Message Scheduling on Trees under a Generalized Line-Communication Model

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Abstract

In this paper, we generalize the line-communication model by relaxing the notion of conflictness between paths. We show that the problem of finding optimal schedules to route any set of messages under both our generalized line-communication model and the bufferless routing model is NP-hard even if restricted to binary trees. Finally, a simple offline 2-approximation algorithm for our model on trees is presented.

Key words: *Message Scheduling, Line-Communications, Bufferless Routing, Tree Networks, NP-completeness, Approximation Algorithms.*

1. Introduction

Efficient routing of messages is a fundamental task in parallel and distributed systems. Many theoretical works about message routing are based on both the *line-communication* model [4, 7, 11] and the *bufferless routing* model [3, 12].

It is natural to divide the problem of message routing into two parts : (i) *Path selection*, in which each message is assigned to a path along which it moves, and (ii) *Scheduling* the movement of messages along the path assigned to it. The goal of the scheduling part is to compute a schedule for moving messages as quickly as possible. For some simple networks, such as trees, the message routing problem is simpler, as there is always a unique path for each message from its source to its destination. In this case, the problem reduces to simply schedule the movements of messages.

In a bufferless routing model, the messages move along the path step by step from the source to the destination in a greedy manner, i.e. once any message starts moving it does not stop along the way.

In a line-communication model, communications are scheduled in phases. During such a phase, some messages move from their source to their destinations (each message follows a given path). So, In this last model, we don't take care of the length of the path assigned to each message. The main constraint is that two messages can not move on their respective assigned paths during a same phase if there is a *conflict* between these paths. There are different ways to consider a conflict between two paths. The most classical one consists in saying that two paths are in conflict if they share some link of the network (see [4, 7, 11] and ref.). This seems sometimes to be too restrictive, for example if the link shared by two long paths is the first link in a path and the last one in the other path. In this paper, we give a natural generalization of the classical notion of conflictness between paths. We say that two paths are in conflict if and only if they share some link of the network which occurs in the i^{th} position on one path and in the j^{th} position on the other path, and if $|i-j|$ is smaller than some given positive integer k. We consider this integer k, that we call the *allowance*, as a parameter of the line-communication model. The allowance could be an indication for the degree of asynchronicity of the network. Note that in [1], the case k=1 was studied about some emulation problems.

This paper is concerned with the question of scheduling message movement on trees under these two communication models.

Previous and related work. The bufferless routing model is introduced in [3]. Ranade et al. [12] give for this model, an offline 2-approximation algorithm to the problem of message scheduling in trees.

In the line-communication model, Kumar et al. [10] show that the problem of finding optimal schedules in binary trees is NP-hard, and Kaklamanis et al. [9] give a $\frac{5}{3}$-approximation algorithm for directed trees.

Our results. We generalize the line-communication model by relaxing the notion of conflictness between paths. We show in Section 2 that even in our relaxed line-communication model, the problem of finding optimal schedules to route any set of messages in binary trees is NP-hard. We also show in this section that the message schedul-

ing problem in binary trees under the bufferless routing model is NP-hard, which was not yet known. In Section 3 we give an offline 2-approximation algorithm for the message scheduling problem in trees under the generalized line-communication model.
We begin by clarifying the generalized line-communication routing model.

1.1. Model and preliminary definitions

A routing network is a symmetric directed graph (see [2] for classical definitions), where processors and switches are nodes and links are modeled by two arcs in opposite directions [8]. It takes unit time to cross any arc and no two messages may traverse the same arc at the same time.
We denote by $\mathcal{P}$ the collection of elementary paths associated to a set $\mathcal{M}$ of messages, where each $m_j \in \mathcal{M}$ has an associated path $p_j = (v_0^j, v_1^j, \ldots, v_{d_j}^j)$, $1 \le j \le |\mathcal{M}|$. So, the message m_j is transmitted along the path p_j whose length is d_j. We say that (v_{i-1}^j, v_i^j) is the i^{th} arc of p_j, $1 \le i \le d_j$.
For all $(u,w) \in p_j$, $1 \le j \le |\mathcal{P}|$, we denote $l_j(u,w) = i$ if $(u,w) = (v_{i-1}^j, v_i^j)$, $1 \le i \le d_j$.
The communication in the network is made by *phases*. A communication phase is a set of directed paths pairwise non conflicting, on which messages have to move (in the sequel, directed paths are refered simply as paths). In other words, during each phase, messages move along the paths *without* buffering in the intermediate nodes. In the generalized line-communication model with allowance $k \ge 1$ ($GLC(k)$ model for short), a conflict between two paths is defined as follows.

Definition 1 *Given an integer $k \ge 1$ and any collection $\mathcal{P}$ of paths on a network, there exists a k-conflict between two paths p and p' in $\mathcal{P}$ iff there exists an arc $a \in p \cap p'$ such that a is the i^{th} and j^{th} arc in p and p' respectively, and $|i-j| < k$.*

Thus, during any communication phase in the $GLC(k)$ model, there exists no k-conflict between any two paths in it.

Definition 2 *Given any collection $\mathcal{P}$ of paths associated to a set of messages, then the k-*phases number *of $\mathcal{P}$ under the $GLC(k)$ model is the minimum cardinality r of any partition $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_r$ of $\mathcal{P}$ such that no two paths in $\mathcal{P}_i$, $1 \le i \le r$, have a k-conflict.*

Note that, given any collection $\mathcal{P}$ of paths on a network H, if the allowance k in the $GLC(k)$ model is chosen sufficiently large (for example, equal to $D(H)$, i.e. the diameter of H), then the k-phases number of $\mathcal{P}$ is clearly at least as large as the congestion of $\mathcal{P}$ (i.e. the maximum number of paths traversing any arc of the network) since no two paths sharing an arc must be used in a same phase. When k is small compared to $D(H)$, the congestion of $\mathcal{P}$ is not a lower bound for the k-phases number of $\mathcal{P}$. More precisely, the maximum number of paths in $\mathcal{P}$ such that (i) they share a same arc $a \in A(H)$, and (ii) they are pairwise in k-conflict, can be considered w.l.o.g., as the maximum taken over any position i, $1 \le i \le D(H)$, of the number of paths that share a in a position j, $i \le j < i+k$. Therefore, we define the k-*congestion* of $\mathcal{P}$ as follows.

Definition 3 *Let $\mathcal{P}$ be any collection of paths on a network H, and let k be an integer, $k \ge 1$. Then, the k-congestion of $\mathcal{P}$ is defined by*

$$\max_{(u,v)\in A(H)} \left(\max_{1 \le i \le D(H)} \left| \left\{ p_j \in \mathcal{P} : \begin{array}{l} (u,v) \in p_j, \\ i \le l_j(u,v) < i+k, \\ 1 \le j \le |\mathcal{P}| \end{array} \right\} \right| \right)$$

Clearly, the k-congestion of $\mathcal{P}$ is a lower bound for the k-phases number of $\mathcal{P}$ under the $GLC(k)$ model.

2. Complexity results on trees

In this section we show that, under the $GLC(k)$ model with allowance $k \ge 1$, the problem of finding an offline schedule to route any set of messages on trees with optimal k-phases number is NP-hard even if restricted to binary trees and k fixed. We outline a polynomial-time transformation of an instance of the EDGE-COLORING problem [5] into an instance of the routing schedule problem on binary trees under the $GLC(k)$ model. An instance of the EDGE-COLORING problem is given by a graph $G = (V,E)$ with maximum degree Δ. The question is whether the edges of G can be colored with Δ colors such that edges are assigned different colors if they share an endpoint. This problem is NP-complete even for *cubic graphs* [6], i.e. for Δ-regular graphs with $\Delta = 3$.

Theorem 1 *Under the $GLC(k)$ model with fixed allowance $k \ge 1$, the problem of finding an offline schedule with optimal k-phases number on binary trees is NP-hard.*

Proof :
Let I be an instance of the EDGE-COLORING problem given by any 3-regular graph $G = (V,E)$. We transform the instance I into an instance I' of the routing scheduling problem on binary trees under the $GLC(k)$ model, given by a symmetric directed binary tree T and by a collection $\mathcal{P}$ of paths on T. We begin by giving the principal ideas of the transformation of the instance I into the instance I' before to present it in a formal manner. Such a transformation is achieved in three parts as follows. In the first part, we construct a rooted binary tree T' whose leaves are the vertices of G, with the property that no two paths on T' beginning at any two different leaves of T' have a k-conflict. In the

second part, we transform the rooted binary tree T' into a binary tree T'' by replacing each leaf v of T' by an isomorphic binary subtree rooted at v on 18 vertices (see Fig. 2). Each one of these binary subtrees rooted at each leaf-vertex v of T' contains 9 leaves, 3 for each one of the three edges of G adjacent to vertex v (since G is a 3-regular graph). Moreover, for each edge of G, we add two particular paths on T'' to $\mathcal{P}$. Finally, in the third part, in order to ensure that each of the two paths on T'' in $\mathcal{P}$ associated to each edge of G are scheduled in the same phase of communication in any schedule with k-phases number equal to 3, we transform the binary tree T'' into a binary tree T by adding new vertices to T'' (as will be described in the following), and adding, for each edge of G, five new paths on T to $\mathcal{P}$, ending the transformation of the instance I into an instance I'.
A formal presentation of the transformation follows.

• *Part I :* Let $M = \lceil \log_2 |V| \rceil$ and let $h(i) = \lceil \frac{|V|}{2^{M-i}} \rceil$, $0 \leq i \leq M$. The construction of the rooted binary tree T' in this part is made in three steps :
- *step 1 :* first, we consider $M+1$ disjoint ordered-sets of vertices, denoted by $CL(i)$, $0 \leq i \leq M$, where each ordered-set $CL(i)$ is composed of $h(i)$ different new vertices. Formally, $CL(i) = (v_1^i, v_2^i, \ldots, v_{h(i)}^i)$, $0 \leq i \leq M$, where the ordered-set $CL(M)$ represents the vertex set V of G, i.e. $CL(M) = (v_1^M, v_2^M, \ldots, v_{|V|}^M)$, with $v_j^M \in V$, $1 \leq j \leq |V|$. We define the vertex $v_1^0 \in CL(0)$ as the root of T'.
- *step 2 :* next, for each vertex $v_j^i \in CL(i)$ with j even, $1 \leq i \leq M$, we construct a new directed symmetric line, denoted by $\mathcal{L}_j^i$, of length $l(i) = k2^{M-i+1} - 1$ (i.e., a line with $k2^{M-i+1}$ new vertices). Let c_j^i and $\bar{c}_j^i$ denote the end-vertices of the line $\mathcal{L}_j^i$, thus we connect the vertex $\bar{c}_j^i$ of $\mathcal{L}_j^i$ to vertex v_j^i.
- *step 3 :* finally, for each i, $1 \leq i \leq M$, and for each j, $1 \leq j \leq h(i)$, we connect the vertex $v_j^{i-1} \in CL(i-1)$ to the vertex $v_{2j-1}^i \in CL(i)$ and to the vertex c_{2j}^i of the line $\mathcal{L}_{2j}^i$ (if $\mathcal{L}_{2j}^i$ exists).

In Figure 1 we show the construction of the binary tree T' rooted at vertex $v_1^0 \in CL(0)$ from a given 3-regular graph G on 6 vertices, assuming $k = 1$. By the construction of the rooted binary tree T' in the Part I, we claim the following (the proofs of the two claims are given in Appendix).

Claim 1 *Given any vertex $v_j^i \in CL(i)$, $0 \leq i \leq M$, $1 \leq j \leq h(i)$, then its level in the binary tree T' rooted at $v_1^0 \in CL(0)$ (i.e. the number of arcs on the unique path from v_j^i to v_1^0), denoted by $N(v_j^i)$, is given by $N(v_j^i) = i + k(j-1)2^{M-i+1}$.*

Claim 2 *Given any two paths $v_l^i \longrightarrow \alpha$ and $v_j^i \longrightarrow \beta$ on the rooted binary tree T' beginning at any two vertices $v_l^i, v_j^i \in CL(i)$, $1 \leq i \leq M$, $j \neq l$, and ending at any two vertices α, β of T' respectively, then these two paths have no a k-conflict.*

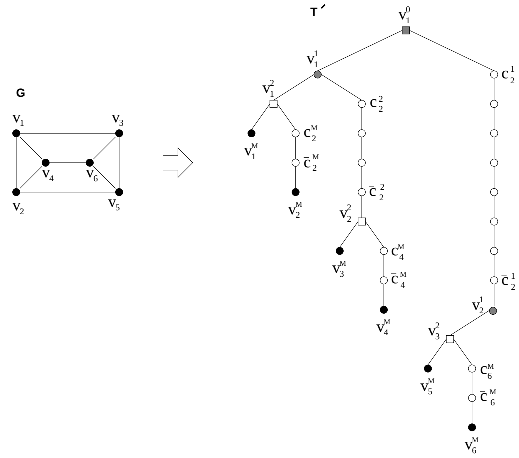

Figure 1. Construction of the rooted binary tree T' in the *Part I*.

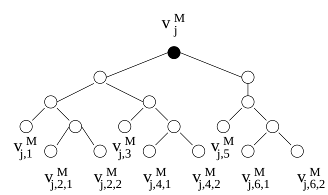

Figure 2. Isomorphic binary subtree rooted at each vertex v_j^M of the *Part II*.

• *Part II :* Now, we transform the rooted binary tree T' into a binary tree T'' by replacing each leaf $v_j^M \in CL(M)$ of T', $1 \leq j \leq |V|$, by an isomorphic binary subtree rooted at v_j^M on 18 vertices (see Fig. 2), 9 of which being leaves. The 9 leaves of the binary subtree rooted at each vertex v_j^M are denoted by $v_{j,1}^M$, $v_{j,2,1}^M$, $v_{j,2,2}^M$, $v_{j,3}^M$, $v_{j,4,1}^M$, $v_{j,4,2}^M$, $v_{j,5}^M$, $v_{j,6,1}^M$ and $v_{j,6,2}^M$, as is showed in Fig. 2. Moreover, we add to the collection $\mathcal{P}$ of paths which is still empty, two particular paths on T'' for each edge $e = [v_i, v_j]$ of G, denoted by p_1^e and p_2^e, which are $p_1^e = v_{i,m}^M \longrightarrow v_{j,l,2}^M$ and $p_2^e = v_{j,l-1}^M \longrightarrow v_{i,m+1,2}^M$, where the values of m (resp. l) are selected from $\{1, 3, 5\}$ (resp. $\{2, 4, 6\}$) such that a different value of m (resp. l) is chosen for each edge incident to v_i (resp. v_j). In Figure 3, we show these two paths corresponding to an edge e of an underlying 3-regular graph G, where the values of m and l are equal to 3 and 2 respectively.
• *Part III :* Finally, in order to ensure that the paths p_1^e and

p_2^e on T'' in $\mathcal{P}$, associated to each edge e of G, are scheduled in the same phase of communication (i.e. colored with the same color) in any schedule with k-phases number equal to 3, we transform the binary tree T'' into a binary tree T by adding other new vertices to T'', and for each edge e of G, we add to $\mathcal{P}$ other five new paths on T as follows. Let $e = [v_i, v_j]$ be any edge of G and let $p_1^e = v_{i,m}^M \longrightarrow v_{j,l,2}^M$ and $p_2^e = v_{j,l-1}^M \longrightarrow v_{i,m+1,2}^M$ be the paths on T'' in $\mathcal{P}$ associated to e, with $m \in \{1,3,5\}$ and $l \in \{2,4,6\}$. It is easy to verify that, by the construction of T'' in Part II, the lengths of the paths p_1^e and p_2^e (i.e., the number of arcs in p_1^e and p_2^e respectively) are equals. Let d_e be length of the path p_1^e, then it is easy to verify that d_e is odd, with $d_e \geq 2k+9$. Let $l_e = d_e - 3$ and let $n_e = (l_e/2) - 1$. Then, for each path p_1^e we construct a different directed symmetric line of length l_e, and we denote its vertices by $w_1^e, w_2^e, \ldots, w_{l_e+1}^e$, where w_1^e and $w_{l_e+1}^e$ are the end-vertices of such a line (see Fig. 3). Next, we connect the vertex w_1^e of such a line to vertex $v_{i,m+1,1}^M$ of T''. We add a new vertex denoted by z_e and we connect it to vertex $w_{n_e}^e$ of the added line. Finally, we add to $\mathcal{P}$ other five new paths on T associated to edge e of G, which are $p_3^e = p_4^e = v_{i,m}^M \longrightarrow z_e$, $p_5^e = w_{l_e+1}^e \longrightarrow z_e$, and $p_6^e = p_7^e = w_{l_e+1}^e \longrightarrow v_{i,m+1,2}^M$ (see Fig. 3). At the end of this part, the binary tree T and the collection $\mathcal{P}$ of paths on T constitute the desired instance I' of the routing schedule problem on binary trees under the $GLC(k)$ model. As remarked above, the paths p_r^e, $3 \leq r \leq 7$, are blockers that make sure that p_1^e and p_2^e are scheduled in the same phase in any schedule with k-phases number equal to 3. A 3-regular graph and part of the resulting instance I' of the message routing schedule problem on binary trees under the $GLC(k)$ model with allowance $k = 1$ are sketched in Figure 3. The vertices of the graph G on the left side of the Figure 3 correspond to the black vertices of the binary tree T on the right side. The dotted edge between the nodes v_1 and v_4 of G corresponds to the seven dotted paths indicated on T. The subtrees rooted at the vertices $v_j^M \in CL(M)$ are shown only for the two relevant vertices.

Now we show that there is a schedule with k-phases number equal to 3 for $\mathcal{P}$ in T if and only if a proper edge-coloring with 3 colors exists for G. Assume that we have a 3-coloring for G. For each edge $e \in E$, we schedule the paths p_1^e and p_2^e in the phase given by the color of the edge e. Since any two paths in each one of the sets $\{p_1^e, p_3^e, p_4^e\}$, $\{p_2^e, p_6^e, p_7^e\}$, $\{p_5^e, p_6^e, p_7^e\}$ and $\{p_3^e, p_4^e, p_5^e\}$ have a k-conflict, then the paths p_r^e, $3 \leq r \leq 7$, are scheduled as follows. The paths p_3^e and p_4^e are scheduled in the two phases that are still available. The path p_5^e is scheduled in the same phase that p_1^e and p_2^e. Finally, the paths p_6^e and p_7^e can be also scheduled in the two phases that are still available. Moreover, let $e = [v_i, v_j]$ be any edge of G, thus by Claim 2, it is clear that only the paths beginning at the leaves $v_{i,m}^M$ (resp. $v_{j,n}^M$) of

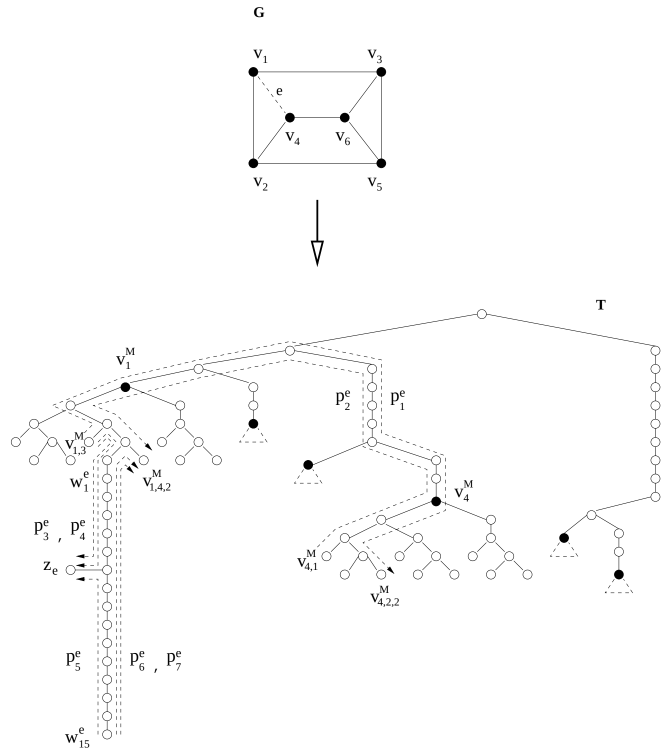

Figure 3. Partial construction of I'.

T, with $m \in \{1,3,5\}$ (resp. $n \in \{1,3,5\}$), of the subtree rooted at v_i^M (resp. v_j^M) have a k-conflict between them, because they share in the 4^{th} position, the arc that joins the vertex v_i^M (resp. v_j^M) to its vertex father, but they have no k-conflict with any other path beginning at the leaves $v_{a,r}^M$ of T, with $r \in \{1,3,5\}$, of the subtree rooted at any vertex v_a^M, with $a \neq i$ (resp. $a \neq j$), which yields a valid scheduled in T with k-phases number of $\mathcal{P}$ equal to 3.
Conversely, assume that there is a schedule in T with k-phases number of $\mathcal{P}$ equal to 3. The blockers make sure that in such a schedule the paths p_1^e and p_2^e in $\mathcal{P}$ (for each edge e of G) are assigned into the same phase of communication. Let $e = [v_i, v_j]$, thus by the transformation of I into I', the paths p_1^f and p_2^f in $\mathcal{P}$ associated to any edge f of G, $f \neq e$, which has as end-vertex also the vertex v_i (resp. v_j) should be assigned in a different phase of communication of the one assigned to the paths p_1^e and p_2^e, because these paths have a k-conflict. Moreover, if f is not adjacent to e, by Claim 2, no any two paths in $\{p_1^f, p_2^f, p_1^e, p_2^e\}$ have a k-conflict. As consequence, if we give to each edge $e \in E$ the color corresponding to the phase assigned to the paths p_1^e and p_2^e, we obtain a proper 3-coloring for the edges of G. We have shown that computing an optimal schedule for $\mathcal{P}$ yields an answer to the NP-complete problem EDGE-COLORING. Since the construction of T and $\mathcal{P}$ can be done in polynomial time, the theorem follows. $\square$

Theorem 2 *The problem of finding an optimal offline schedule on binary trees under the bufferless routing model is NP-hard.*

Proof : (by reduction from EDGE-COLORING).
Consider the instance I' of the routing scheduling problem on binary trees under the $GLC(k)$ model obtained in Theorem 1. Let $k = 2$ and let d be the length of the longest path in I'. We transform the instance I' into an instance I'' of the routing scheduling problem on binary trees under the bufferless routing model in which all paths in it have length equal to d as follows. Let $\mathcal{P}_v$ be the set of paths in I' having as final vertex the vertex v of T. Construct a new directed line and add it to vertex v in such a way that any path $p \in \mathcal{P}_v$ can be extended in order to have a length equal to d. It is clear that the above transformation preserves the 2-conflictness of the paths in I', and therefore in a similar way as in Theorem 1, it is easy to see that there is an schedule of length $d+2$ for the instance I'' if and only if a proper edge-coloring with 3 colors exists for G. □

3. A 2-approximation algorithm for trees

In this section, we give an offline 2-approximation algorithm to compute a message scheduling problem on tree networks under the $GLC(k)$ model, for any fixed value of k, $k \geq 1$. The main ideas of the algorithm uses an approach similar to that used by Ranade, Schleimer and Wilkerson [12] in the context of the bufferless routing model.

Let $\mathcal{P}$ be a collection of paths associated to any set $\mathcal{M}$ of messages to be routed on a tree T under the $GLC(k)$ model.

Definition 4 *A total order $<$ on the paths of $\mathcal{P}$ is* 2-entrant *if it is possible to associate to each path p of $\mathcal{P}$ a set* entrance(p) *of arcs lying on p such that :*
(i) entrance(p) contains at most 2 arcs and
(ii) if $p < p'$ and p and p' have a k-conflict, then some arc in entrance(p') lies on p.

Clearly, any collection of paths on a tree has a total order $<^*$ 2-entrant as follows. Pick some node of the tree T to be the root and for any path $p \in \mathcal{P}$ let its *highest point* be the node on p closest to the root. Again for any path $p \in \mathcal{P}$ let entrance(p) be the two arcs in p adjacent to the *highest* point of p if such two arcs exist, otherwise let entrance(p) be the only one arc in p beginning or ending at the *highest* point of p. For any two paths p and p' in $\mathcal{P}$, say $p <^* p'$ iff the highest point of p is higher than that of p', breaking ties arbitrarily. Thus, by Definition 4, $<^*$ is a total order 2-entrant on any collection of paths on a tree.

Now we present the algorithm which uses a greedy strategy as follows.

Algorithm
1. Choose a total order $<^*$ on the paths of $\mathcal{P}$ as it is showed above. Initialize all paths of $\mathcal{P}$ to have no assigned phase of communication.
2. Consider the paths of $\mathcal{P}$ in increasing order by $<^*$ and assign to each path the smaller possible phase of communication such that there are no k-conflicts with paths that have already been assigned a phase of communication.

Theorem 3 *Let $\mathcal{P}$ be a collection of paths associated to any set of messages to be routed on a tree T under the $GLC(k)$ model, with $k \geq 1$. Let c_k be the k-congestion of $\mathcal{P}$. Using the greedy algorithm above, the k-phases number of $\mathcal{P}$ is at most equal to $2c_k - 1$.*

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Appendix

Proof of Claim 1 :
We prove this claim showing that the level of any vertex $v_j^i \in CL(i)$ of the rooted binary tree T' verifies the following recurrence relation :

$$\begin{cases} N(v_1^i) = i \\ N(v_j^i) = N(v_{j-1}^i) + k2^{M-i+1} \text{ , if } j \geq 2 \end{cases}$$

By the construction of T' in Part I (see Section 2), the above recurrence relation is trivially true for $j = 1$ and for j even. Thus, we only need to show that this recurrence relation is true for any j odd, $j \geq 3$. Let j be an odd integer, $j \geq 3$, and let $j - 1 = 2^{\alpha_1} + 2^{\alpha_2} + \ldots + 2^{\alpha_{p(j-1)}}$, with $\alpha_1 < \alpha_2 < \ldots < \alpha_{p(j-1)}$, where $p(j-1)$ is the number of $1's$ in the binary representation of the even integer $j-1$. Let $v_c^{m-1} \in CL(m-1)$ be the vertex-root of the minimal subtree in T' containing the vertices v_{j-1}^i and v_j^i, $0 \leq m-1 < i$. Let v_a^m and v_{a+1}^m be two consecutive vertices in the ordered-set $CL(m)$ such that v_a^m is the left children of v_c^{m-1}. Thus, by construction of T', it is clear that a is odd and the vertices v_c^{m-1} and v_{a+1}^m are connected by a line of length $k2^{M-m+1}+1$. Again, by construction of T', since $j-1$ is even and equal to $2^{\alpha_1}+2^{\alpha_2}+\ldots+2^{\alpha_{p(j-1)}}$, with $\alpha_1 < \alpha_2 < \ldots < \alpha_{p(j-1)}$, then the value of a is equal to $\frac{j-1}{2^{\alpha_1}}$. Moreover, the vertices v_{a+1}^m and v_j^i are necessarily connected by a line of length α_1. The following relationships can be easily obtained from the construction of T'.
(r_1) $m = i - \alpha_1$
(r_2) $N(v_{j-1}^i) = N(v_a^m) + \sum_{r=1}^{\alpha_1}(k2^{M-(i-\alpha_1+r)+1} + 1)$
(r_3) $N(v_{a+1}^m) = N(v_a^m) + k2^{M-m+1}$
(r_4) $N(v_j^i) = N(v_{a+1}^m) + \alpha_1$

Computing the sum of the right hand of (r_2), we get
(r_5) $N(v_{j-1}^i) = N(v_a^m) + k2^{M-m+1} - k2^{M-i+1} + \alpha_1$,
and replacing (r_3) and (r_5) in (r_4), we get $N(v_j^i) = N(v_{j-1}^i) + k2^{M-i+1}$, which ends the proof. □

Proof of Claim 2 :
Let $v_l^i \longrightarrow \alpha$ and $v_j^i \longrightarrow \beta$ be any two paths on the rooted binary tree T' constructed in Part I of the Theorem 1 (see Section 2) beginning at vertices $v_l^i, v_j^i \in CL(i)$, $1 \leq i \leq M$, $j \neq l$, and ending at any two vertices α and β of T' respectively. Let $v_a^m \in CL(m)$ be the vertex-root of the minimal subtree in T' containing the vertices v_j^i and v_l^i, and let $v_c^q \in CL(q)$ be an end-vertex of the first possible arc shared for the paths $v_l^i \longrightarrow \alpha$ and $v_j^i \longrightarrow \beta$ respectively. Thus, by the construction of T', it is sufficient to consider only the two following cases :
- *case 1* : $v_c^q = v_a^m$ (see Fig. 4.a). By construction of T',

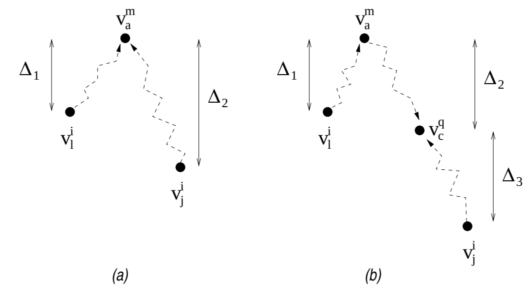

Figure 4. (a) (resp. (b)) Case 1 (resp. 2) of Claim 2.

it is clear that $N(v_a^m) < N(v_j^i)$ and $N(v_a^m) < N(v_l^i)$. Let $\Delta_1 = N(v_l^i) - N(v_a^m)$ and let $\Delta_2 = N(v_j^i) - N(v_a^m)$. Assume that the paths $v_l^i \longrightarrow \alpha$ and $v_j^i \longrightarrow \beta$ have a k-conflict. Thus, assuming w.l.o.g. that $j > l$, the k-conflictness between these two paths implies that $\Delta_2 - \Delta_1 < k$, and by using the Claim 1, it is analogous to the condition $k(j-l)2^{M-i+1} < k$. However, $j - l \geq 1$ and $2^{M-i+1} \geq 2$ by construction of T', and therefore $k(j-l)2^{M-i+1} > k$, which gives a contradiction.
- *case 2* : $v_c^q \neq v_a^m$ (see Fig. 4.b). Again we assume w.l.o.g. that $j > l$. Thus, by construction of T', we have that $i > q > m$, $N(v_a^m) < N(v_l^i)$, $N(v_a^m) < N(v_c^q)$ and $N(v_c^q) < N(v_j^i)$. Let $\Delta_1 = N(v_l^i) - N(v_a^m)$, $\Delta_2 = N(v_c^q) - N(v_a^m)$ and $\Delta_3 = N(v_j^i) - N(v_c^q)$. Assume that the paths $v_l^i \longrightarrow \alpha$ and $v_j^i \longrightarrow \beta$ have a k-conflict. The k-conflictness of these two paths implies that $\Delta_1 + \Delta_2 - \Delta_3 < k$. However, by construction of T', it is easily verify that
$(r1)$ $\Delta_1 \geq i - m$
$(r2)$ $\Delta_2 \geq k2^{M-q+1} + 1$
$(r3)$ $\Delta_3 \leq \sum_{r=1}^{i-q}(k2^{M-(q+r)+1} + 1) = k2^{M-q+1} - k2^{M-i+1} + i - q$

Therefore, by using $(r1)$, $(r2)$ and $(r3)$, we have that $\Delta_1 + \Delta_2 - \Delta_3 \geq q - m + 1 + k2^{M-i+1}$. Moreover, by hypothesis, $q - m + 1 > 1$ and $2^{M-i+1} \geq 2$, which implies that $\Delta_1 + \Delta_2 - \Delta_3 > k$, giving a contradiction to the assumption of k-conflictness and ending the proof. □