On lower bounds for the b-chromatic number of connected bipartite graphs

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Abstract

A b-coloring of a graph G by k colors is a proper k-coloring of the vertices of G such that in each color class there exists a vertex having neighbors in all the other $k-1$ color classes. The b-chromatic number $\chi_b(G)$ of a graph G is the largest integer k such that G admits a b-coloring by k colors. We present some lower bounds for the b-chromatic number of connected bipartite graphs. We also discuss some algorithmic consequences of such lower bounds on some subfamilies of connected bipartite graphs.

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1 Introduction

We consider finite undirected graphs without loops or multiple edges. A *coloring* (i.e. *proper coloring*) of a graph $G = (V, E)$ is an assignment of colors to the vertices of G, such that any two adjacent vertices have different colors. A coloring is called a *b-coloring*, if for each color i there exists a vertex x_i of color i such that for every color $j \neq i$, there exists a vertex y_j of color j adjacent to x_i (such a vertex x_i is called a *dominating* vertex for the color class i). The *b-chromatic number* $\chi_b(G)$ of a graph G is the largest number k such that G has a b-coloring with k colors. The b-chromatic number of a graph was introduced by R.W. Irving and D.F. Manlove [1] when considering minimal proper colorings with respect to a partial order defined on the set of all partitions of the vertices of a graph. They proved that determining $\chi_b(G)$ is NP-hard for general graphs, but polynomial-time solvable for trees. Kratochvil et al. [2] have shown that determining $\chi_b(G)$ is NP-hard even for connected bipartite graphs. Some bounds for the b-chromatic number of a graph are given in [1,3]. Our paper is organized as follows. In the next section we introduce some definitions. In Section 3, we give two lower bounds for the b-chromatic number of connected bipartite graphs. We also discuss some algorithmic consequences of such lower bounds on some subfamilies of connected bipartite graphs.

2 Preliminaries

Let $G = (V, E)$ be a graph and let $W \subseteq V$ be a subset of vertices. The subgraph of G induced by W is denoted by $G[W]$.

Let $K_{p,p}$ denote a complete bipartite graph on $2p$ vertices, that is, a bipartite graph $G = (A \cup B, E)$, where $|A| = |B| = p$ and $E = \{\{x, y\} : x \in A, y \in B\}$. We denote by $K_{p,p}^{-M}$ a complete bipartite graph $K_{p,p}$ without a perfect matching M.

Let $G = (A \cup B, E)$ be a bipartite graph. Let x be a vertex in G. We denote by $N(x)$ the set of neighbors of x, that is, $N(x) = \{y : xy \in E\}$. Moreover, if $x \in A$ (resp. $x \in B$), we denote by $\tilde{N}(x)$ the set of non-neighbors of x in B (resp. in A), that is, $\tilde{N}(x) = \{y : y \in B \text{ and } xy \notin E\}$ (resp. $\tilde{N}(x) = \{y : y \in A \text{ and } xy \notin E\}$).

Let $G = (A \cup B, E)$ be a bipartite graph. Let $A = A_0 \cup A_1$ and let $B = B_0 \cup B_1$, where $A_0 \cap A_1 = B_0 \cap B_1 = \emptyset$. We say that A_1 *dominates* B_0 (resp. B_1 *domi-*

nates A_0) if there exists at least one vertex $x \in A_1$ (resp. $y \in B_1$) such that $B_0 \subseteq N(x)$ (resp. $A_0 \subseteq N(y)$). Finally, we say that an edge $xy \in E$ is a dominating edge in G if $N(x) \cup N(y) = A \cup B$.

The following result is easy to deduce.

Remark 2.1 Let G be a connected bipartite graph. If G has a dominating edge, then $\chi_b(G) = 2$.

So, in the sequel we consider only connected bipartite graphs without dominating edges.

3 Main results

3.1 First lower bound

Theorem 3.1 *Let $G = (A \cup B, E)$ be a connected bipartite graph. If there are subsets $A_0 \subseteq A$ and $B_0 \subseteq B$ such that :*

(c_1) *the induced subgraph $G[A_0 \cup B_0]$ is isomorph to $K_{p,p}^{-M}$ for some positive integer p,*

(c_2) *$A \setminus A_0$ does not dominate B_0 or $B \setminus B_0$ does not dominate A_0,*

then $\chi_b(G) \geq p$.

Proof. Assume $G = (A \cup B, E)$ verifies Conditions (c_1) and (c_2). So, there are subsets $A_0 \subseteq A$ and $B_0 \subseteq B$ such that, by Condition (c_1), $G[A_0 \cup B_0]$ is isomorph to $K_{p,p}^{-M}$ for some positive integer p. Let $A_1 = A \setminus A_0$ and $B_1 = B \setminus B_0$. By Condition (c_2), we have that A_1 and B_1 does not dominate simultaneously B_0 and A_0 respectively. Now, let $A_0 = \{x_1, x_2, \ldots, x_p\}$ and let $B_0 = \{y_1, y_2, \ldots, y_p\}$. We want to construct a b-coloring of G with at least p colors. For this, we assign to vertices x_i and y_i the color i for each $i = 1, 2, \ldots, p$. In order to complete the coloring, we need to consider the following cases :

- Case 1 : B_1 dominates A_0. We color the vertices in B_1 with color $p+1$. By Condition (c_2), we have that A_1 does not dominate B_0 which implies that we can assign to each vertex in A_1 the color of one of its non-neighbor vertices in B_0. Let $v \in B_1$ be a vertex adjacent to all vertices in A_0. Clearly, the previous coloring is a b-coloring of G with $p+1$ colors, being the vertices $x_1, x_2, \ldots, x_p, v$ the dominant vertices for the color classes $1, 2, \ldots, p, p+1$ respectively.

- Case 2 : A_1 dominates B_0. This case is analogous to the previous one.
- Case 3 : A_1 does not dominate B_0 and B_1 does not dominate A_0. We assign to each vertex in A_1 the color $p+1$ and each vertex $v \in B_1$ is colored with the smallest integer $i \in \{1, 2, \ldots, p\}$ such that v has no neighbors in A_0 colored with color i. At this point, the previous coloring is proper but not necessarily it is a b-coloring. Therefore, we consider the following cases :
  * Case 3.1 : the color class $p+1$ has no dominant vertex. This means that each vertex in A_1 misses at least one color in the set $\{1, 2, \ldots, p\}$. Therefore, we can recolor each vertex in A_1 with one of its missing colors in $\{1, \ldots, p\}$, converting the previous coloring into a new coloring using p colors. Notice that after such recoloring, each vertex x_i is a dominant vertex for the color class i, for $i = 1, \ldots, p$, which is a b-coloring of G with p colors.
  * Case 3.2 : the color class $p+1$ has at least one dominant vertex. Consider the following process :

    (a) Let i be the smallest positive integer, with $1 \leq i \leq p$, such that vertex $y_i \in B_0$ has no neighbors in A_1. Notice that if such i does not exist, then the current coloring is a b-coloring with $p+1$ colors. In fact, let $v \in A_1$ be a dominant vertex for the color class $p+1$. Then, the vertices $y_1, y_2, \ldots, y_p, v$ are dominant vertices for the color classes $1, 2, \ldots, p, p+1$ respectively. So, assume that such $i \leq p$ exists. Let $W_i \subseteq B_1$ be the subset of vertices in B_1 colored with color i and such that each one of them has at least one neighbor in the set A_1. Clearly, $|W_i| > 0$ because, there is at least one dominant vertex in A_1 for the color class $p+1$ and thus, it has at least one neighbor in B_1 colored with color i. Now, if there is a vertex $w \in W_i$ such that $A_0 \setminus \{x_i\} \subseteq N(w)$, then we swap vertices y_i and w and we repeat Step (a). Otherwise, we have that :

    (b) Each vertex $w_k \in W_i$ is non-adjacent to at least one vertex of $A_0 \setminus \{x_i\}$, say x_{t_k}. So, recolor w_k with color t_k, for each $w_k \in W_i$. Notice that at this point, no vertex in A_1 has a neighbor colored with color i. The last fact implies that there is no dominant vertex for the color class $p+1$. Therefore, we can recolor each vertex in A_1 with a missing color in the set $\{1, 2, \ldots, p\}$, obtaining in this way, a b-coloring with p colors.

In all cases, we obtain a b-coloring of G with at least p colors. □

3.2 Second lower bound

Definition 3.2 Let $G = (A \cup B, E)$ be a connected bipartite graph. Let $S = (a_1, B_1), \ldots, (a_p, B_p)$ be a sequence of vertices in $A \cup B$, with $a_i \in A$,

$B_i \subset B$, where $a_i \neq a_j$ and $B_i \cap B_j = \emptyset$ whenever $i \neq j$, constructed as follows :

(i) $a_1 \in A$ is such that $|\tilde{N}(a_1)| = \min\{|\tilde{N}(a_i)| : a_i \in A\}$. Set $B_1 := \tilde{N}(a_1)$.

(ii) Assume that we have chosen $(a_1, B_1), \ldots, (a_i, B_i)$. We choose (a_{i+1}, B_{i+1}) as follows :

- $B_j \not\subseteq \tilde{N}(a_{i+1})$, for all $j \leq i$.
- $|\tilde{N}(a_{i+1}) \setminus \cup_{j=1}^{i} B_j|$ is minimum and not equal to zero. Set $B_{i+1} := \tilde{N}(a_{i+1}) \setminus \cup_{j=1}^{i} B_j$.

Then, we say that S is a good-sequence of size p for G.

Theorem 3.3 *Let $G = (A \cup B, E)$ be a connected bipartite graph without dominating edges. Let $(a_1, B_1), \ldots, (a_p, B_p)$ be a maximal good-sequence of size $p \geq 2$ for G. Then, $\chi_b(G) \geq p$.*

Proof. We will construct a b-coloring of G with at least p colors. For this, for each $i = 1, \ldots, p$, we color the vertices in $\{a_i\} \cup B_i$ with color i. Let $A' = A \setminus \{a_1, \ldots, a_p\}$ and $B' = B \setminus \cup_{i=1}^{p} B_i$. Given such a precoloring, we will extend it to the whole graph G as follows. If $B' \neq \emptyset$ then, we color each vertex in B' with color $p+1$. Notice that, by construction, $\{a_1, \ldots, a_p\} \subseteq N(x)$, for all $x \in B'$. In fact, suppose that $x \in B'$ is non-adjacent to some a_i. Then, x should be in B_i, as $x \notin \bigcup_{j=1}^{i-1} B_j$, a contradiction. Before extending such a precoloring to the vertices in A', we will show that vertices a_i are dominating vertices for the color i, with $1 \leq i \leq p$. Clearly, each vertex in B' is a dominating vertex for the color $p+1$. By construction, there exists $x \in B_j$ adjacent to a_i, for all $j < i$, and also, $B \setminus \bigcup_{j=1}^{i} B_j \subseteq N(a_i)$. Therefore, vertex a_i is a dominating vertex for color i. Now, let $a \in A'$. As there is no dominating edge in G, $\tilde{N}(a)$ is not empty. We need to consider the following cases :

- There exists B_i, with $1 \leq i \leq p$, such that $B_i \subseteq \tilde{N}(a)$.
  In such a case, we color vertex a with color i.
- For all i, with $1 \leq i \leq p$, $B_i \not\subseteq \tilde{N}(a)$.
  In such a case, by maximality of the good-sequence, $|\tilde{N}(a) \setminus \cup_{i=1}^{p} B_i| = 0$. Moreover, by hypothesis, $N(a) \cap B_i \neq \emptyset$, for all $1 \leq i \leq p$. Let $j_0 = \min\{j : \tilde{N}(a) \subset \cup_{i=1}^{j} B_j\}$. Clearly, $1 \leq j_0 \leq p$. However, $|\tilde{N}(a) \setminus \cup_{i=1}^{j_0-1} B_i| \neq 0$ and $\tilde{N}(a) \cap B_{j_0} \neq \emptyset$. Therefore, $|\tilde{N}(a) \setminus \cup_{i=1}^{j_0-1} B_i| < |\tilde{N}(a_{j_0}) \setminus \cup_{i=1}^{j_0-1} B_i|$, which is a contradiction with the choice of a_{j_0} instead of a in the construction. So, this case there does not exist.

As all the cases have been considered, we have that G admits a b-coloring with at least p colors. □

The following results are direct consequences of Theorem 3.3.

Corollary 3.4 *Let $G = (A \cup B, E)$ be a connected d-regular bipartite graph, with $|A| = |B| = n$ and $d < n$. If $n - d$ is equal to a constant $c \geq 1$ then, there is a c-approximation algorithm for b-coloring G with the maximum number of colors.*

Proof. Let $(a_1, B_1), \ldots, (a_p, B_p)$ be a maximal good-sequence of G constructed as in Definition 3.2. Notice that $|B_1| = n - d$ and $|B_i| \leq n - d$ for $i \in \{2, \ldots, p\}$. Therefore, $p \geq \frac{n}{n-d} = \frac{n}{c} > \frac{d}{c}$. By using Theorem 3.3, we know that we can construct in polynomial time a b-coloring of G with at least p colors. Therefore, $\chi_b(G) \geq p \geq \frac{d+c}{c} \geq \frac{d+1}{c}$. Indeed, it is easy to deduce that $\chi_b(G) \leq d + 1$, which proves the result. □

Corollary 3.5 *Let $G = (A \cup B, E)$ be a connected bipartite graph, with $|A| = |B| = n$. Let δ (resp. Δ) be the minimum (resp. maximum) degree of G, with $\delta \leq \Delta < n$ and $n - \delta$ equal to a constant $c \geq 1$. Then, there is a c-approximation algorithm for b-coloring G with the maximum number of colors.*

Proof. Let $(a_1, B_1), \ldots, (a_p, B_p)$ be a maximal good-sequence of G constructed as in Definition 3.2. Clearly, $|B_1| = n - \Delta$ and $|B_i| \leq n - \Delta$ for $i \in \{2, \ldots, p\}$. Therefore, $p \geq \frac{n}{n-\Delta} \geq \frac{n}{n-\delta} = \frac{n}{c}$. Indeed, as $\Delta \leq n - 1$ and $\chi_b(G) \leq \Delta + 1 \leq n$ then, by using Theorem 3.3, the result holds. □

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