be expressed as follows

$$\mathrm{H}^-_{s_1,\dots,s_r}(N) := \sum_{N \geq n_1 > \dots > n_r > 0} n_1^{s_1} \dots n_r^{s_r}. \tag{2}$$

Then it can be checked that $\mathrm{H}^-_w(N) \in \mathbb{C}[N]$ and $\mathrm{Li}^-_w(z) \in \mathbb{C}[z]$ for any $w \in Y_0^*$. Moreover, the sets $\{\mathrm{H}^-_w(N)\}_{w\in Y_0^*}$ and $\{\mathrm{Li}^-_w(N)\}_{w\in Y_0^*}$ are basis of the vector spaces $\mathbb{Q}_+[N]$ and $\mathbb{Q}[z]$ respectively. In this talk, we discuss some combinatorics on Harmonic sums and the polylogarithms at the negative integer multi-indices.

References

[1] Costermans C., Hoang Ngoc Minh, *Noncommutative algebra, multiple harmonic sums and applications in discrete probability*, Journal of Symbolic Computation (2009), pp. 801-817.

[2] Faulhaber J., *Darinnen die miraculosische Inventiones zu den hochsten Cossen weiters continuirt und profitiert werden*, Academia Algebrae (1631).

[3] Foata D., Marcel-P. Schützenberger, *Théorie Géométrique des Polynômes Eulériens*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, Heidelberg, New York, (1970), 138(45 pp).