

On the bend number of circular-arc graphs as edge intersection graphs of paths on a grid

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**Dedicated to Martin Charles Golumbic
on the occasion of his 65th birthday**

Abstract

Golumbic, Lipshteyn and Stern [12] proved that every graph can be represented as the edge intersection graph of paths on a grid (EPG graph), i.e., one can associate with each vertex of the graph a nontrivial path on a rectangular grid such that two vertices are adjacent if and only if the corresponding paths share at least one edge of the grid. For a nonnegative integer k , B_k -EPG graphs are defined as EPG graphs admitting a model in which each path has at most k bends. Circular-arc graphs are intersection graphs of open arcs of a circle. It is easy to see that every circular-arc graph is a B_4 -EPG graph, by embedding the circle into a rectangle of the grid. In this paper, we prove that every circular-arc graph is B_3 -EPG, and that there exist circular-arc graphs which are not B_2 -EPG. If we restrict ourselves to rectangular representations (i.e., the union of the paths used in the model is contained in a rectangle

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of the grid), we obtain EPR (edge intersection of path in a rectangle) representations. We may define B_k -EPR graphs, $k \geq 0$, the same way as B_k -EPG graphs. Circular-arc graphs are clearly B_4 -EPR graphs and we will show that there exist circular-arc graphs that are not B_3 -EPR graphs. We also show that normal circular-arc graphs are B_2 -EPR graphs and that there exist normal circular-arc graphs that are not B_1 -EPR graphs. Finally, we characterize B_1 -EPR graphs by a family of minimal forbidden induced subgraphs, and show that they form a subclass of normal Helly circular-arc graphs.

Keywords. edge intersection graphs, paths on a grid, forbidden induced subgraphs, (normal, Helly) circular-arc graphs, powers of cycles.

1 Introduction

Let \mathcal{G} be a rectangular grid of size $(\ell + 1) \times (\ell + 1)$. The horizontal grid lines will be referred to as *rows* and denoted by x_0, x_1, \dots, x_ℓ , and the vertical grid lines will be referred to as *columns* and denoted by y_0, y_1, \dots, y_ℓ . A grid point lying on row x and column y is referred to as (x, y) . Let \mathcal{P} be a collection of nontrivial simple paths on \mathcal{G} . The edge intersection graph of \mathcal{P} (denoted by $\text{EPG}(\mathcal{P})$) is the graph whose vertices correspond to the paths of \mathcal{P} and two vertices are adjacent in $\text{EPG}(\mathcal{P})$ if and only if the corresponding paths in \mathcal{P} share at least one edge in \mathcal{G} . A graph G is called an *edge intersection graph of paths on a grid (EPG graph)* if $G = \text{EPG}(\mathcal{P})$ for some \mathcal{P} . Every graph G satisfies $G = \text{EPG}(\mathcal{P})$ for some \mathcal{P} on a large enough grid and allowing an arbitrary number of *bends* (turns on a grid point) for each path [12]. In recent years, the subclasses for which the number of bends of each path is bounded by some integer $k \geq 0$, known as B_k -EPG graphs, were widely studied [2, 3, 4, 8, 12, 13, 14]. The *bend number* of a graph G (resp. a graph class \mathcal{H}), is the smallest integer $k \geq 0$ such that G (resp. every graph in \mathcal{H}) is a B_k -EPG graph. We denote by B_k -EPG, $k \geq 0$, the class of B_k -EPG graphs.

In [13], it was shown that for every integer $k \geq 0$ there exists a graph with bend number k , and that recognizing B_1 -EPG graphs is NP-complete. The bend number of classical graph classes was investigated as well. In [14], it was shown that outerplanar graphs are B_2 -EPG graphs and that planar graphs are B_4 -EPG graphs. For planar graphs, it is still an open question whether their bend number is equal to 3 or 4. On the other hand, it is easy to see that B_0 -EPG graphs exactly correspond to interval graphs (i.e., intersection graphs of intervals on a line) [12]. A generalization of interval graphs are circular-arc graphs (i.e., intersection graphs of open arcs on a circle). It is natural to see circular-arc graphs as EPG graphs by identifying the circle with a rectangle of the grid. Hence, circular-arc graphs form a subclass of B_4 -EPG graphs. This leads to some natural questions. For example, the bend number of circular-arc graphs or the characterization of circular-arc graphs that are B_k -EPG graphs, for some $k < 4$. One of the main results of this paper is that the bend number of circular-arc graphs is 3.

Another interesting question is how many bends per path are needed for a circular-arc graph to be represented in a rectangle of the grid, i.e., in such a way that the union of

the paths is contained in a rectangle of the grid. We call such graphs *edge intersection graphs of paths on a rectangle (EPR graphs)*. It is easy to see that EPR graphs are exactly the circular-arc graphs. We will study the classes B_k -EPR, for $0 \leq k \leq 4$, in which the paths on the grid that represent the vertices of the graph have at most k bends. As before, we denote by B_k -EPR, $k \geq 0$, the class of B_k -EPR graphs. Similar to the case of EPG graphs, one can define for a circular-arc graph G the bend number with respect to an EPR representation as the smallest integer k such that G is a B_k -EPR graph. Notice that $CA = EPR = B_4$ -EPR. We strengthen this observation by showing that the bend number for circular-arc graphs with respect to EPR representations is 4. Furthermore, we focus on B_1 -EPR graphs and B_2 -EPR graphs (B_0 -EPR graphs correspond again to interval graphs), and relate these classes with the class of normal Helly circular-arc graphs. In summary, we obtain the following results: we prove that the bend number of normal circular-arc graphs with respect to EPR representations is 2; moreover, we characterize B_1 -EPR graphs by a family of minimal forbidden induced subgraphs, and show that they are exactly the normal Helly circular-arc graphs containing no powers of cycles C_{4k-1}^k , with $k \geq 2$, as induced subgraphs.

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2 Preliminaries

All graphs that we consider in this paper are connected, finite and simple. For all graph theoretical terms and notations not defined here, we refer the reader to [5].

We denote by C_n , $n \geq 3$, the chordless cycle on n vertices. A graph is called *chordal*, if it does not contain any chordless cycle of length at least four. Given a graph G and an integer $k \geq 0$, the *power graph* G^k has the same vertex set as G , two vertices being adjacent in G^k if their distance in G is at most k .

Let $G = (V, E)$ be a graph and let $X \subseteq V$. We denote by $G - X$ the graph obtained from G by deleting all vertices in X .

A *clique* (resp. a *stable set*) is a subset of vertices that are pairwise adjacent (resp. nonadjacent). A *thick spider* S_n is the graph whose $2n$ vertices can be partitioned into a clique $K = \{c_1, \dots, c_n\}$ and a stable set $S = \{s_1, \dots, s_n\}$ such that, for $1 \leq i, j, \leq n$, c_i is adjacent to s_j if and only if $i \neq j$. Notice that S_{n_1} is an induced subgraph of S_{n_2} if $n_1 \leq n_2$.

We say that a vertex v *dominates* a vertex w if they are adjacent and every neighbor of w is also a neighbor of v .

Let G be a circular-arc graph (*CA graph*). If \mathcal{C} denotes the corresponding circle and \mathcal{A} the corresponding set of open arcs, then $(\mathcal{A}, \mathcal{C})$ is called a *circular-arc model* of G [17]. A graph G is a *Helly circular-arc graph (HCA graph)* if it is a circular-arc graph having a circular-arc model such that any subset of pairwise intersecting arcs has a common point on the circle [10]. A circular-arc graph having a circular-arc model without two arcs covering the whole circle is called a *normal circular-arc graph (NCA graph)*. Circular-arc models

that are at the same time normal and Helly are precisely those without three or less arcs covering the whole circle. A graph that admits such a model is called a *normal Helly circular-arc graph* (*NHCA graph*) [15]. We will denote by NCA (resp. NHCA) the class of normal (resp. normal Helly) circular-arc graphs.

In [6], the authors present a characterization of NHCA graphs by a family of minimal forbidden induced subgraphs. Recent surveys on circular-arc graphs are given in [7, 16]. A very recent characterization of circular-arc graphs by forbidden structures is presented in [9].

3 Circular-arc graphs as EPG graphs

In this section, we show that the bend number of circular-arc graphs with respect to EPG representations is equal to 3. We start with the following result.

Theorem 1 *Every circular-arc graph is a B_3 -EPG graph.*

Proof. Let G be a circular-arc graph and let $(\mathcal{A}, \mathcal{C})$ be a circular-arc model of G . Without loss of generality, we may assume that the endpoints of the arcs are all distinct and we can number them clockwise in the circle from 1 to $2n$ (with n being the number of vertices of G). We also define a point 0 in the circle between $2n$ and 1 (clockwise). The arc (a, b) , $1 \leq a, b \leq 2n$, denotes the arc traversing the circle clockwise from endpoint a to endpoint b . In particular, an arc (a, b) contains point 0 of \mathcal{C} if and only if $a > b$. Let X be the set of vertices in G corresponding to arcs containing point 0 of \mathcal{C} . Clearly, these vertices form a clique in G . Moreover, $G - X$ is an interval graph that can be represented on a line by taking, for each vertex, the interval (a, b) defined by the endpoints of its corresponding arc, since $a < b$ for vertices in $G - X$. We will construct the following EPG representation of G on a grid. For each vertex in $G - X$ corresponding to an arc (a, b) , assign the 3-bends path on the grid whose endpoints are (x_0, y_b) and (x_b, y_0) and whose bend points correspond to the grid points (x_0, y_a) , (x_a, y_a) , (x_a, y_0) . For each vertex of X corresponding to an arc (a, b) (in this case $a > b$), assign the 3-bends path on the grid whose endpoints are (x_0, y_0) and (x_{2n+1}, y_0) and whose bend points correspond to the grid points (x_0, y_b) , (x_a, y_b) , (x_a, y_0) . Since all the endpoints of the arcs in \mathcal{A} are different, the edge intersections of the paths are either on row x_0 or on column y_0 of the grid. Clearly, two paths corresponding to vertices of $G - X$ intersect if and only if the corresponding arcs intersect on \mathcal{C} . Two paths corresponding to vertices of X intersect at least on the edge of the grid going from $(0, 0)$ to $(0, 1)$. The path corresponding to a vertex in $G - X$ with endpoints (a, b) and the path corresponding to a vertex in X with endpoints (c, d) intersect if and only if either $d > a$ or $c < b$, and the same condition holds for the corresponding arcs in \mathcal{C} . Thus, we obtain a representation of G as a B_3 -EPG graph. \square

Combining Theorem 1 with the following result shows that the bend number of circular-arc graphs with respect to EPG representations is 3.

Proposition 2 *The thick spider S_{40} is in B_3 -EPG \setminus B_2 -EPG.*

Proof. Since all thick spiders are circular-arc graphs, it follows from Theorem 1 that S_{40} is a B_3 -EPG graph. Suppose there exists a B_2 -EPG representation of S_{40} . Let us consider the path \mathcal{P}_c corresponding to a vertex c of the clique and the paths corresponding to its 39 neighbors in the stable set S . The path \mathcal{P}_c uses at most three lines (rows and/or columns) of the grid since it has at most 2 bends. Thus, \mathcal{P}_c intersects at least 13 paths, $\mathcal{P}_1, \dots, \mathcal{P}_{13}$, corresponding to 13 of its neighbors in S on a same line x . Without loss of generality, we may assume that x is a row of the grid. Notice that, since the paths have at most 2 bends, the edges of each path on a same row or column form a connected subpath. Consider now the 13 connected subpaths on x corresponding to the paths $\mathcal{P}_1, \dots, \mathcal{P}_{13}$ that \mathcal{P}_c intersects on row x . Since these paths correspond to vertices of S , they are edge-disjoint and thus their subpaths on x can be ordered. We may assume then that P_1, \dots, P_{13} , the subpaths of $\mathcal{P}_1, \dots, \mathcal{P}_{13}$ on row x , are ordered by index from left to right. Let $s_j, j \in \{1, \dots, 39\}$, be the vertex in S corresponding to the path \mathcal{P}_7 , i.e., $\mathcal{P}_{s_j} = \mathcal{P}_7$. The path \mathcal{P}_{c_j} , corresponding to vertex c_j of the clique that is not adjacent to s_j , cannot intersect the subpaths corresponding to P_1, \dots, P_{13} on both sides of P_7 on x since it has at most two bends. Thus, it intersects at least 6 of them on some other row or column. So we may assume, without loss of generality, that it intersects the paths $\mathcal{P}_8, \dots, \mathcal{P}_{13}$ on some other row or column. But since these paths have at most two bends, are edge-disjoint and all use row x , it follows that \mathcal{P}_{c_j} intersects at most 2 paths among $\mathcal{P}_8, \dots, \mathcal{P}_{13}$ on a column. Therefore \mathcal{P}_{c_j} necessarily uses two rows and one column, and it intersects at least 4 paths among $\mathcal{P}_8, \dots, \mathcal{P}_{13}$ on a row $x' \neq x$. Since these 4 paths use both rows x, x' and they are edge disjoint, it follows that each of them uses a different column. Also notice that the order of their corresponding subpaths on the two rows x, x' from left to right must be the same, since they have at most 2 bends and therefore they cannot swap the order without intersecting. Let $s_k, k \in \{1, \dots, 39\}$, be the vertex corresponding to one of the two paths of these four whose subpaths on row x' are located in the middle (i.e., the second or third subpath). It is now easy to see that, given that fixed configuration for the four paths, it is impossible for the path \mathcal{P}_{c_k} , corresponding to vertex c_k of the clique that is not adjacent to s_k , to avoid the path \mathcal{P}_{s_k} while intersecting the paths corresponding to the remaining three vertices using only two bends. Thus S_{40} does not admit a B_2 -EPG representation. \square

Finding a characterization of $CA \cap B_2$ -EPG and $CA \cap B_1$ -EPG by minimal forbidden induced subgraphs is left as an open problem.

4 Circular-arc graphs as EPR graphs

In this section, we focus on circular-arc graphs that can be represented as edge intersection graphs of paths on a rectangle of the grid, i.e., we restrict ourselves to 2 rows and 2 columns of the grid. Obviously, $CA = B_4$ -EPR since we can embed the CA model into a rectangle of the grid in a natural way. A strengthening of this observation is that the bend number of

CA graphs with respect to EPR representations is equal to 4. To show this, it is sufficient to prove the following.

Proposition 3 *The thick spider S_{13} is not in B_3 -EPR.*

Proof. By contradiction, suppose that S_{13} admits a B_3 -EPR representation. Clearly, at most four of the paths corresponding to vertices in the stable set contain a corner of the rectangle, since they are pairwise non adjacent. So, from the remaining 9 paths, representing vertices in the stable set, at least three of them are intervals completely contained in one side of the rectangle. Let us denote these paths in order by $\mathcal{P}_i, \mathcal{P}_j, \mathcal{P}_k$, representing vertices s_i, s_j, s_k . The path corresponding to vertex c_j in the clique has to intersect \mathcal{P}_i , and \mathcal{P}_k avoiding \mathcal{P}_j , so it necessarily needs four bends, a contradiction. \square

We will now consider normal circular-arc graphs and show that their bend number with respect to EPR representations is equal to 2.

Theorem 4 *Every NCA graph is a B_2 -EPR graph.*

Proof. Let $(\mathcal{A}, \mathcal{C})$ be a NCA model of a circular-arc graph. Without loss of generality, we may assume that the endpoints of the arcs are pairwise different. Let p be a point of \mathcal{C} that is not the endpoint of an arc of \mathcal{A} . Since the model is normal, the union of the arcs of \mathcal{A} that contain p does not cover \mathcal{C} . Thus, there exists a point q in \mathcal{C} that is not the endpoint of an arc of \mathcal{A} and is not contained in the union of the arcs of \mathcal{A} containing p . We can then embed our model on a rectangle of the grid in the following way (arcs will correspond to paths and \mathcal{C} will correspond to the rectangle): two consecutive corners correspond to point p of the circle and the remaining two corners correspond to point q of the circle. In this way, since no arc of \mathcal{A} contains both p and q , paths corresponding to arcs containing either p or q have two bends, while paths corresponding to arcs containing neither p nor q have no bend. \square

In order to show that the bend number of NCA graphs with respect to EPR representations is equal to 2, consider the graph S_3 . S_3 is not in NHCA [6], so in particular it is not in B_1 -EPR (see Theorem 11), but it is in B_1 -EPG [12] and it is also in NCA (see for example [6]).

Let us also show that there exist B_2 -EPR graphs that are not in NCA.

Proposition 5 *The thick spider S_6 is in B_2 -EPR \setminus NCA.*

Proof. Let $(\mathcal{A}, \mathcal{C})$ be a circular-arc model of S_6 . Without loss of generality, we may assume that the disjoint arcs $\mathcal{A}_1, \dots, \mathcal{A}_6$ representing the vertices s_1, \dots, s_6 in the stable set are in clockwise order. It is not difficult to see that the arcs representing vertices c_1 and c_4 cover the circle. Thus, S_6 is not in NCA. A B_2 -EPR representation of S_6 is given in Figure 2. \square

We will now focus on B_1 -EPR graphs and show that they are NHCA graphs.

Lemma 6 $B_1\text{-EPR} \subseteq \text{NHCA}$.

Proof. Consider a $B_1\text{-EPR}$ representation of a graph G and let \mathcal{P} be the set of paths corresponding to the vertices of G . We will consider the natural bijection between the rectangle \mathcal{R} and a circle \mathcal{C} , that maps the paths in \mathcal{P} to open arcs \mathcal{A} of \mathcal{C} . Notice that two open arcs intersect if and only if the corresponding paths of \mathcal{P} intersect on an least one edge of the grid. Thus, $(\mathcal{A}, \mathcal{C})$ is a circular-arc representation of G . Now, since each path has at most one bend and the arcs are open, the union of three (resp. two) arcs of \mathcal{A} contains at most three (resp. two) points of \mathcal{C} corresponding to corners of \mathcal{R} . Hence $(\mathcal{A}, \mathcal{C})$ is a NHCA model for G . \square

Next, we will present a family of NHCA graphs that are not in the class of $B_1\text{-EPG}$ graphs. First we need to introduce some more definitions. It was shown in [12] that an induced cycle C_4 in a graph G corresponds to either a *true pie* or a *false pie* or a *frame* in any $B_1\text{-EPG}$ representation of G (see Figure 1). In a true pie or a false pie, the paths representing the vertices of the induced C_4 use one common grid point which is defined as the *center of the pie*. A frame is a model of C_4 such that each of the four corresponding paths has a bend in one of the four corners of a rectangle of the grid. Some examples of frame models are shown in Figure 1.

Lemma 7 *Powers of cycles C_{4k-1}^k , with $k \geq 2$, are not in $B_1\text{-EPG}$.*

Proof. Let $k \geq 2$ and let G be the graph C_{4k-1}^k , where the vertices of the cycle are denoted by v_1, \dots, v_n , with $n = 4k - 1$. Suppose, by contradiction, that G admits a $B_1\text{-EPG}$ representation. Note that v_1, v_{k+1}, v_{2k+1} and v_{3k+1} induce a C_4 in G .

We will consider the possible representations of the 4-cycle induced by v_1, v_{k+1}, v_{2k+1} and v_{3k+1} . Let us denote by \mathcal{P}_i the path corresponding to v_i , for $i = 1, \dots, n$. We will inductively show that, for each of the possible representations, the path \mathcal{P}_{i+1} has to share with \mathcal{P}_i a special point p_i and its two incident edges of the grid. The point p_i will be either the bend of \mathcal{P}_i or the center of the pie in case of pie representations.

Suppose first that the cycle is represented by a true pie using row x and column y of the grid. Let p be the intersection point of x and y . Vertex v_2 is adjacent to v_1, v_{k+1} , and v_{3k+1} in G . Since its corresponding path \mathcal{P}_2 must intersect $\mathcal{P}_1, \mathcal{P}_{k+1}$ and \mathcal{P}_{3k+1} and it cannot intersect \mathcal{P}_{2k+1} , it is forced to have a bend at p and use the same semi-row and semi-column as \mathcal{P}_1 . The same argument can then be applied to v_{k+2} and \mathcal{P}_{k+2} with respect to \mathcal{P}_{k+1} , to v_{2k+2} and \mathcal{P}_{2k+2} with respect to \mathcal{P}_{2k+1} and to v_{3k+2} and \mathcal{P}_{3k+2} with respect to \mathcal{P}_{3k+1} . Considering now the cycle v_2, v_{k+2}, v_{2k+2} and v_{3k+2} , we can repeat the process and, after $k - 1$ iterations, we will reach a contradiction because \mathcal{P}_1 should not use the same semi-row and semi-column as \mathcal{P}_{3k+1} .

Suppose now that the 4-cycle is represented by a false pie with center $p = (x, y)$. By symmetry, we may assume that \mathcal{P}_1 has a bend at p and \mathcal{P}_{k+1} uses edges on row x on both sides of p . Clearly, no path with at most one bend uses edges on two different rows or columns. Since v_2 is adjacent to v_{3k+1}, v_1 , and v_{k+1} , the path \mathcal{P}_2 must have a bend at p

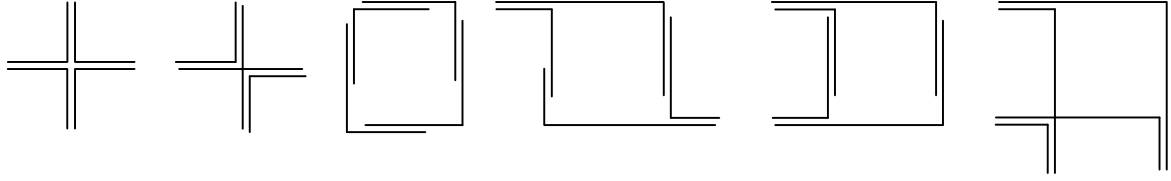


Figure 1: From left to right: a true pie, a false pie and four examples of a frame [12].

and use the same semi-row and semi-column as \mathcal{P}_1 . A similar argument shows that \mathcal{P}_{k+2} must use edges on row x on both sides of p , as \mathcal{P}_{k+1} . Symmetrically, an analogous situation holds for \mathcal{P}_{2k+2} with respect to \mathcal{P}_{2k+1} and for \mathcal{P}_{3k+2} with respect to \mathcal{P}_{3k+1} . As above, we can repeat the process and, after $k - 1$ iterations, we will reach again a contradiction.

Finally, suppose that the 4-cycle is represented by a frame on the rectangle defined by rows x and x' and columns y and y' . Suppose that \mathcal{P}_1 has edges on x and y , \mathcal{P}_{k+1} on y and x' , \mathcal{P}_{2k+1} on x' and y' , and \mathcal{P}_{3k+1} on y' and x . In order to have edge intersections with \mathcal{P}_1 , \mathcal{P}_{k+1} , and \mathcal{P}_{3k+1} , the only possible row-column combination for the path \mathcal{P}_2 is to use edges on x and y . Now repeating similar, iterative arguments as previously, we will reach again a contradiction. \square

In the next theorem, we will introduce five equivalent statements for B_1 -EPR graphs that are not chordal, two of them using structural properties, other two in terms of its NHCA models, and one in terms of forbidden induced subgraphs.

Theorem 8 *Let $G = (V, E)$ be a graph which is not chordal. Then the following statements are equivalent:*

- (i) $G \in B_1$ -EPR;
- (ii) $G \in NHCA$ and G contains no C_{4k-1}^k , with $k \geq 2$, as induced subgraph;
- (iii) $G \in NHCA$ and admits a NHCA model $(\mathcal{A}, \mathcal{C})$ with the following property: there are four points of \mathcal{C} , different from the endpoints of the arcs of \mathcal{A} , such that no arc of \mathcal{A} contains two of these points;
- (iii') $G \in NHCA$ and in every NHCA model $(\mathcal{A}, \mathcal{C})$ of G there are four points of \mathcal{C} , different from the endpoints of the arcs of \mathcal{A} , such that no arc of \mathcal{A} contains two of these points;
- (iv) $G \in NHCA$ and G has four disjoint connected subgraphs H_1, H_2, H_3, H_4 , such that H_1 and H_3 are in different connected components of $G \setminus (V(H_2) \cup V(H_4))$ and H_2 and H_4 are in different connected components of $G \setminus (V(H_1) \cup V(H_3))$.
- (iv') $G \in NHCA$ and G has four disjoint complete subgraphs H_1, H_2, H_3, H_4 , such that H_1 and H_3 are in different connected components of $G \setminus (V(H_2) \cup V(H_4))$ and H_2 and H_4 are in different connected components of $G \setminus (V(H_1) \cup V(H_3))$.

Proof. We will prove $(iii) \Rightarrow (iv')$, $(iv) \Rightarrow (iii')$, $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$, $(iii) \Rightarrow (i)$. The implications $(iii') \Rightarrow (iii)$ and $(iv') \Rightarrow (iv)$ are straightforward.

(iii) \Rightarrow (iv'): Let $(\mathcal{A}, \mathcal{C})$ be a circular-arc model of G such that there exist four points p_1, \dots, p_4 (clockwise) of \mathcal{C} satisfying that no arc of \mathcal{A} contains two of these points. Define H_i as the subgraph induced by the vertices corresponding to arcs of \mathcal{A} containing p_i , for $i = 1, \dots, 4$. Since G is not chordal, the four graphs H_1, \dots, H_4 are clearly non empty complete subgraphs, and since no arc contains two of the four points, they are disjoint. By the topology of the circle and the order of the points on it, it follows that every path connecting a vertex corresponding to an arc containing p_1 and a vertex corresponding to an arc containing p_3 necessarily contains either a vertex corresponding to an arc containing p_2 or a vertex corresponding to an arc containing p_4 . So, H_1 and H_3 are in different connected components of $G \setminus (V(H_2) \cup V(H_4))$ and, analogously, H_2 and H_4 are in different connected components of $G \setminus (V(H_1) \cup V(H_3))$.

(iv) \Rightarrow (iii'): Let H_1, H_2, H_3, H_4 be disjoint connected subgraphs of G such that H_1 and H_3 are in different connected components of $G \setminus (V(H_2) \cup V(H_4))$ and H_2 and H_4 are in different connected components of $G \setminus (V(H_1) \cup V(H_3))$. Let $(\mathcal{A}, \mathcal{C})$ be a NHCA model of G , and let \mathcal{A}^i be the set of arcs corresponding to vertices of H_i , for $i = 1, \dots, 4$.

The union of the arcs in \mathcal{A}^i induces a connected sector of \mathcal{C} , for each $i = 1, \dots, 4$. Moreover, by our assumptions, $\mathcal{A} \setminus (\mathcal{A}^1 \cup \mathcal{A}^3)$ induces more than one connected sector on \mathcal{C} , such that arcs in \mathcal{A}^2 and in \mathcal{A}^4 are in different connected sectors A_2 and A_4 . These sectors can be represented as two disjoint arcs of \mathcal{C} with endpoints t_2, h_2 , and t_4, h_4 (respectively) in clockwise order (the arc corresponding to A_i is obtained by taking the union of all arcs in belonging to the same sector as \mathcal{A}^i , for $i = 2, 4$). Define A_1 and A_3 analogously, and let us represent them by two disjoint arcs of \mathcal{C} with t_1, h_1 , and t_3, h_3 being their endpoints, respectively. Notice that since the graph is not chordal, and thus \mathcal{A} covers the circle, either A_1 covers $(h_2 - \varepsilon, t_4 + \varepsilon)$ and A_3 covers $(h_4 - \varepsilon, t_2 + \varepsilon)$, or A_1 covers $(t_2 - \varepsilon, h_4 + \varepsilon)$ and A_3 covers $(t_4 - \varepsilon, h_2 + \varepsilon)$, for some $\varepsilon > 0$. Without loss of generality, we may assume that the first case holds.

Let p_1 be a point in the clockwise open arc (h_2, t_4) , p_2 in (h_1, t_3) , p_3 in (h_4, t_2) , and p_4 in (h_3, t_1) . No arc of $\mathcal{A}^1, \mathcal{A}^2, \mathcal{A}^3$ or \mathcal{A}^4 contains two of these points, since these arcs are contained in A_1, A_2, A_3 and A_4 , respectively. Furthermore, no arc in $\mathcal{A} \setminus \bigcup_{i=1}^4 \mathcal{A}^i$ contains two of these points, because A_1, A_2, A_3 and A_4 are connected sectors of either $\mathcal{A} \setminus (\mathcal{A}^1 \cup \mathcal{A}^3)$ or $\mathcal{A} \setminus (\mathcal{A}^2 \cup \mathcal{A}^4)$.

At this point, we know (iii) \Rightarrow (iii').

(i) \Rightarrow (ii): This follows directly from Lemmas 6 and 7.

(ii) \Rightarrow (iii): Let G be a minimal counterexample to (iii), and let $(\mathcal{A}, \mathcal{C})$ be an arbitrary NHCA model of G . Then for every choice of four points of \mathcal{C} , there is an arc of \mathcal{A} that contains two of these points.

As usual, we will assume that the arcs in \mathcal{A} are open and their endpoints are pairwise distinct. We can number the endpoints clockwise in the circle from 1 to $2n$ (n being the

number of vertices of G). For an arc $\mathcal{A}_i \in \mathcal{A}$, its endpoints will be referred to as *tail* and *head* in such a way that \mathcal{A}_i is the open arc traversing \mathcal{C} clockwise from the tail to the head.

Claim 9 *No arc of \mathcal{A} is properly contained in another.*

Suppose there is an arc \mathcal{A}_i which is properly contained in an arc \mathcal{A}_j , $i \neq j$. By minimality, and since (iii) \Rightarrow (iii'), the model $(\mathcal{A} \setminus \{\mathcal{A}_i\}, \mathcal{C})$ admits four points p_1, p_2, p_3, p_4 such that no arc contains two of them. But if \mathcal{A}_j does not contain two of these points, neither does \mathcal{A}_i that is properly contained in \mathcal{A}_j . Thus, $(\mathcal{A}, \mathcal{C})$ satisfies the property as well, a contradiction to our hypothesis. \diamond

Claim 10 *No vertex is dominated by another.*

Suppose that vertex v dominates vertex w . Let \mathcal{A}_v and \mathcal{A}_w be their corresponding arcs in the NHCA model $(\mathcal{A}, \mathcal{C})$. If there is an arc \mathcal{A}_z , corresponding to a vertex z , that intersects \mathcal{A}_w only on $\mathcal{A}_w \setminus \mathcal{A}_v$, then, since z is also adjacent to v , \mathcal{A}_z intersects \mathcal{A}_v only in $\mathcal{A}_v \setminus \mathcal{A}_w$. But then $\mathcal{A}_v, \mathcal{A}_w$ and \mathcal{A}_z cover \mathcal{C} , a contradiction. Thus, such an arc does not exist. But then we can replace \mathcal{A}_w by $\mathcal{A}_w \cap \mathcal{A}_v$ obtaining a NHCA model of the same graph with an arc properly contained in another, a contradiction with the previous claim. \diamond

It was shown by Golumbic and Hammer [11] that the last claim implies that, when traversing the endpoints of the arcs on the circle clockwise, heads and tails necessarily alternate. Moreover, they proved that G is the k -th power of the cycle C_n , for some value of k . Since G is a counterexample to (iii), it follows that $k \geq 2$.

Let $1, \dots, 2n$ denote the endpoints of the arcs, where odd numbers correspond to tails and even numbers to heads. Thus, every arc is of the form $(2i - 1, 2i + 2k)$, for $i = 1, \dots, n$, where the sums are taken modulo $2n$. In particular, every arc properly contains $2k$ of the $2n$ endpoints. Since the model is normal and Helly, $6k < 2n$, otherwise arcs $(1, 2k + 2)$, $(2k + 1, 4k + 2)$, and $(4k + 1, 6k + 2)$ cover the circle. On the other hand, $8k > 2n$, otherwise points $2, 2k + 2, 4k + 2$, and $6k + 2$ would be such that no arc of \mathcal{A} contains two of them.

We will show now that $6k < 2n < 8k$ is also a sufficient condition for the k -th power of the cycle C_n to be a counterexample to (iii) (not necessarily minimal). It is clear that $6k < 2n$ ensures that C_n^k is a NHCA graph. Consider a NHCA model of C_n^k . Now, suppose $2n < 8k$ and let p_1, p_2, p_3, p_4 be four points of \mathcal{C} . We may assume that they correspond to endpoints of arcs, otherwise we can move each of them to its closest endpoint without creating a new containment relation between arcs and points. If there are two points at distance at most $2k - 2$, i.e., in the closed interval $[i, i + 2k - 2]$ for some $i = 1, \dots, n$, then they are both contained either in the arc $(i - 1, i + 2k)$ or in the arc $(i - 2, i + 2k - 1)$, depending on the parity of i . So we may assume now that p_1, p_2, p_3, p_4 are pairwise at distance at least $2k - 1$. It follows from the inequality $2n < 8k$ that at least two pairs of vertices are at distance exactly $2k - 1$ on \mathcal{C} , and at least one of these pairs corresponds to

endpoints $i, i + 2k - 1$ with i even. Thus, these two points are both contained in the arc $(i - 1, i + 2k)$.

Since $k \geq 2$, the inequality $6k < 2n$ implies that $n \geq 7$ thus, by the property above, C_7^2 is a minimal counterexample to (iii). Indeed, $C_n^k, k \geq 2$, contains C_7^2 as induced subgraph if and only if $12k < 4n \leq 14k$ (it can be verified that the arcs $(1, 2k + 2), (2k + 1, 4k + 2), (4k + 1, 6k + 2), \dots, (12k + 1, 14k + 2)$, where the operations are done modulo $2n$, induce C_7^2).

More in general and inductively, we can prove that C_{4t-1}^t , with $t \geq 2$, is a minimal NHCA counterexample to (iii) and that $C_n^k, k \geq 2$, contains C_{4t-1}^t as induced subgraph if and only if $2(4t - 5)k < 2(t - 1)n$ and $2tn \leq 2(4t - 1)k$, or equivalently, $(4t - 5)/(t - 1) < n/k \leq (4t - 1)/t$.

As $(4t - 1)/t$ converges to 4 as t tends to infinity, every C_n^k with $k \geq 2$ and such that $3 < n/k < 4$ contains a power of a cycle C_{4t-1}^t as induced subgraph, for some $t \geq 2$, and this completes the proof.

(iii) \Rightarrow (i): Let $(\mathcal{A}, \mathcal{C})$ be a circular-arc model of G such that there exist four points on \mathcal{C} satisfying that no arc of \mathcal{A} contains two of these points.

We will place the corners of the rectangle in those four points. Since the arcs in \mathcal{A} are open and we are assuming, without loss of generality, that the endpoints of the arcs are pairwise distinct, we may assume as well that the four points are different from all the arc endpoints. It is easy to see then that we can find a big enough rectangle in the grid such that we can represent the arcs as paths in the grid, maintaining the order of their endpoints, and placing the four corners at the desired points, and this will give us a B_1 -EPR representation of G . \square

Now we are able to prove the following.

Theorem 11 *Let $G = (V, E)$ be a graph. Then $G \in B_1$ -EPR if and only if $G \in NHCA$ and G has no $C_{4k-1}^k, k \geq 2$, as induced subgraph.*

Proof. One implication follows immediately from Lemmas 6 and 7, since the class is hereditary. For the converse, if G is not a chordal graph, then the result follows from Theorem 8. Suppose now that G is chordal. It is shown in [6] that a chordal NHCA graph is indeed an interval graph, thus a B_0 -EPR graph and, in particular, a B_1 -EPR graph. \square

Since $C_{4k-1}^k, k \geq 2$, is not in B_1 -EPG (see Lemma 7), it follows that B_1 -EPG \cap NHCA = B_1 -EPR = NHCA \cap $\{C_{4k-1}^k\}_{k \geq 2}$ -free.

It is easy to develop a polynomial-time algorithm to recognize B_1 -EPR graphs, based on the polynomial-time recognition algorithm of NHCA graphs [6], that outputs a NHCA model of the graph if there is one, and property (iii') of Theorem 8, since the four points can be chosen from the endpoints of the arcs of the model.

We leave as an open problem the characterization of $CA \cap B_2$ -EPR and $CA \cap B_3$ -EPR by minimal forbidden induced subgraphs.

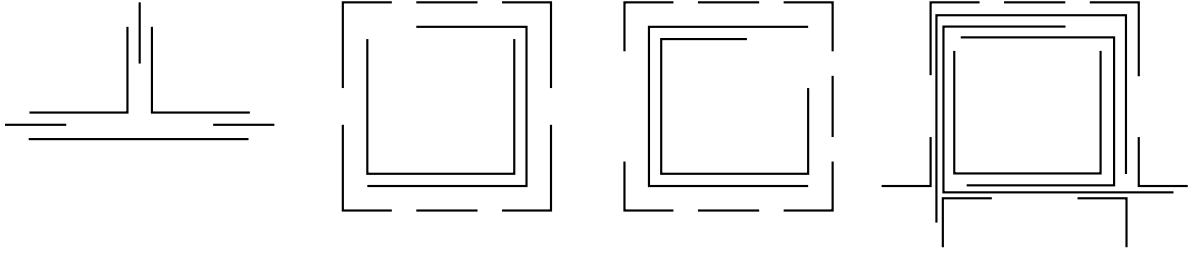


Figure 2: From left to right: a B_1 -EPG representation of S_3 ; a sketch of a B_2 -EPR representation of S_6 (we draw all the vertices of the stable set and representative vertices of the clique; other vertices in the clique can be represented symmetrically); sketches of B_3 -EPR and B_2 -EPG representations of S_7 (again, we draw all the vertices of the stable set and representative vertices of the clique; other vertices in the clique can be represented symmetrically).

5 Further results

The thick spider S_3 is one of the minimal forbidden induced subgraphs for the class NHCA [6], but all the thick spiders are CA graphs and, by Theorem 1, B_3 -EPG graphs. Thick spiders allow us to distinguish classes in the families B_k -EPR ($k \leq 4$), B_k -EPG ($k \leq 3$), NHCA and NCA. In fact, S_6 is not NCA but it is in B_2 -EPR (see Figure 2 and Proposition 5); S_7 is in B_3 -EPR $\setminus B_2$ -EPR (Proposition 12); S_{40} is in B_3 -EPG $\setminus B_2$ -EPG (Proposition 2); S_7 is in B_2 -EPG $\setminus B_1$ -EPG (Proposition 13).

We complete the proof of these claims in the following propositions.

Proposition 12 *The thick spider S_7 is not a B_2 -EPR graph.*

Proof. By contradiction, suppose that S_7 admits a B_2 -EPR representation. Let us consider only the paths corresponding to vertices in the stable set. If there are three of them whose union is contained in two adjacent sides of the rectangle (or in one side of the rectangle), let us say in order $\mathcal{P}_i, \mathcal{P}_j, \mathcal{P}_k$, representing vertices s_i, s_j, s_k , respectively, then the path corresponding to vertex c_j in the clique has to intersect \mathcal{P}_i and \mathcal{P}_k avoiding \mathcal{P}_j , and so it needs at least three bends, a contradiction. So, we may assume that this situation does not occur. Consider two opposite corners of the rectangle. If they are not covered by two different paths, then there are at least three paths whose union is contained in two adjacent sides of the rectangle, a contradiction. So, we may assume that the four corners are covered by paths, corresponding to (at most) four vertices of the stable set. If no such path uses two corners, then, from the remaining three paths, there are two of them that are intervals contained in the same side or in adjacent sides of the rectangle. But then, their union together with one path on a corner are contained in two adjacent sides of the

rectangle, a contradiction. If exactly one of the paths uses two corners (and one side of the rectangle), from the remaining four paths, there are two of them that are intervals contained in the same side of the rectangle, so as before we obtain a contradiction. Finally, if there are two paths using two corners each, and two sides of the rectangle, from the remaining five paths, there are three of them that are intervals contained in the same side of the rectangle, a contradiction. This completes the proof. \square

Proposition 13 *The thick spider S_7 is not a B_1 -EPG graph.*

Proof. By contradiction, suppose that S_7 admits a B_1 -EPG representation. Let us consider the path \mathcal{P}_c corresponding to a vertex c of the clique and the paths corresponding to its 6 neighbors in the stable set S . The path \mathcal{P}_c has edges on at most two lines (one row and one column) of the grid. Thus, it intersects at least 3 paths corresponding to neighbors in S on a same line x . Without loss of generality, we may assume that x corresponds to a row of the grid. Consider now the 3 intervals on row x belonging to the 3 paths mentioned above. Since the corresponding vertices are pairwise non adjacent, they admit some order on row x , say $\mathcal{I}_i, \mathcal{I}_j, \mathcal{I}_k$, corresponding to vertices s_i, s_j, s_k , respectively. Having at most one bend each, the paths corresponding to s_i and s_k do not have edges in a common column (and they do not have edges in a row different from x , or more than one interval on x). So there is no way for the path \mathcal{P}_{c_j} corresponding to the vertex c_j of the clique that is not adjacent to s_j of avoiding the interval \mathcal{I}_j while intersecting the paths corresponding to s_i and s_k using only one bend. \square

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