On the chromatic polynomial of some graphs *

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Abstract. In this note, we compute the chromatic polynomial of some circulant graphs via elementary combinatorial techniques.

Key words. Chromatic polynomial, circulant graphs, complement graphs.

1. Introduction

The *chromatic polynomial* $P(G,\lambda)$ gives the number of ways of coloring a graph G when λ colors are used. The most recent results (see [1,2] and ref.) use algebraic methods to compute chromatic polynomials for some class of graphs called *bracelets*. In this note, we compute the chromatic polynomial for some circulant graphs using pure and elementary combinatorial techniques, avoiding the deletion/contraction method.

Let $\chi(G)$ denote the *chromatic number* of a graph G, and let $P_{n,\lambda} = P(K_n,\lambda)$ denote the chromatic polynomial of the complete graph K_n, that is, $P_{n,\lambda} = \lambda(\lambda-1)(\lambda-2)\dots(\lambda-n+1)$, in particular, $P_{n,n+i} = \frac{(n+i)!}{i!}$ for any integer $i \geq 0$, and $P_{0,\lambda} = 1$. Notice that the chromatic polynomial $P(G,\lambda)$ of a graph G on n vertices can be expressed in the *complete graph basis*, i.e., $P(G,\lambda) = \sum_{k=0}^{n} C(G,k)P_{n-k,\lambda}$, where $C(G,k)$ is the number of *color partitions* of the vertices of G (i.e. partitions of the vertices of G induced by proper colorings) into exactly $n-k$ non-empty indistinguishable classes. Given a graph G on n vertices, a matching in G is a set of pairwise non-adjacent edges of G. For any integer $k \geq 0$ we denote by $M(G,k)$ the number of different matchings of size k in G. In particular, $M(G,0) = 1$. The *matching polynomial* $\mu(G,\lambda)$ of G is defined by $\mu(G,\lambda) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k M(G,k)\lambda^{n-2k}$ (see [3,6] and ref.). Let p_n and c_n be the simple path graph and the cycle graph with n edges respectively. The following lemma is a well known result (see [3]).

Lemma 1. *Let n be a positive integer. Then,*

(1) $M(p_n,k) = \binom{n-k+1}{k}$ *if* $1 \leq k \leq \lceil \frac{n}{2} \rceil$. *Otherwise,* $M(p_n,k) = 0$.
(2) $M(c_n,k) = \left(\frac{n}{n-k}\right)\binom{n-k}{k}$ *if* $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$. *Otherwise,* $M(c_n,k) = 0$.

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Farrell and Whitehead shown in [4] a connection between the matching and chromatic polynomials of a graph. The results given in [4] generalize previous ones given by Frucht and Giudici in [5]. In particular, it is shown in [4] the following result.

Lemma 2. *(Corollary 1.1 in [4]) Let G be a graph and let $\bar{G}$ denote its complement graph. Then, $M(G,k) = C(\bar{G},k)$, for any non-negative integer k, if and only if G is triangle-free.*

The following result is a direct consequence of the Lemma 2.

Corollary 1. *Let G be a graph on n vertices such that $\bar{G}$ is triangle-free. Then, $\chi(G) = n - t$ where t is the maximum cardinality of a matching in $\bar{G}$.*

As an example of application of Lemma 2, by using Lemma 1, we can compute the chromatic polynomials for the graphs $K_n \setminus p_i$ and $K_n \setminus c_i$, the graphs obtained from K_n deleting the edges of a path p_i or a cycle c_i respectively, with $i \leq n$.

Lemma 3. *Let n, i be positive integers with $i \leq n$. Then,*

(1) $P(K_n \setminus p_i, \lambda) = \displaystyle\sum_{j=0}^{\lceil \frac{i}{2} \rceil} \binom{i-j+1}{j} P_{n-j,\lambda}.$

(2) $P(K_n \setminus c_i, \lambda) = \displaystyle\sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} \frac{i}{i-j} \binom{i-j}{j} P_{n-j,\lambda}$, *for* $i > 3$.

Our first result is a formula to compute the coefficients of the matching polynomial of a triangle-free graph from the chromatic polynomial of its complement graph as follows.

Theorem 1. *Let G be a triangle-free graph on n vertices. Let $\chi = \chi(\bar{G})$. Then,*

$$M(G,k) = \sum_{j=0}^{n-\chi-k} \frac{P(\bar{G}, n-k-j)}{j!(n-k-j)!}(-1)^j, \text{ where } 0 \leq k \leq n-\chi.$$

Let $G = (A \cup B, E)$ be a bipartite graph and let $\bar{G}$ its complement graph. Thus, $\bar{G}$ can be formed by two disjoint complete graphs that we denote by K_A and K_B respectively, joined by the set of edges $E' = \{\{a,b\} : a \in A, b \in B, \{a,b\} \notin E\}$, and let $G' = (A \cup B, E')$. Also let $|A| = n_1$ and $|B| = n_2$ and assume that $n_1 \leq n_2$.

Theorem 2. $P(\bar{G}, \lambda) = \sum_{k=0}^{n_1} (-1)^k M(G', k) P_{n_1,\lambda} P_{n_2-k,\lambda-k}$.

Notice that as G is a bipartite graph, by using Lemma 2, we can compute the chromatic polynomial of $\bar{G}$ via the matchings of G. However, if G has so many edges, the result given in Theorem 2 is a more efficient method to compute the chromatic polynomial of $\bar{G}$.

Finally, we start the study of the chromatic polynomial of circular graphs.

2. Main results

Proof of Theorem 1: Let i be an integer such that $0 \leq i \leq n - \chi$. We will prove by induction on i that $M(G, n-(\chi+i)) = \sum_{j=0}^{i} \frac{P(\bar{G},\chi+i-j)}{j!(\chi+i-j)!}(-1)^j$. By Lemma 2 and Corollary 1, we know that $P(\bar{G}, \lambda) = \sum_{k=0}^{n-\chi} M(G,k) P_{n-k,\lambda}$. Thus, $P(\bar{G}, \chi) = M(G, n-\chi) P_{\chi,\chi}$ and so, $M(G, n-\chi) = \frac{P(\bar{G},\chi)}{\chi!}$ which proves the basis case. Now, let $i \geq 0$. We assume that for any $j \leq i$ the theorem holds. By Lemma 2 and Corollary 1 we have that

$$\begin{aligned} P(\bar{G}, \chi+i+1) &= \sum_{j=0}^{i+1} M(G, n-(\chi+i+1-j)) P_{\chi+i+1-j,\chi+i+1} \\ &= (\chi+i+1)! \sum_{j=0}^{i+1} \frac{M(G, n-(\chi+i+1-j))}{j!}, \end{aligned}$$

and therefore,

$$M(G, n-(\chi+i+1)) = \frac{P(\bar{G}, \chi+i+1)}{(\chi+i+1)!} - \sum_{j=1}^{i+1} \frac{M(G, n-(\chi+i+1-j))}{j!}.$$

Then, by induction hypothesis, we have

$$\begin{aligned} M(G, n-(\chi+i+1)) &= \frac{P(\bar{G}, \chi+i+1)}{0!(\chi+i+1)!} - \sum_{s=0}^{i} \left[\frac{1}{(s+1)!} \sum_{k=0}^{i-s} \frac{P(\bar{G}, \chi+i-k-s)}{k!(\chi+i-k-s)!} (-1)^k \right] \\ &= \frac{P(\bar{G}, \chi+i+1)}{0!(\chi+i+1)!} - \sum_{j=0}^{i} \left[(-1)^j \frac{P(\bar{G}, \chi+i-j)}{(\chi+i-j)!} \sum_{s=0}^{j} (-1)^s \frac{1}{(s+1)!(j-s)!} \right] \\ &= \frac{P(\bar{G}, \chi+i+1)}{0!(\chi+i+1)!} + \sum_{j=1}^{i+1} \left[(-1)^j \frac{P(\bar{G}, \chi+i+1-j)}{(\chi+i+1-j)!} \sum_{s=1}^{j} (-1)^{s-1} \frac{1}{s!(j-s)!} \right]. \end{aligned}$$

For any positive integer j we have that,

$$\sum_{s=1}^{j} (-1)^{s-1} \frac{1}{s!(j-s)!} = \frac{1}{j!} \sum_{s=1}^{j} (-1)^{s-1} \binom{j}{s} = \frac{1}{j!},$$

and thus,

$$M(G, n-(\chi+i+1)) = \sum_{j=0}^{i+1} \frac{P(\bar{G}, \chi+i+1-j)}{j!(\chi+i+1-j)!} (-1)^j.$$

□

Proof of Theorem 2: To compute the chromatic polynomial of $\bar{G}$ we use an inclusion-exclusion technique. We begin by computing N_k, the number of (non necessarily proper) colorings of $\bar{G}$ where K_A and K_B are properly colored and there are at least k monochromatic edges in E'. Fix one of such colorings, by assumption any monochromatic edge belongs to E' and there are not two adjacent ones, otherwise there will be two vertices in one of the complete graphs having the same color. Therefore the set of monochromatic edges correspond to a matching in G'. On the other hand, given any matching M in G' of size k we can produce a coloring for $\bar{G}$ were all the edges in M are monochromatic and there is not monochromatic edges in each one of the graphs K_A and K_B. For this, just do a proper coloring of K_A, fix the color of the k endpoints of M in K_B, and next color the remaining $n_2 - k$ vertices in B using the remaining $\lambda - k$ colors. From this argument we get $N_k = M(G', k)P_{n_1,\lambda}P_{n_2-k,\lambda-k}$. □

As an example we consider the Cayley graph for the cyclic group $\mathbb{Z}_{2n}$ with connector set $\{\pm 1, \pm 2, \pm 4, \pm 6, \ldots, \pm 2(n-1)\}$ where n is a positive integer. Let G_{2n} denote such a Cayley graph.

Corollary 2. $P(G_{2n}, \lambda) = P_{n,\lambda} \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} P_{n-k,\lambda-k}$.

Proof : Notice that, by construction, the graph G_{2n} can be formed from two disjoint copies of K_n (one induced by the odd integers and the other one induced by the even integers in $\{0, 1, \ldots, 2n-1\}$) joined by the edges $\{i, i+1\}$ for $0 \le i \le 2n-1$. Notice that the complement of G_{2n} is $K_{n,n} \setminus E(c_{2n})$, the complete (n, n)-bipartite graph without the edges of a cycle of length $2n$. Applying Theorem 2 and Lemma 1 the result follows. □

Let n, a be positive integers such that $n \ge 2a$. The *circular graph* C^n_a is the Cayley graph for the cycle group $\mathbb{Z}_n$ with connector set $\{a, a+1, \ldots, n-a\}$. These graphs play an important role in the definition of the star chromatic number defined by Vince in [7]. It is well known that $\chi(C^n_a) = \lceil \frac{n}{a} \rceil$. Moreover, if $a > 1$ then, the complement graph of C^n_a is the $(a-1)$-th power of a cycle C_n, that is, a Cayley graph for the cycle group $\mathbb{Z}_n$ with connector set $\{\pm 1, \pm 2, \ldots, \pm(a-1)\}$, which we denote by $C(n, a-1)$. Figure 1 shows an example of a circular graph and its complement graph.

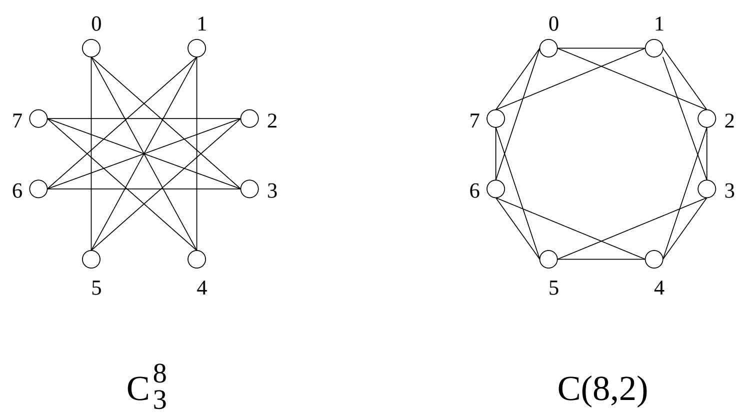

Fig. 1. Circular graph C^8_3 and its complement graph $C(8, 2)$.

First, note that the graph C^n_2 is isomorphic to a complete graph K_n without the edges of a Hamiltonian cycle. Therefore, the following result is a consequence of the part (2) of Lemma 3.

Corollary 3. $P(C^n_2, \lambda) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i} \binom{n-i}{i} P_{n-i,\lambda}$.

In order to compute the chromatic polynomial for the graph C^n_3, we first generalize Lemma 1 as follows.

Definition 1. *Let G be a graph. We denote by $\mathbb{M}^G(m_1, m_2, \ldots, m_k)$ the number of ways of choosing k pairwise disjoint subsets of vertices from $V(G)$, in such a way that for $1 \le i \le k$, the ith subset of vertices contains m_i disjoint paths each one with i edges.*

Lemma 4. $\mathbb{M}^{p_n}(m_1, m_2, \ldots, m_k) = \binom{n+1-\sum_{i=1}^{k} i.m_i}{m_1, m_2, \ldots, m_k}$.

Proof : The result follows from the fact that $\mathbb{M}^{p_n}(m_1, m_2, \ldots, m_k)$ is equal to number of ways of choosing k pairwise disjoint sets of vertices from a path of size $n+1-\sum_{i=1}^{k} i.m_i$, where the ith subset has size m_i. To prove this fact, fix a selection $A_1, \ldots, A_k$ of pairwise disjoint subsets of $V(p_n)$, such that for each $1 \leq i \leq k$, A_i contains m_i disjoint paths each one with i edges. Now let B the path obtained by contracting the vertices in each of the elements in each of the A_i's, and let B_i be the set of vertices of B corresponding to the contractions of the elements of A_i. The B_i's correspond to a selection of k pairwise disjoint sets of vertices from B, with $|B_i| = m_i$. It is straight forward to see that this procedure corresponds to a bijection. $\square$

Lemma 5. $\mathbb{M}^{c_n}(m_1, m_2, \ldots, m_k) = \frac{n}{n-\sum_{i=1}^{k} i.m_i}.\mathbb{M}^{p_{n-1}}(m_1, m_2, \ldots, m_k)$.

Proof : We are going to prove

$$\left(n - \sum_{i=1}^{k} i.m_i\right) \mathbb{M}^{c_n}(m_1, m_2, \ldots, m_k) = n\mathbb{M}^{p_{n-1}}(m_1, m_2, \ldots, m_k),$$

by a counting argument. For this we consider the number of ways of choosing k pairwise disjoint subsets of vertices $A_1, \ldots, A_k$ and an edge e from c_n in such a way that for $1 \leq i \leq k$, the A_i contains m_i disjoint paths each one with i edges all different from e. We count this number in two ways. First we choose e and then the A_i's (in the path $c_n \setminus e$). There are $n\mathbb{M}^{p_{n-1}}(m_1, m_2, \ldots, m_k)$ ways to do this. Second we first choose the A_i's and then choose e between the edges no appearing in the A_i's. There are $\mathbb{M}^{c_n}(m_1, m_2, \ldots, m_k)\left(n - \sum_{i=1}^{k} i.m_i\right)$ ways to do this. $\square$

Now, we are ready to compute the chromatic polynomial of C_3^n.

Theorem 3. *Let $n \geq 7$,*

$$P(C_3^n, \lambda) = \sum_{k=0}^{n} \sum_{(n_1,n_2,n_3,n_4)\in\Gamma_k} \mathbb{M}^{c_n}(n_1, n_2+n_3, n_4)\binom{n_2+n_3}{n_2} P_{n-k+n_3,\lambda},$$

where $\Gamma_k = \{(n_1, n_2, n_3, n_4) \in \mathbb{N}^4 : n_1 + n_2 + 3n_3 + 2n_4 = k\}$.

Proof : First, we define the *exterior cycle* of $C(n, 2)$ as the set of edges of the form $\{i, i+1\}$, $0 \leq i < n$. We also define the *interior* of $C(n, 2)$ as the set of edges in $C(n, 2)$ that are not in the exterior cycle. Let $k \geq 0$ be an integer and consider the proper colorings of C_3^n with λ colors for which there are exactly k monochromatic edges in the complement graph $C(n, 2)$. We are interested in counting these proper colorings. For this purpose, we use the following property of the graph $C(n, 2)$:

Property 1. The clique number of $C(n, 2)$ is 3 and all triangles in $C(n, 2)$ have two edges in the exterior cycle.

Property 1 implies that any path of monochromatic edges in $C(n, 2)$ of length greater than 1 can be completed to a triangle, with two edges in the exterior cycle. In particular any path of monochromatic edges in the interior has length 1.

Now, we define *the projection* of an edge $e \in C(n,2)$ into the exterior cycle; if e is in the exterior cycle, the projection of e is e. If e is in the interior, the projection of e is the path formed by the two consecutive edges in the exterior, which are adjacent to e.

Fix a proper coloring of C_3^n with k monochromatic edges in $C(n,2)$. Let P be the union of all the projections of these monochromatic edges into C, the exterior cycle of $C(n,2)$. By Property 1, P can not have paths of length greater than 3, thus, P is the union of disjoint paths in C of length less than 4. For $i = 1,2,3$ let A_i the set of such paths of length i. Again, by Property 1 we have,

- All the elements of A_1 are the projection of an isolated monochromatic edge in the exterior.
- All the elements of A_2 correspond to either the projection of an isolated interior edge or the projection of a triangle (see Figure 2a).
- All the element of A_3 correspond to the projection of two crossing non-consecutive interior edges (see Figure 2b) .

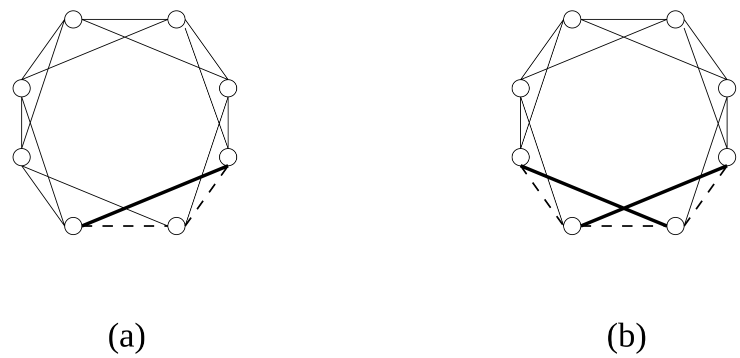

Fig. 2. (a) Projection of an isolated interior edge of $C(n,2)$. (b) Projection of a pair of crossing interior edges of $C(n,2)$.

Thus, if n_1 is the number of isolated monochromatic edges belonging to the exterior cycle, n_2 is the number of isolated monochromatic interior edges, n_3 is the number of monochromatic triangles, and n_4 is the number of pairs of crossing monochromatic interior edges, then $|A_1| = n_1$, $|A_2| = n_2 + n_3$, $|A_3| = n_4$ and $n_1 + n_2 + 3n_3 + 2n_4 = k$.

Therefore, the number of ways of choosing k monochromatic edges in $C(n,2)$ derived from proper colorings of C_3^n is equal to

$$\sum_{(n_1,n_2,n_3,n_4)\in\Gamma_k} \mathbb{M}^{c_n}(n_1, n_2+n_3, n_4)\binom{n_2+n_3}{n_2},$$

where the binomial term $\binom{n_2+n_3}{n_2}$ is used to distinguish a monochromatic triangle from an isolated interior monochromatic edge. Therefore, the total number of proper colorings of C_3^n with λ colors is equal to $\sum_{k=0}^{n}\sum_{(n_1,n_2,n_3,n_4)\in\Gamma_k} \mathbb{M}^{c_n}(n_1, n_2+n_3, n_4)\binom{n_2+n_3}{n_2}P_{n-k+n_3,\lambda}$. $\square$

An interesting problem is to compute the chromatic polynomial of circular graphs C_a^n for $a > 3$.

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References

1. N. L. Biggs. *A matrix method for chromatic polynomials*, J. Combin. Theory Ser. B 82 (2001) 19-29.

2. N. L. Biggs, M. H. Klin, P. Reinfeld. *Algebraic methods for chromatic polynomials*, Europ. J. Combinatorics, 25 (2004) 147-160.

3. E. J. Farrell. *An introduction to matching polynomials*, J. Combin. Theory Ser. B 27 (1979) 75-86.

4. E. J. Farrell, E. G. Whitehead. *Connections between the matching and chromatic polynomials*, Internat. J. Math. and Math. Sci., Vol. 12, No. 4 (1992) 757-766.

5. R. W. Frucht, R. E. Giudici. *A note on the matching numbers of triangle-free graphs*, J. Graph Theory, 9 (1985) 455-458.

6. C. D. Godsil, I. Gutman. *On the theory of the matching polynomial*, J. Graph Theory, 5 (1981) 137-144.

7. A. Vince. *Star chromatic number*, J. Graph Theory, 12 (1988) 551-559.