A Functorial Bridge between the Infinitary Affine Lambda-Calculus and Linear Logic

Damiano Mazza[1] and Luc Pellissier[2]

[1] CNRS, UMR 7030, LIPN, Université Paris 13, Sorbonne Paris Cité
Damiano.Mazza@lipn.univ-paris13.fr
[2] Université Paris 13, Sorbonne Paris Cité, LIPN, CNRS, UMR 7030
Luc.Pellissier@lipn.univ-paris13.fr

Abstract. It is a well known intuition that the exponential modality of linear logic may be seen as a form of limit. Recently, Melliès, Tabareau and Tasson gave a categorical account for this intuition, whereas the first author provided a topological account, based on an infinitary syntax. We relate these two different views by giving a categorical version of the topological construction, yielding two benefits: on the one hand, we obtain canonical models of the infinitary affine lambda-calculus introduced by the first author; on the other hand, we find an alternative formula for computing free commutative comonoids in models of linear logic with respect to the one presented by Melliès et al.

1 Introduction

The exponential modality of linear logic as a limit. Following the work of Girard [5], linearity has become a central notion in computer science and proof theory: it provides a finer-grained analysis of cut-elimination, which in turn, via Curry-Howard, gives finer tools for the analysis of the execution of programs. It is important to observe that the expressiveness of strictly linear or affine calculi is severely restricted, because programs in these calculi lack the ability to duplicate their arguments. The power of linear logic (which, in truth, is not linear at all!) resides in its so-called *exponential modalities*, which allow duplication (and erasing, if the logic is not already affine).

A possible approach to understand exponentials is to see the non-linear part of linear logic as a sort of limit of its purely linear part. The following old result morally says that, in the propositional case, exponential-free linear logic is "dense" in full linear logic:

Theorem 1 (Approximation [5]). *Define the* bounded exponential

$$!_p A := \overbrace{(A \mathbin{\&} 1) \otimes \cdots \otimes (A \mathbin{\&} 1)}^{p\ \text{times}},$$

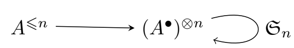
and define $?_p A := (!_p A^{\perp})^{\perp}$. *Note that these formulas are exponential-free (if* A *is). Let* A *be a propositional formula with* m *occurrences of the* $!$ *modality* *and n occurrences of the $?$ modality. If A is provable in full linear logic, then for every $p_1, \ldots, p_m \in \mathbf{N}$ there exist $q_1, \ldots, q_n \in \mathbf{N}$ such that A' is provable in exponential-free linear logic, where A' is obtained from A by replacing the i-th occurrence of $!$ with $!_{p_i}$ and the j-th occurrence of $?$ with $?_{q_j}$.*

For example, from the canonical proof of $?A^\perp \parr (!A \otimes !A)$ (contraction, *i.e.* duplication), we get proofs of $?_{p_1+p_2}A^\perp \parr (!_{p_1}A \otimes !_{p_2}A)$ for all $p_1, p_2 \in \mathbf{N}$.

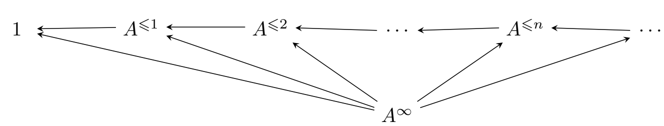
Remember that, if a linear formula A says "A exactly once", then $!A$ stands for "A at will". The formula $A \,\&\, 1$ is an affine version of A: it says "A at most once". This is a very specialized use of additive conjunction, in the sequel we prefer to avoid additive connectives and denote the affine version of A by $A^\bullet$, which may or may not be defined as $A \,\&\, 1$ (for instance, in affine logic, $A^\bullet = A$). Therefore, $!_pA = (A^\bullet)^{\otimes p}$ stands for "A at most p times", hence the name bounded exponential. So the Approximation Theorem supports the idea that $!A$ is somehow equal to $\lim_{p\to\infty} !_pA$.

Categories vs. topology. This idea was recently formalized in two quite different ways. The first is due to Melliès, Tabareau and Tasson [14], who rephrased the question in categorical terms. It is well known [3] that a $*$-autonomous category admitting the free commutative comonoid A^∞ on every object A is a model of linear logic (a so-called *Lafont category*). So, given a Lafont category, how does one compute A^∞? Using previous work by the first two authors [13], Melliès et al. showed that one may proceed as follows:

– compute the free co-pointed object $A^\bullet$ on A (which is $A \,\&\, 1$ if the category has binary products);
– compute the symmetric versions of the tensorial powers of $A^\bullet$, *i.e.* the following equalizers, where $\mathfrak{S}_n$ is the set of canonical symmetries of $(A^\bullet)^{\otimes n}$:



– compute the following projective limit, where $A^{\leqslant n} \longleftarrow A^{\leqslant n+1}$ is the canonical arrow "throwing away" one component:



At this point, for A^∞ to be the commutative comonoid on A it is enough that all relevant limits (the equalizers and the projective limit) commute with the tensor. Although not valid in general, this condition holds in several Lafont categories of very different flavor, such as Conway games and coherence spaces.

The second approach, due to the first author [10], is topological, and is based directly on the syntax. One considers an affine λ-calculus in which variables are treated as bounded exponentials: in a term of this calculus, a variable x may appear any number of times, each occurrence appears indexed by an integer (each instance, noted x_i, is labelled with a distinct $i \in \mathbf{N}$). The argument of applications is not a term but a sequence of terms, and to reduce the redex $(\lambda x.t)\langle u_0, \ldots, u_{n-1}\rangle$ one replaces each free x_i in t with u_i (a special term $\perp$ is substituted if $i \geq n$). The calculus is therefore affine, in the sense that no duplication is performed, and in fact it strongly normalizes even in absence of types (the size of terms strictly decreases with reduction).

At this point, the set of terms is equipped with the structure of uniform space[3], the Cauchy-completion of which, denoted by $\Lambda_\infty^{\mathrm{aff}}$, contains infinitary terms, *i.e.* allowing infinite sequences $\langle u_1, u_2, u_3, \ldots\rangle$. The original calculus embeds (and is dense) in $\Lambda_\infty^{\mathrm{aff}}$ by considering a finite sequence as an almost-everywhere $\perp$ sequence. Reduction, which is continuous, is defined as above, except that infinitely many substitutions may occur. This yields non-termination, in spite of the calculus still being affine: if $\Delta_n := \lambda x.x_0\langle x_1, \ldots, x_n\rangle$, then $\Delta := \lim_{n\to\infty} \Delta_n = \lambda x.x_0\langle x_1, x_2, x_3, \ldots\rangle$ and $\Omega := \Delta\langle\Delta, \Delta, \Delta, \ldots\rangle \to \Omega$.

Ideally, these infinitary terms should correspond to usual λ-terms. But there is a continuum of them, definitely too many. The solution is to consider a partial equivalence relation $\approx$ such that, in particular, $x_i \approx x_j$ for all i, j and $t\langle u_1, u_2, u_3, \ldots\rangle \approx t'\langle u'_1, u'_2, u'_3, \ldots\rangle$ whenever $t \approx t'$ and, for all $i, i' \in \mathbf{N}$, $u_i \approx u'_{i'}$. After introducing a suitable notion of reduction $\Rightarrow$ on the equivalence classes of $\approx$, one finally obtains the isomorphism for the reduction relations

$$(\Lambda_\infty^{\mathrm{aff}}/{\approx}, \Rightarrow) \quad \cong \quad (\Lambda, \to_\beta),$$

where $(\Lambda, \to_\beta)$ is the usual pure λ-calculus with β-reduction. Similar infinitary calculi (also with a notion of partial equivalence relation) were considered by Kfoury [7] and Melliès [11], although without a topological perspective. The indices identifying the occurrences of exponential variables are also reminiscent of Abramsky, Jagadeesan and Malacaria's games semantics [1].

Reconciling the two approaches. The contribution of this paper is to draw a bridge between the two approaches presented above. Indeed, we develop a categorical version of the topological construction of [10], which turns out to:

1. give a canonical way of building denotational models of the infinitary affine λ-calculus;
2. provide an alternative formula for computing the free commutative comonoid in a Lafont category.

Drawing inspiration from [13,14], we base our work on functorial semantics in the sense of Lawvere, computing free objects as Kan extensions.

Functorial semantics. The idea of functorial semantics is to describe an algebraic theory as a certain category constituted of the different powers of the domain of the theory as the objects, the operations of the theory as morphisms, and encode the relations between the operations in the composition operation. We will not consider algebraic theories as Lawvere did, but the more general symmetric monoidal theories, or PROPs [8] (*product and permutation categories*).

[3] The generalization of a metric space, still allowing one to speak of Cauchy sequences.

Definition 1 (symmetric monoidal theory). *An* n-sorted symmetric monoidal theory *is defined as a symmetric monoidal category* $\mathbb{T}$ *whose objects are* n*-tuples of natural numbers and with a tensorial product defined as the point-wise arithmetical sum.*

A model *of* $\mathbb{T}$ *in a symmetric monoidal category (SMC)* $\mathcal{C}$ *is a symmetric strong monoidal functor* $\mathbb{T} \to \mathcal{C}$*.*

A morphism *of models of* $\mathbb{T}$ *in* $\mathcal{C}$ *is a monoidal natural transformation between models of* $\mathbb{T}$ *in* $\mathcal{C}$*. We will denote as* $\mathrm{Mod}(\mathbb{T}, \mathcal{C})$ *the category with models of* $\mathbb{T}$ *in* $\mathcal{C}$ *as objects and morphisms between models as morphisms.*

The simplest symmetric monoidal theory, denoted by $\mathbb{B}$, has as objects the natural numbers seen as finite ordinals and as morphisms the bijections between them (the permutations). Alternatively, $\mathbb{B}$ can be seen as the free symmetric monoidal category on one object (the object 1, with monoidal unit 0). As such, a model of $\mathbb{B}$ is nothing but an object A in a symmetric monoidal category $\mathcal{C}$, and the categories $\mathcal{C}$ and $\mathrm{Mod}(\mathbb{B}, \mathcal{C})$ are equivalent.

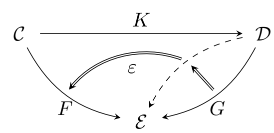
The key non-trivial example in our context is that of commutative (co)monoids. We remind that a commutative monoid in a SMC $\mathcal{C}$ is a triple $(A, \mu : A \otimes A \to A, \eta : \mathbf{1} \to A)$, with A an object of $\mathcal{C}$, such that the arrows μ and η interact with the associator, unitors and symmetry of $\mathcal{C}$ to give the usual laws of associativity, neutrality and commutativity (see *e.g.* [9]). A morphism of monoids $f : (A, \mu, \eta) \to (A', \mu', \eta')$ is an arrow $f : A \to A'$ such that $f \circ \mu = \mu' \circ (f \otimes f)$ and $f \circ \eta = \eta'$. We denote the category of monoids of $\mathcal{C}$ and their morphisms as $\mathrm{Mon}(\mathcal{C})$. The dual notion of comonoid, and the relative category $\mathrm{Comon}(\mathcal{C})$, is obtained by reversing the arrows in the above definition. Now, consider the symmetric monoidal theory $\mathbb{F}$ whose objects are the natural numbers seen as finite ordinals and its morphisms are the functions between them (*i.e.* $\mathbb{F}$ is the skeleton of the category of finite sets). We easily check that $\mathrm{Mod}(\mathbb{F}, \mathcal{C}) \simeq \mathrm{Mon}(\mathcal{C})$ and $\mathrm{Mod}(\mathbb{F}^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{Comon}(\mathcal{C})$. Indeed, a strict symmetric monoidal functor from $\mathbb{F}$ to $\mathcal{C}$ picks an object of $\mathcal{C}$ and the image of any arrow $m \to n$ of $\mathbb{F}$ is unambiguously obtained from the images of the unique morphisms $0 \to 1$ and $2 \to 1$ in $\mathbb{F}$, which are readily verified to satisfy the monoid laws.

Summing up, finding the free commutative comonoid A^∞ on an object A of a SMC $\mathcal{C}$ is the same thing as turning a strict symmetric monoidal functor $\mathbb{B} \to \mathcal{C}$ into a strict symmetric monoidal functor $\mathbb{F}^{\mathrm{op}} \to \mathcal{C}$ which is universal in a suitable sense. This is where Kan extensions come into the picture.

Free comonoids as Kan extensions. Kan extensions allow to extend a functor along another. Let $K : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{C} \to \mathcal{E}$ be two functors. If we think of K as an inclusion functor, it seems natural to try to define a functor $\mathcal{D} \to \mathcal{E}$ that would in a sense be universal among those that extend F. There are two ways of formulating this statement precisely, yielding left and right Kan extensions. We only describe the latter, because it is the case of interest for us:

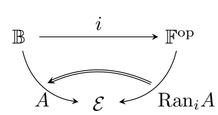
Definition 2 (Kan extension). *Let* $\mathcal{C}, \mathcal{D}, \mathcal{E}$ *be three categories and* $F : \mathcal{C} \to \mathcal{E}$*,* $K : \mathcal{C} \to \mathcal{D}$ *two functors. The* right Kan extension of F along K *is a functor* $\mathrm{Ran}_K F : \mathcal{D} \to \mathcal{E}$ *together with a natural transformation* $\varepsilon : \mathrm{Ran}_K F \circ K \Rightarrow F$

such that for any other pair $(G : \mathcal{D} \to \mathcal{E}, \gamma : G \circ K \Rightarrow F)$*,* γ *factors uniquely through* ε*:*



It is easy to check that $\mathbf{Cat}(G, \mathrm{Ran}_K F) \simeq \mathbf{Cat}(G \circ K, F)$, where by $\mathbf{Cat}(f, g)$ (f and g being functors with same domain and codomain) we mean the 2-homset of the 2-category $\mathbf{Cat}$, *i.e.* the set of all natural transformations from f to g. In other words, Ran_K is right adjoint to U_K, the functor precomposing with K (whence the terminology "right"—the left adjoint to U_K is the left Kan extension). This observation is important because it tells us that Kan extensions may be relativized to any 2-category. In particular, we may speak of *symmetric monoidal Kan extensions* by taking the underlying 2-category to be **SymMonCat** (symmetric monoidal categories, strict symmetric monoidal functors and monoidal natural transformations).

Now, there is an obvious inclusion functor $i : \mathbb{B} \to \mathbb{F}^{\mathrm{op}}$ (bijections are particular functions), which is strictly symmetric monoidal. So if $\mathcal{E}$ is symmetric monoidal and A is an object of $\mathcal{E}$, we are in the situation described above with $\mathcal{C} = \mathbb{B}$, $\mathcal{D} = \mathbb{F}^{\mathrm{op}}$, $K = i$ and F the strict symmetric monoidal functor corresponding to A, which we abusively denote by A. The fundamental difference is that the diagram lives in **SymMonCat** instead of **Cat**. It is an instructive exercise to verify that the free commutative comonoid on A, if it exists, is $A^\infty = \mathrm{Ran}_i A(1)$, *i.e.* the right symmetric monoidal Kan extension of A along i, computed in 1:



Remember that the free commutative comonoid on A is a commutative comonoid A^∞ with an arrow $\mathrm{d} : A^\infty \to A$ such that, whenever C is a commutative comonoid and $f : C \to A$, there is a unique comonoid morphism $u : C \to A^\infty$ such that $f = \mathrm{d} \circ u$. The arrow d is ε, where $\varepsilon : \mathrm{Ran}_i A \circ i \Rightarrow A$ is the natural transformation coming with the Kan extension.

More generally, if $\mathbb{T}_1$ and $\mathbb{T}_2$ are two symmetric monoidal theories, a symmetric monoidal functor $i : \mathbb{T}_1 \to \mathbb{T}_2$ induces a forgetful functor $U_i : \mathrm{Mod}(\mathbb{T}_2, \mathcal{E}) \to \mathrm{Mod}(\mathbb{T}_1, \mathcal{E})$ such that $M \mapsto M \circ i$. So we may reformulate the problem of finding the "free $\mathbb{T}$-model" on an object A of $\mathcal{E}$ as finding a left monoidal adjoint to U_i with $i : \mathbb{B} \to \mathbb{T}$. That is precisely what we did above, with $\mathbb{T} = \mathbb{F}^{\mathrm{op}}$.

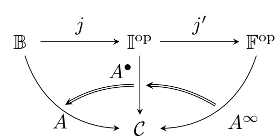
Computing monoidal Kan extensions. The above discussion is interesting because it provides a way of explicitly computing A^∞ from A. In fact, there is a well-known formula for computing Kan extensions [9]. When applied to the above special case, it gives

$$A^\infty = \prod_n A^{\otimes n}/\!\sim,$$

where $A^{\otimes n}/\!\sim$ is the symmetric tensor product. However, this formula works only for Kan extensions in **Cat** and there are no known formulas in other 2-categories. The main contribution of [13] was to find a sufficient condition under which the formula is correct also in **SymMonCat**. The condition is, roughly speaking, a commutation of the tensor with certain limits depending on the Kan extension at stake. In the above case, it requires the tensor to commute with countable products, which, in models of linear logic, boils down to having countable biproducts. Lafont categories of this kind do exist (*e.g.* the category **Rel** of sets and relations), but they are a little degenerate and not very representative.

The idea of [14] was to decompose the Kan extension in two, so that the commutation condition is weaker and satisfied by more Lafont categories. The intermediate step uses a symmetric monoidal theory denoted by $\mathbb{I}$, whose objects are natural numbers (seen as finite ordinals) and morphisms are the injections. Note that $\mathrm{Mod}(\mathbb{I}^{\mathrm{op}}, \mathcal{C})$ is equivalent to the slice category $\mathcal{C} \downarrow \mathbf{1}$. By definition, this is the category of *copointed objects* of $\mathcal{C}$: pairs $(A, w : A \to \mathbf{1})$ (with $\mathbf{1}$ the tensor unit, not necessarily terminal), with morphisms $f : (A, w) \to (A', w')$ arrows $f : A \to A'$ such that $w = w' \circ f$[4].

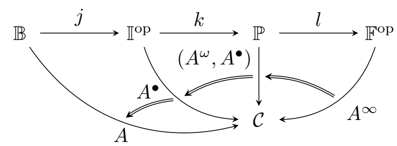
There are of course strict symmetric monoidal injections $j : \mathbb{B} \to \mathbb{I}^{\mathrm{op}}$ and $j' : \mathbb{I}^{\mathrm{op}} \to \mathbb{F}^{\mathrm{op}}$, such that $j' \circ j = i$. Unsurprisingly, $\mathrm{Ran}_j A(1)$ is the free copointed object on A, which we denoted by $A^\bullet$ above. Since Kan extensions compose (assuming they exist), we have $A^\infty = \mathrm{Ran}_{j'} A^\bullet(1)$:



For the second Kan extension to be computed in **SymMonCat** using the **Cat** formula, a milder commutation condition than requiring countable biproducts suffices. It is the commutation condition we mentioned above when we recalled the three-step computation of A^∞ (free copointed object, equalizers, projective limit), which indeed results from specializing the general Kan extension formula.

One more intermediate step. The bridge between the categorical and the topological approach will be built upon a further decomposition of the Kan extension: in the second step, we interpose a 2-sorted theory, denoted by $\mathbb{P}$ (this is why we introduced multi-sorted theories, all theories used so far are 1-sorted):

[4] The w stands for weakening.



We will call the models of $\mathbb{P}$ *partitionoids*. Intuitively, the free partitionoid on A allows to speak of infinite streams on $A^\bullet$, from which one may extract arbitrary elements and substreams via maps of type $A^\omega \to (A^\bullet)^{\otimes m} \otimes (A^\omega)^{\otimes n}$. Such maps are the key to model the infinitary affine λ-calculus. This intuition is especially evident in **Rel** (the category of sets and relations), where A^ω is the set of all functions $\mathbf{N} \to A^\bullet$ which are almost everywhere $*$ (in **Rel**, $A^\bullet = A \uplus \{*\}$).

2 The Infinitary Affine Lambda-Calculus

We consider three pairwise disjoint, countable sets of *linear*, *affine* and *exponential* variables, ranged over by $k, l, m \ldots$, $a, b, c \ldots$ and $x, y, z \ldots$, respectively. The terms of the infinitary affine λ-calculus belong to the following grammar:

$$\begin{array}{ll} t, u ::= l \mid \lambda l.t \mid tu \mid \mathsf{let}\, k \otimes l = u \,\mathsf{in}\, t \mid t \otimes u & \text{linear} \\ \quad \mid a \mid \mathsf{let}\, a^\bullet = u \,\mathsf{in}\, t \mid \bullet t & \text{affine} \\ \quad \mid x_i \mid \mathsf{let}\, x^\omega = u \,\mathsf{in}\, t \mid \langle u_0, u_1, u_2, \ldots \rangle & \text{exponential} \end{array}$$

The linear part of the calculus comes from [2]. It is the internal language of symmetric monoidal closed categories. As usual, let constructs are binders. The notation $\langle u_0, u_1, u_2, \ldots \rangle$ stands for an infinite sequence of terms. We use $\mathbf{u}$ to range over such sequences and write $\mathbf{u}(i)$ for u_i. Note that each u_i is inductively smaller than $\mathbf{u}$, so terms are infinite but well-founded. The usual linearity/affinity constraints apply to linear/affine variables, with the additional constraint that if x_i, x_j are distinct occurrences of an exponential variable in a term, then $i \neq j$. Furthermore, the free variables of a term of the form $\mathbf{u}$ (resp. $\bullet t$) must all be exponential (resp. exponential or affine).

The reduction rules are as follows:

$$\begin{array}{cc} (\lambda l.t)u \to t[u/l] & \mathsf{let}\, k \otimes l = u \otimes v \,\mathsf{in}\, t \to t[u/k][v/l] \\ \mathsf{let}\, a^\bullet = \bullet u \,\mathsf{in}\, t \to t[u/a] & \mathsf{let}\, x^\omega = \mathbf{u} \,\mathsf{in}\, t \to t[\mathbf{u}(i)/x_i] \end{array}$$

In the exponential rule, i ranges over $\mathbf{N}$, so there may be infinitely many substitutions to be performed. There are also the usual commutative conversions involving let binders, which we omit for brevity. The reduction is confluent, as the rules never duplicate any subterm.

The results of [10] are formulated in an infinitary calculus with exponential variables only, whose terms and reduction are defined as follows:

$$t, u ::= x_i \mid \lambda x.t \mid t\langle u_0, u_1, u_2, \ldots \rangle, \qquad (\lambda x.t)\mathbf{u} \to t[\mathbf{u}(i)/x_i]$$

$$\frac{}{\Gamma;\Delta;l:A\vdash l:A}\ \textsf{lin-ax}\qquad\frac{}{\Gamma;\Delta,a:A;\vdash a:A}\ \textsf{aff-ax}\qquad\frac{i\in\mathbf{N}}{\Gamma,x:A;\Delta;\vdash x_i:A^\bullet}\ \textsf{exp-ax}$$

$$\frac{\Gamma;\Delta;\Sigma,l:A\vdash t:B}{\Gamma;\Delta;\Sigma\vdash\lambda l.t:A\multimap B}\ {\multimap}I\qquad\frac{\Gamma;\Delta;\Sigma\vdash t:A\multimap B\quad\Gamma;\Delta';\Sigma'\vdash u:A}{\Gamma;\Delta,\Delta';\Sigma,\Sigma'\vdash tu:B}\ {\multimap}E$$

$$\frac{\Gamma;\Delta;\Sigma\vdash t:A\quad\Gamma;\Delta';\Sigma'\vdash u:B}{\Gamma;\Delta,\Delta';\Sigma,\Sigma'\vdash t\otimes u:B}\ {\otimes}I\qquad\frac{\Gamma;\Delta;\Sigma\vdash u:A\otimes B\quad\Gamma;\Delta';\Sigma',k:A,l:B\vdash t:C}{\Gamma;\Delta,\Delta';\Sigma,\Sigma'\vdash\mathsf{let}\,k\otimes l=u\,\mathsf{in}\,t:C}\ {\otimes}E$$

$$\frac{\Gamma;\Sigma;\vdash t:A}{\Gamma;\Sigma;\vdash\bullet t:A^\bullet}\ {\bullet}I\qquad\frac{\Gamma;\Delta;\Sigma\vdash u:A^\bullet\quad\Gamma;\Delta',a:A;\Sigma'\vdash t:C}{\Gamma;\Delta,\Delta';\Sigma,\Sigma'\vdash\mathsf{let}\,a^\bullet=u\,\mathsf{in}\,t:C}\ {\bullet}E$$

$$\frac{\dots\quad\Gamma;;\vdash\mathbf{u}(i):A^\bullet\quad\dots}{\Gamma;;\vdash\mathbf{u}:A^\omega}\ \omega I\qquad\frac{\Gamma;\Delta;\Sigma\vdash u:A^\omega\quad\Gamma,x:A;\Delta';\Sigma'\vdash t:C}{\Gamma;\Delta,\Delta';\Sigma,\Sigma'\vdash\mathsf{let}\,x^\omega=u\,\mathsf{in}\,t:C}\ \omega E$$

Fig. 1. The simply-typed infinitary affine λ-calculus. In every non-unary rule we require that t, u (or, for the ωI rule, $\mathbf{u}(i), \mathbf{u}(j)$ for all $i \neq j \in \mathbf{N}$) contain pairwise disjoint sets of occurrences of the exponential variables in Γ.

(the abstraction binds all occurrences of x). Such a calculus may be embedded in the one introduced above, as follows:

$$\begin{aligned} x_i^\circ &:= \mathsf{let}\,a^\bullet = x_i\,\mathsf{in}\,a \\ (\lambda x.t)^\circ &:= \lambda l.\mathsf{let}\,x^\omega = l\,\mathsf{in}\,t^\circ \\ (t\langle u_0, u_1, u_2, \dots\rangle)^\circ &:= t^\circ\langle \bullet u_0^\circ, \bullet u_1^\circ, \bullet u_2^\circ, \dots\rangle \end{aligned}$$

and we have $t \to t'$ implies $t^\circ \to^* t'^\circ$, so we do not lose generality. However, the categorical viewpoint adopted in the present paper naturally leads us to consider a simply-typed version of the calculus, given in Fig. 1. It is for this calculus that our construction provides denotational models. The types are generated by

$$A, B ::= X \mid A \multimap B \mid A \otimes B \mid A^\bullet \mid A^\omega,$$

where X is an atomic type. Note that the context of typing judgments has three *finite* components: exponential (Γ), affine (Δ) and linear (Σ). Although it may appear additive, the treatment of contexts is multiplicative also in the exponential case, as enforced by the condition in the caption of Fig. 1. The typing system enjoys the subject reduction property, as can be proved by an induction on the depth of the reduced redex.

3 Denotational Semantics

Definition 3 (reduced fpp, monoidal theory $\mathbb{P}$). *A* finite partial partition *(*fpp*) is a finite (possibly empty) sequence $(S_1, \dots, S_k)$ of non-empty, pairwise disjoint subsets of $\mathbf{N}$. Fpp's may be composed as follows: let $\beta := (S_1, \dots, S_k)$, with S_i infinite, and let $\beta' := (S'_1, \dots, S'_{k'})$; we define $\beta' \circ_i \beta := (S_1, \dots, S_{i-1}, T_1, \dots, T_{k'}, S_{i+1}, \dots, S_k)$, where each T_j is obtained as follows: let $n_0 < n_1 < n_2 < \cdots$ be the elements of S_i in increasing order; then,*

$T_j := \{n_m \mid m \in S'_j\}$. *It must be noted that endowed with this composition, fpp's form an operad.*

We will only consider reduced *fpp's, in which each S_i is either a singleton or infinite. We will use the notation $(S_1, \ldots, S_m; T_1, \ldots, T_n)$ to indicate that the S_i are singletons and the T_j are infinite, and we will say that such an fpp has size $m + n$. Note that the composition of reduced fpp's is reduced. The set of all reduced fpp's will be denoted by $\mathcal{P}$.*

Reduced fpp's induce a 2-sorted monoidal theory $\mathbb{P}$, as follows: each $\beta \in \mathcal{P}$ of size $m+n$ induces an arrow $\beta : (0,1) \to (m,n)$ of $\mathbb{P}$. There is also an arrow $w : (1,0) \to (0,0)$ to account for partiality. Composition is defined as above.

For example, let $\beta := (E, O)$, where E and O are the even and odd integers, and let $\beta' := (\{0\}, \mathbf{N} \setminus \{0\})$ (these are actually total partitions). Then $\beta' \circ_1 \beta = (\{0\}, E \setminus \{0\}, O)$, whereas $\beta \circ_2 \beta' = (\{0\}, O, E \setminus \{0\})$.

Definition 4 (partitionoid). *A* partitionoid *in a symmetric monoidal category $\mathcal{C}$ is a strict symmetric monoidal functor*[5] *$G : \mathbb{P} \to \mathcal{C}$. Spelled out, it is a tuple $(G_0, G_1, w, (r_\beta)_{\beta \in \mathcal{P}}))$ with (G_0, w) a copointed object and $r_\beta : G_1 \to G_0^{\otimes m} \otimes G_1^{\otimes n}$ whenever β is of size $m+n$, such that the composition of compatible w and r_β satisfies the equations induced by $\mathbb{P}$.*

A morphism of partitionoids $G \to G'$ is a pair of arrows $f_0 : G_0 \to G'_0$, $f_1 : G_1 \to G'_1$ such that f_0 is a morphism of copointed objects and $r'_\beta \circ f_1 = (f_0^{\otimes m} \otimes f_1^{\otimes n}) \circ r_\beta$ for all $\beta \in \mathcal{P}$ of size $m + n$.

We say that F is the free partitionoid on A *if it is endowed with an arrow $e : F_0 \to A$ such that, for every partitionoid G with an arrow $f : G_0 \to A$, there exists a unique morphism of partitionoids $(u_0, u_1) : G \to F$ such that $f = e \circ u$.*

For example, for any set X, $(X, X^{\mathbf{N}}, !_X, (r_\beta)_{\beta \in \mathcal{P}})$ is a partitionoid in **Set**, where $!_X$ is the terminal arrow $X \to 1$ and, if $\beta = (\{i_1\}, \ldots, \{i_m\}; \{j_1^1 < j_2^1 < \cdots\}, \ldots, \{j_1^n < j_2^n < \cdots\})$ and $f : \mathbf{N} \to X$, $r_\beta(f) := (f(i_1), \ldots, f(i_m), k \mapsto f(j_k^1), \ldots, k \mapsto f(j_k^n)) \in X^m \times (X^{\mathbf{N}})^n$.

Lemma 1. *If (F_0, F_1) is the free partitionoid on A, then $F_0 = A^\bullet$, the free co-pointed object on A.*

Proof. This follows from observing that $(A^\bullet, F_1)$ is also a partitionoid on A. $\square$

Definition 5 (infinitary affine category). *Let A be an object in a symmetric monoidal category. We denote by $\dagger_A$ the following diagram:*

$$\mathbf{1} \xleftarrow{\varepsilon_1} A^\bullet \xleftarrow{\varepsilon_2} (A^\bullet)^{\otimes 2} \longleftarrow \cdots \xleftarrow{\varepsilon_n} (A^\bullet)^{\otimes n} \xleftarrow{\varepsilon_{n+1}} (A^\bullet)^{\otimes n+1} \longleftarrow \cdots$$

where $\varepsilon_1 = \varepsilon$ is the copoint of $A^\bullet$ and $\varepsilon_{n+1} := (\mathrm{id})^{\otimes n} \otimes \varepsilon$, i.e., the arrow erasing the rightmost component. We set $A^\omega := \lim \dagger_A$ (if it exists).

An infinitary affine category *is a symmetric monoidal closed category such that, for all A, the free partitionoid on A exists and is $(A^\bullet, A^\omega)$.*

[5] An algebra for the fpp operad.

Several well-known categories are examples of affine infinitary categories: sets and relations, coherence spaces and linear maps, Conway games. Finiteness spaces are a non-example. We give the relational example here, which is a bit degenerate but easy to describe and grasp. For the others, we refer to the extended version.

The category **Rel** has sets as objects and relations as morphisms. It is symmetric monoidal closed: the Cartesian product (which, unlike in **Set**, is not a categorical product in **Rel**!) acts both as $\otimes$ (with unit the singleton $\{*\}$) and $\multimap$. Let A be a set and let us assume that $* \notin A$. The free copointed object on A is (up to iso) $A \cup \{*\}$, with copoint the relation $\{(*, *)\}$. The F_1 part of the free partitionoid on A in **Rel** is (up to iso) the set of all functions $\mathbf{N} \to A^\bullet$ which are almost everywhere $*$. Given a reduced fpp $\beta := (\{i_1\}, \ldots, \{i_m\}; \{j_0^1 < j_1^1 < \ldots\}, \ldots, \{j_0^n < j_1^n < \ldots\})$, the corresponding morphism of type $A^\omega \to (A^\bullet)^{\otimes m} \otimes (A^\omega)^{\otimes n}$ is

$$r_\beta := \{(\mathbf{a}, (a_{i_1}, \ldots, a_{i_m}, \langle a_{j_0^1}, a_{j_1^1}, \ldots\rangle, \ldots, \langle a_{j_0^n}, a_{j_1^n}, \ldots\rangle)) \mid \mathbf{a} \in A^\omega\},$$

where we wrote $\langle a_0, a_1, a_2, \ldots\rangle$ for the function $\mathbf{a} : \mathbf{N} \to A^\bullet, i \mapsto a_i$.

Theorem 2. *An infinitary affine category is a denotational model of the infinitary affine λ-calculus.*

Proof. The interpretation of types is parametric in an assignment of an object to the base type X, and it is straightforward (notations are identical). In fact, we will confuse types and the objects interpreting them.

Let now $\Gamma; \Delta; \Sigma \vdash t : A$ be a typing judgment. The type of the corresponding morphism will be of the form $C_1 \otimes \cdots \otimes C_n \longrightarrow A$, where the C_i come from the context and are defined as follows. If it comes from $l : C \in \Sigma$ (resp. $a : C \in \Delta$), then $C_i := C$ (resp. $C_i := C^\bullet$). If it comes from $x : C \in \Gamma$, then $C_i := C^\omega$ if x appears infinitely often in t, otherwise, if it appears k times, $C_i := (C^\bullet)^{\otimes k}$.

The morphism interpreting a type derivation of $\Gamma; \Delta; \Sigma \vdash t : A$ is defined as customary by induction on the last typing rule. The lin-ax rule and all the rules concerning $\otimes$ and $\multimap$ are modeled in the standard way, using the symmetric monoidal closed structure. The only delicate point is modeling the seemingly additive behavior of the exponential context Γ in the binary rules (the same consideration will hold for the elimination rules of $\bullet$ and ω as well). Let us treat for instance the $\otimes I$ rule, and let us assume for simplicity that $\Gamma = x : C, y : D, z : E$, with x (resp. z) appearing infinitely often (resp. m and n times) in t and u, whereas y appears infinitely often in t but only k times in u. Let us also disregard the affine and linear contexts, which are unproblematic. The interpretation of the two derivations gives us two morphisms

$$[t] : C^\omega \otimes D^\omega \otimes (E^\bullet)^{\otimes m} \longrightarrow A, \qquad [u] : C^\omega \otimes (D^\bullet)^{\otimes k} \otimes (E^\bullet)^{\otimes n} \longrightarrow B.$$

Now, we seek a morphism of type $C^\omega \otimes D^\omega \otimes (E^\bullet)^{\otimes(m+n)} \longrightarrow A \otimes B$, because x and y appear infinitely often in $t \otimes u$, whereas z appears $m + n$ times. This is obtained by precomposing $[t] \otimes [u]$ with the morphisms $r_\beta : C^\omega \to C^\omega \otimes C^\omega$ and

$r_{\beta'} : D^\omega \to (D^\bullet)^{\otimes k} \otimes D^\omega$ associated with the fpp's $\beta = (; T_t, T_u)$ such that T_t (resp. T_u) contains all i such that x_i is free in t (resp. in u), and $\beta' = (S'_u; T'_t)$ is defined in a similar way with the variable y.

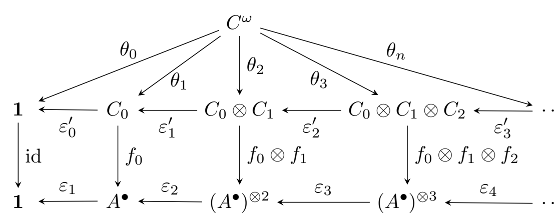
The weakening on exponential and affine variables in all axiom rules is modeled by the canonical morphisms $A^\bullet \to \mathbf{1}$ and $A^\omega \to \mathbf{1}$. For the rules aff-ax and exp-ax, we use the canonical morphism $A^\bullet \to A$ and the identity on $A^\bullet$, respectively.

The $\bullet I$ rule is modeled by observing that objects of the form $\Gamma^\omega \otimes \Delta^\bullet$ are copointed (from tensoring their copoints), so from an arrow $\Gamma^\omega \otimes \Delta^\bullet \longrightarrow A$ we obtain a unique arrow $\Gamma^\omega \otimes \Delta^\bullet \longrightarrow A^\bullet$ by universality of $A^\bullet$. The $\bullet E$ rule is just composition.

For what concerns the ωI rule, let us assume for simplicity that $\Gamma = x : C$. This defines a sequence of objects $(C_i)_{i\in\mathbf{N}}$ such that C_i is either C^ω or $(C^\bullet)^{\otimes k_i}$ according to whether x appears in $\mathbf{u}(i)$ infinitely often or k_i many times. Let now $S_i := \{j \in \mathbf{N} \mid x_j \text{ is free in } \mathbf{u}(i)\}$, define the fpp $\beta_i = (S_0, \ldots, S_i)$ and let

$$\varepsilon'_i := (\mathrm{id})^{\otimes i} \otimes w_i : C_0 \otimes \cdots \otimes C_{i-1} \otimes C_i \longrightarrow C_0 \otimes \cdots \otimes C_{i-1},$$

where $w_i : C_i \to \mathbf{1}$ is equal to $r_\emptyset$ if $C_i = C^\omega$ (with $\emptyset$ the empty fpp) or it is equal to $\varepsilon^{\otimes k_i}$ if $C_i = (C^\bullet)^{\otimes k_i}$. Let $\widehat{\beta}_i$ be the reduced fpp obtained from β_i by "splitting" its finite sets into singletons. If we set $\theta_i := r_{\widehat{\beta}_i}$, we have that for all $i \in \mathbf{N}$, $\varepsilon'_i \circ \theta_{i+1} = \theta_i$. Let now f_i be the interpretations of the derivations of $x : C; ; \vdash \mathbf{u}(i) : A^\bullet$ and consider the diagram



We showed above that all the upper triangles commute. It is easy to check that the bottom squares commute too, making $(C^\omega, ((f_0 \otimes \cdots \otimes f_{i-1}) \circ \theta_i)_{i\in\mathbf{N}})$ a cone for $\dagger_A$. Since $A^\omega = \lim \dagger_A$, this gives us a unique arrow $f : C^\omega \to A^\omega$, which we take as the interpretation of the derivation. The ωE rule is just composition, modulo the interposition of the canonical arrow $A^\omega \to (A^\bullet)^{\otimes k}$ in case x appears k times in t.

It remains to check that the above interpretation is stable under reduction, which may be done via elementary calculations. □

4 Computing Symmetric Monoidal Kan Extensions

We mentioned that there is a well-known formula for computing regular Kan extensions (*i.e.* in **Cat**). This requires some notions coming from enriched category theory, which we recall next (although here the enrichment will be trivial, *i.e.* on **Set**).

Definition 6 (cotensor product of an object by a set). *Let $\mathcal{C}$ be a (locally small) category. Let A be an object in $\mathcal{C}$ and E a set. The* cotensor product $E \circ A$ *of A by E is defined by:*

$$\forall B \in \mathcal{C}, \mathcal{C}(B, E \circ A) \simeq \mathbf{Set}(E, \mathcal{C}(B, A))$$

Any locally small category with products is cotensored over **Set** *(all of its objects have cotensor products with any set) and the cotensor product is given by:*

$$E \circ A = \prod_E A$$

We will write $\langle f_e \rangle_{e \in E} : B \to E \circ A$ for the infinite pairing of arrows $f_e : B \to A$ and $\pi_e : E \circ A \to A$ the projections.

Definition 7 (end). *Let $\mathcal{C}$, $\mathcal{E}$ be two categories and $H : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{E}$ a functor. The* end *of H, denoted by $\int_{\mathcal{C}} H$, is defined as the universal object endowed with projections $\int_{\mathcal{C}} H \to H(c, c)$ for all $c \in \mathcal{C}$ making the following diagram commute:*

$$\begin{array}{ccc} \int_{c \in \mathcal{C}} H(c,c) & \longrightarrow & H(c',c') \\ \downarrow & & \downarrow{\scriptstyle f^*} \\ H(c,c) & \xrightarrow{f_*} & H(c,c') \end{array}$$

for all arrows $f : c \to c'$ in $\mathcal{C}$.

Finally, here is the formula computing Kan extensions:

Theorem 3 ([9, X.4, Theorem 1]). *With the notations of Definition 2, whenever the objects exist:*

$$\mathrm{Ran}_K F(d) = \int_{c \in \mathcal{C}} \mathcal{D}(d, Kc) \circ Fc.$$

However, as mentioned in the introduction, the formula of Theorem 3 is only valid in **Cat** and we do not have any formula for computing a Kan extension in an arbitrary 2-category, or even in **SymMonCat**, our case of interest. Fortunately, Melliès and Tabareau proved a very general result [13, Theorem 1] giving sufficient conditions under which the Kan extension in **Cat** (something *a priori* worthless for our purposes) is actually the Kan extension in **SymMonCat** (what we want to compute). What follows is a specialized version of their result.

Theorem 4 ([13]). *Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be three symmetric monoidal categories and $F : \mathcal{C} \to \mathcal{E}$, $K : \mathcal{C} \to \mathcal{D}$ two monoidal symmetric functors. If (all the objects considered exist and) the canonical morphism*

$$X \otimes \int_{c \in \mathcal{C}} \mathcal{D}(d, Kc) \circ Fc \longrightarrow \int_{c \in \mathcal{C}} X \otimes \mathcal{D}(d, Kc) \circ Fc$$

is an isomorphism for every object X, then the right monoidal Kan extension (in the 2-category **SymMonCat***) of F along K may be computed as in Theorem 3.*

We may now give the abstract motivation behind Definition 5. The key property therein is that the free partitionoid on A is equal to $(A^\bullet, A^\omega)$. We now instantiate Theorem 4 to give a sufficient condition for that to be the case.

Proposition 1. *Let $\mathcal{C}$ be a symmetric monoidal closed category with all free partitionoids. If, for every objects X and A of $\mathcal{C}$, the canonical morphism*

$$X \otimes \int_{n\in\mathbb{I}^{\mathrm{op}}} \mathbb{P}((0,1),(n,0)) \circ (A^\bullet)^{\otimes n} \longrightarrow \int_{n\in\mathbb{I}^{\mathrm{op}}} X \otimes \left(\mathbb{P}((0,1),(n,0)) \circ (A^\bullet)^{\otimes n}\right)$$

is an isomorphism, then $\mathcal{C}$ is an infinitary affine category.

Proof. In what follows, when denoting the objects of the theory $\mathbb{P}$, we use the abbreviation $n^\bullet := (n,0)$ and $n^\omega := (0,n)$.

Let A be an object of $\mathcal{C}$, seen as a strict monoidal functor $A : \mathbb{B} \to \mathcal{C}$. We let the reader check that, if $(A^\bullet, F_1)$ is the free partitionoid on A, then $F_1 = \mathrm{Ran}_{k'} A(1^\omega)$, where $k' : \mathbb{B} \to \mathbb{P}$ is the strict monoidal functor mapping $n \mapsto n^\bullet$ (indeed, Definition 4 is just this Kan extension spelled out). This functor may be written as $k \circ j$, with $j : \mathbb{B} \to \mathbb{I}^{\mathrm{op}}$ the inclusion functor and $k : \mathbb{I}^{\mathrm{op}} \to \mathbb{P}$ mapping $n \mapsto n^\bullet$, which induces a decomposition of the Kan extension, yielding $F_1 = \mathrm{Ran}_k A^\bullet(1^\omega)$. Now, the hypothesis is exactly the condition allowing us to apply Theorem 4, which gives us

$$F_1 = \int_{n\in\mathbb{I}^{\mathrm{op}}} \mathbb{P}(1^\omega, n^\bullet) \circ (A^\bullet)^{\otimes n},$$

so it is enough to prove that $\lim \dagger_A = \int_{n\in\mathbb{I}^{\mathrm{op}}} \mathbb{P}(1^\omega, n^\bullet) \circ (A^\bullet)^{\otimes n}$.

We start with showing that $\int_{n\in\mathbb{I}^{\mathrm{op}}} \mathbb{P}(1^\omega, n^\bullet) \circ (A^\bullet)^{\otimes n}$ is a cone for $\dagger_A$. Let $\psi_n : (0,1) \to (n,0)$ be the morphism corresponding to the fpp $(\{0\},\dots,\{n-1\};)$. By composing the canonical projection with π_{ψ_n} (see Definition 6) we get an arrow

$$p_n : \int_{n\in\mathbb{I}^{\mathrm{op}}} \mathbb{P}(1^\omega, n^\bullet) \circ (A^\bullet)^{\otimes n} \to \mathbb{P}(1^\omega, n^\bullet) \circ (A^\bullet)^{\otimes n} \to (A^\bullet)^{\otimes n}.$$

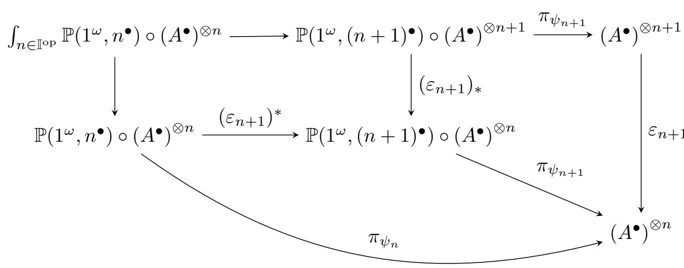
Observe now that the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{P}(1^\omega, n^\bullet) \circ (A^\bullet)^{\otimes n} & \xrightarrow{(\varepsilon_{n+1})^*} & \mathbb{P}(1^\omega, (n+1)^\bullet) \circ (A^\bullet)^{\otimes n} \\
& \searrow^{\pi_{\psi_n}} & \downarrow \pi_{\psi_{n+1}} \\
& & (A^\bullet)^{\otimes n}
\end{array}$$

because $\varepsilon_{n+1} \circ \psi_{n+1} = \psi_n$. Moreover, the diagram

$$\begin{array}{ccc}
\mathbb{P}(1^\omega, (n+1)^\bullet) \circ (A^\bullet)^{\otimes n+1} & \xrightarrow{\pi_{\psi_{n+1}}} & (A^\bullet)^{\otimes n+1} \\
\downarrow (\varepsilon_{n+1})_* & & \downarrow \varepsilon_{n+1} \\
\mathbb{P}(1^\omega, (n+1)^\bullet) \circ (A^\bullet)^{\otimes n} & \xrightarrow{\pi_{\psi_{n+1}}} & (A^\bullet)^{\otimes n}
\end{array}$$

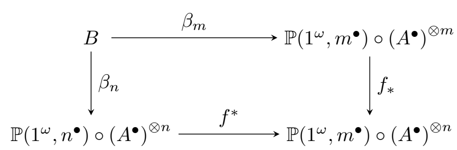
commutes too. So, by pasting them with the defining diagram of $\int_{n\in\mathbb{I}^{\mathrm{op}}} \mathbb{P}(1^\omega, n^\bullet)\circ (A^\bullet)^{\otimes n}$, one gets:



In particular, $(\int_{n\in\mathbb{I}^{\mathrm{op}}} \mathbb{P}(1^\omega, n^\bullet)\circ (A^\bullet)^{\otimes n}, (p_n))$ is a cone for the diagram.

Reciprocally, let $(B, (b_n))$ be any cone for this diagram. (b_n) extends uniquely into a family (β_n) such that:

- $\forall n \in \mathbf{N}, b_n = \pi_{\psi_n} \circ \beta_n$
- (β_n) makes the following diagrams commute:



for all $f : m \to n$ in $\mathbb{P}$.

Indeed, any element s of $\mathbb{P}(1^\omega, n^\bullet)$ is of the form $s = q \circ \psi_m$, where $m \geqslant n$ and $q \in \mathbb{I}^{\mathrm{op}}(m^\bullet, n^\bullet)$. So the family (β_n) is defined by:

$$\forall n \in \mathbf{N}, \beta_n = \langle A^\bullet(q) \circ b_m \rangle_{q\circ\psi_m \in \mathbb{P}(1^\omega, n^\bullet)}$$

is the unique family satisfying

$$\pi_{q\circ\psi_m} \circ \beta_n = q \circ \pi_{\psi_m} \circ \beta_m$$

This definition is sound, as $m > m'$ such that there exists $q, q', \psi_m, \psi_{m'}$ such that $s = q \circ \psi_m = q' \circ \psi_{m'}$, we have

$$q = q' \circ ((\mathrm{id})^{\otimes m'} \otimes (w^\bullet)^{\otimes m-m'})$$

and as such

$$A^\bullet(q) = A^\bullet(q') \circ \varepsilon_{m-m'+1} \circ \cdots \circ \varepsilon_m$$

and, as (b_n) is a cone for the sequential diagram,

$$A^\bullet(q) \circ b_m = A^\bullet(q') \circ b_{m'}.$$

So B makes the defining diagram of $\int_{n\in\mathbb{I}^{\mathrm{op}}} \mathbb{P}(1^\omega, n^\bullet)\circ (A^\bullet)^{\otimes n}$ commute, as such, (β_n) (and thus (b_n)) factors through it. Since all the cones of $\dagger_A$ factor through $\int_{n\in\mathbb{I}^{\mathrm{op}}} \mathbb{P}(1^\omega, n^\bullet)\circ (A^\bullet)^{\otimes n}$, it is its limit. □

Observe that the condition of Proposition 1 is actually quite easy to grasp: it says that the limit of $\dagger_A$ commutes with the tensor, *i.e.*, if we denote by $X \otimes \dagger_A$ the $\dagger_A$ diagram in which each $(A^\bullet)^{\otimes n}$ and ε_n are replaced by $X \otimes (A^\bullet)^{\otimes n}$ and $\mathrm{id}_X \otimes \varepsilon_n$, respectively, then the condition says $\lim(X \otimes \dagger_A) = X \otimes \lim \dagger_A$.

5 From Infinitary Affine Terms to Linear Logic

In [10], it was shown that usual λ-terms may be recovered as *uniform* infinitary affine terms. The categorical version of this result is that, in certain conditions, a model of the infinitary affine λ-calculus is also a model of linear logic.

Theorem 5. *Let $\mathcal{C}$ be an infinitary affine category. If, for every objects X and A in $\mathcal{C}$, the canonical morphism*

$$X \otimes \int_{(n,m)\in\mathbb{P}} (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m} \longrightarrow \int_{(n,m)\in\mathbb{P}} X \otimes (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m}$$

is an isomorphism, then $\mathcal{C}$ is a Lafont category. Moreover, the free commutative comonoid A^∞ on A may be computed as the equalizer of the diagram: where

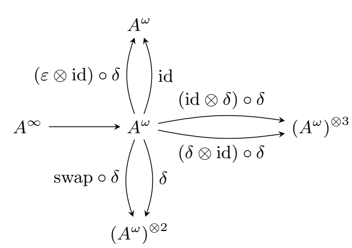


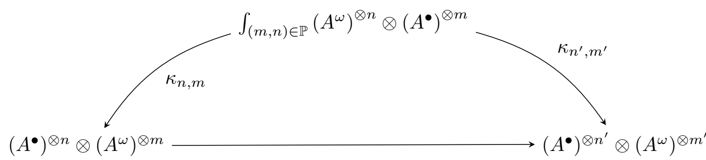
Fig. 2. Recovering the free co-commutative comonoid

$\delta : A^\omega \to A^\omega \otimes A^\omega$ and $\varepsilon : A^\omega \to \mathbf{1}$ are the morphisms induced by the fpp $(;E,O)$ (even and odd numbers) and the empty fpp, respectively, and $\mathrm{swap} : A^\omega \otimes A^\omega \to A^\omega \otimes A^\omega$ *is the symmetry of $\mathcal{C}$.*

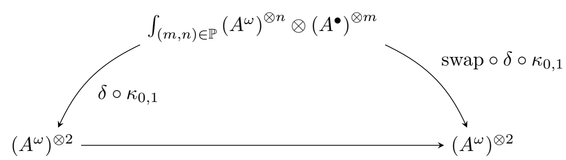
Proof. Let $l : \mathbb{P} \to \mathbb{F}^{\mathrm{op}}$ be the strict monoidal functor mapping $(m,n) \mapsto m+n$ and collapsing every arrow $(0,1) \to (m,n)$ to the unique morphism $1 \to m+n$ in $\mathbb{F}^{\mathrm{op}}$. By composing Kan extensions, we know that $A^\infty = \mathrm{Ran}_l(A^\bullet, A^\omega)(1)$. Remark that $\mathbb{F}^{\mathrm{op}}(1,p)$ is a singleton for all $p \in \mathbf{N}$, so the hypothesis is exactly what allows to apply Theorem 4, giving us

$$A^\infty = \int_{(m,n)\in\mathbb{P}} (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m}.$$

Now, $\int_{(m,n)\in\mathbb{P}} (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m}$ is the universal object making



commute. We are going to show that $\int_{(m,n)\in\mathbb{P}} (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m}$ is a cone for the diagram of Fig. 2. We will only show that



commutes. The family $(\iota_n \otimes \iota_m \circ \delta \circ \kappa_{0,1})_{n,m}$ is a cone for $\dagger_A^{\otimes 2}$. Moreover, the $\theta_{n,m} \circ \delta$ are defined in terms of the operations of $\mathbb{P}$, they actually are the canonical maps, and

$$\forall n, m, \iota_n \otimes \iota_m \circ \delta \circ \kappa_{0,1} = \kappa_{0,n+m}$$

The exact same reasoning gives:

$$\forall n, m, \iota_n \otimes \iota_m \circ \mathrm{swap} \circ \delta \circ \kappa_{0,1} = \kappa_{0,n+m}$$

But $(\kappa_{0,n+m})_{n,m}$ factors uniquely through $(A^\omega)^{\otimes 2}$ (the limit of $\dagger_A^{\otimes 2}$) and as such,

$$\forall n, m, \delta \circ \kappa_{0,1} = \mathrm{swap} \circ \delta \circ \kappa_{0,1}$$

which is what we wanted. So $\int_{(m,n)\in\mathbb{P}} (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m}$ is a cone for the diagram of Fig. 2.

Let us now prove that every cone for the diagram of Fig. 2 is a cone of the diagrams defining $\int_{(m,n)\in\mathbb{P}} (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m}$.

It is easy to verify that any object B making the diagram defining A^∞ commute is endowed with exactly one map $B \to (A^\omega)^{\otimes n}$ for all $n \in \mathbf{N}$, built from δ and ε which, is moreover, stable under all swaps. In particular, by composing these maps $(B \to (A^\omega)^{\otimes n})_{n\in\mathbf{N}}$ with the arrow $A^\omega \to A^\bullet$, it is clear that there is a unique family of arrows

$$\forall n, m \in \mathbf{N}, B \to (A^\bullet)^{\otimes n} \otimes (A^\omega)^{\otimes m}$$

stable under extractions and weakenings. So any cone for the diagram defining A^ω is a cone for the diagram defining $\int_{(m,n)\in\mathbb{P}} (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m}$ and as such, factorizes through it. So $\int_{(m,n)\in\mathbb{P}} (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m}$ is the limit of the diagram of Fig. 2, and thus isomorphic to A^∞. $\square$

Intuitively, this construction amounts to collapsing the family of non-associative and non-commutative "contractions" built with δ, ε and swap.

It should be remarked that the particular δ used is not canonical, other morphisms would yield the same result. Indeed, from [10] we know that recovering usual λ-terms from infinitary affine terms is possible using uniformity which, as recalled in the introduction, amounts to identifying

$$\lambda x.\langle x_0, x_1, x_2, \ldots\rangle \approx \lambda x.\langle x_{\beta(0)}, x_{\beta(1)}, x_{\beta(2)}, \ldots\rangle,$$

for every injection $\beta : \mathbf{N} \to \mathbf{N}$. Theorem 5 amounts to defining a congruence on terms verifying

$$\lambda x.\, \langle x_0, x_1, x_2, \ldots\rangle \simeq \lambda x.\, \langle x_0, x_2, x_4, \ldots\rangle$$
$$\lambda x.\, \langle x_0, x_2, x_4, \ldots\rangle \otimes \langle x_1, x_3, x_5, \ldots\rangle \simeq \lambda x.\, \langle x_1, x_3, x_5, \ldots\rangle \otimes \langle x_0, x_2, x_4, \ldots\rangle$$

which is sufficient to recover $\approx$.

6 Discussion

We saw how the functorial semantic framework provides a bridge between the categorical and topological approaches to expressing the exponential modality of linear logic as a form of limit. This gives a way to construct, under certain hypotheses, denotational models of the infinitary affine λ-calculus. Moreover, it gives us a formula for computing the free exponential which is alternative to that of Melliès et al. Since both formulas apply only under certain conditions, it is natural to ask whether one of them is more general than the other. Although we do not have a general result, we are able to show that, under a mild condition verified in all models of linear logic we are aware of, our construction is applicable in every situation where Melliès et al.'s is.

Indeed, Melliès et al.'s construction amounts to checking that the Kan extension along m (below, left) is a monoidal Kan extension, whereas the one exposed in this article amounts to checking that the two Kan extensions along k, then l are monoidal (below, right):

As Kan extensions compose, it suffices to know that the Kan extension along m is monoidal, that $m = k \circ l$, and that there exists two monoidal natural transformations inside the two upper triangles that can be composed to the last one to be sure that the Kan extensions along k and along l are monoidal too. We thus get:

Proposition 2. *Let $\mathcal{C}$ be a symmetric monoidal category with all free partitionoids. Assume that Melliès et al.'s formula works and that A^ω exists. If there* *exists, for all integers n, m monoidal maps*

$$(A^{\infty})^{\otimes n+m} \to (A^{\omega})^{\otimes n} \otimes (A^{\bullet})^{\otimes m}$$
$$(A^{\omega})^{\otimes n} \otimes (A^{\bullet})^{\otimes m} \to (A^{\bullet})^{\otimes n+m}$$

that composed together are the $n + m$ tensor of the map $A^{\infty} \to A^{\leqslant 1} \to A^{\bullet}$ then $\mathcal{C}$ is an infinitary affine category and a Lafont category.

Actually, in all models we are aware of, either both formulas work, or neither does. For instance, our construction fails for finiteness spaces [4], as does the construction given in [14].

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An extended version of this work is available on the HAL open archive server.

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