On approximating the b-chromatic number *

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Abstract

We consider the problem of approximating the b-chromatic number of a graph. We show that there is no constant $\varepsilon > 0$ for which this problem can be approximated within a factor of $120/113 - \varepsilon$ in polynomial time, unless P = NP. This is the first hardness result for approximating the b-chromatic number.

Keywords: Combinatorial problems, approximation algorithms, graph coloring.

1 Introduction

We consider finite undirected graphs without loops or multiple edges. A *coloring* (i.e. *proper coloring*) of a graph $G = (V, E)$ is an assignment of colors to the vertices of G, such that any two adjacent vertices have different colors. A coloring is called a *b-coloring*, if for each color i there exists a vertex x_i of color i such that for every color $j \neq i$, there exists a vertex y_j of color j adjacent to x_i (such a vertex x_i is called a *dominating* vertex for the color class i). The *b-chromatic number* $\varphi(G)$ of a graph G is the largest number k such that G has a b-coloring with k colors. The b-chromatic number of a graph was introduced by R.W. Irving and D.F. Manlove [3] when considering minimal proper colorings with respect to a partial order defined on the set of all partitions of the vertices of a graph. They proved that determining $\varphi(G)$ is NP-hard for general graphs, but polynomial-time solvable for trees. Recently, Kratochvil et al. [5] have shown that determining $\varphi(G)$ is NP-hard even for bipartite graphs. Some bounds for the b-chromatic number of a graph are given in [3, 6].

In this paper we prove that there is no constant $\varepsilon > 0$ for which this problem can be approximated within a factor of $120/113 - \varepsilon$ in polynomial time, unless P = NP. No hardness of approximation was previously known for this problem.

The organization of the paper is as follows. In Section 2 we give the preliminaries. In Section 3 we present the hardness of approximation result. We end in Section 4 with some concluding remarks.

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2 Preliminaries

Let $\mathcal{P}$ be a maximization problem and let $\alpha \geq 1$. For an instance x of $\mathcal{P}$ let $OPT(x)$ be the optimal value. An *α-approximation algorithm* for $\mathcal{P}$ is a polynomial time algorithm $\mathcal{A}$ such that on each input instance x of $\mathcal{P}$ it outputs a number $\mathcal{A}(x)$ such that $OPT(x)/\alpha \leq \mathcal{A}(x) \leq OPT(x)$.

To show the hardness of approximating the b-chromatic number we relate it to the hardness of approximating the optimization version of the k-ESAT problem. Let k be an integer greater than 1.

k-ESAT problem.
Instance: A set $X = \{x_1, x_2, \ldots, x_n\}$ of boolean variables, a collection $C = \{c_1, c_2, \ldots, c_p\}$ of disjunctive clauses with exactly k different literals, where a literal is a variable or a negated variable in X.
Question: Does there exist a truth assignment for the variables in X such that each clause in C is satisfied?

The decision version of the k-ESAT problem is NP-complete for $k \geq 3$ [1]. Johnson showed in [4] the following result.

Theorem 1 *(Theorem 3 in [4]) Let (X, C) be an instance of the k-ESAT problem. Then, there is a deterministic polynomial time algorithm that finds a truth assignment for variables in X which satisfies at least $|C|(1 - 1/2^k)$ clauses in C.*

The **MAX k-ESAT** problem is the optimization version of the k-ESAT problem in which, given an instance of k-ESAT, the goal consists of finding the maximum number of clauses that can be satisfied simultaneously by any truth assignment of the boolean variables. The MAX k-ESAT problem is NP-hard [1].

Note that in the case $k = 3$, Theorem 1 gives an 8/7-approximation algorithm for the MAX 3-ESAT problem. Moreover, Håstad showed in [2] the following inapproximability result for the MAX 3-ESAT problem.

Theorem 2 *(Theorem 6.1 in [2]) The MAX 3-ESAT problem is not approximable within $8/7 - \varepsilon$ for any $\varepsilon > 0$, unless P = NP.*

In the following section, we use Theorem 2 restricted to a special kind of instances in order to obtain an inapproximability result for the b-chromatic number problem of a graph.

Definition 1 *We say that an instance (X, C) of MAX 3-ESAT is non-trivial if $|C| > 4$, and for all $x \in X$*

- *There is no $c \in C$ such that $x, \overline{x} \in c$,*
- *There are $c, d \in C$ such that $x \in c$ and $\overline{x} \in d$.*

We now show that Theorem 2 holds when restricted to non-trivial instances of MAX 3-ESAT.

Corollary 1 *The MAX 3-ESAT problem is not approximable within $8/7 - \varepsilon$ for any $\varepsilon > 0$, even when restricted to non-trivial instances.*

Proof : We present a proof by contradiction. Assume that there is an $(8/7-\varepsilon)$-approximation algorithm running in polynomial time $p(|X|+|C|)$ for non-trivial instances (X, C) of the MAX 3-ESAT problem, for some $0 < \varepsilon \leq 1/7$. We prove that there is an $(8/7-\varepsilon)$-approximation algorithm for the MAX 3-ESAT problem. This contradicts Theorem 2.

We prove this by induction on $|X|+|C|$. The base case is trivial. Now, let $k > 1$ and assume that the statement holds for all instances (X, C) such that $|X|+|C| < k$, and let (X, C) be an instance of MAX 3-ESAT such that $|X|+|C| = k$. If the instance is non-trivial, the statement follows from our initial assumption. If not we have three possible cases:

- There is $x \in X$ such that there is $c \in C$ with $x, \overline{x} \in c$. Let $C' = C \setminus \{c\}$. By induction hypothesis applied to (X, C'), we can get, in polynomial time, a truth assignment for the variables in X that satisfies at least $\frac{|C'|}{8/7-\varepsilon}$ clauses in C'. This assignment also satisfies c and therefore satisfies at least
$$\frac{|C'|}{8/7-\varepsilon} + 1 \geq \frac{|C|}{8/7-\varepsilon}$$
clauses of C.

- There is $x \in X$ such that no clause $c \in C$ contains $\overline{x}$. Let $X' = X \setminus \{x\}$ and $C' = C \setminus \{c \in C : x \in c\}$. By induction hypothesis we can get, in polynomial time, a truth assignment for the variables in X' that satisfies at least $\frac{|C'|}{8/7-\varepsilon}$ clauses in C'. Now we assign the value True to x, and all clauses in C containing it are satisfied. Therefore we have a truth assignment satisfying at least
$$\frac{|C'|}{8/7-\varepsilon} + |C \setminus C'| \geq \frac{|C|}{8/7-\varepsilon}$$
clauses.

- There is $x \in X$ such that no clause $c \in C$ contains x. This case is analogous to the previous one.

Therefore, there is a $(8/7-\varepsilon)$-approximation algorithm for the MAX 3-ESAT problem running in polynomial-time $O(k^2)p(k)$, where the $O(k^2)$ term represents the time needed to find the desired x and construct X' and C' and is certainly not the best possible. □

3 Hardness of approximation

In this section we prove the hardness result for approximating the b-chromatic number problem of a graph.

Let (X, C) be an instance of the 3-ESAT problem. We define $\mathbf{G}(X, C) = (V, E)$ to be the graph constructed as follows:

Let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of boolean variables, and let $C = \{c_1, c_2, \ldots, c_p\}$ be the collection of disjunctive clauses, with $c_i = \{l_{i,1}, l_{i,2}, l_{i,3}\}$ for $i = 1, 2, \ldots, p$, where $l_{i,j} = x_k$ or $l_{i,j} = \overline{x_k}$ for some $1 \leq k \leq n$.

Let

$$\begin{aligned} V = &\{v\} \cup \{z_i : 1 \leq i \leq p-1\} \cup \{w_j : 1 \leq j \leq 2p\} \\ &\cup \{y_i : 1 \leq i \leq p\} \cup \{x_{i,j}, \overline{x_{i,j}} : 1 \leq i \leq n,\, 1 \leq j \leq p\}, \end{aligned}$$

and let

$$
\begin{aligned}
E =&\{\{z_i, w_j\} : 1 \le i \le p-1,\ 1 \le j \ne i \le 2p\} \\
&\cup \{\{v, z_i\} : 1 \le i \le p-1\} \cup \{\{v, y_i\} : 1 \le i \le p\} \\
&\cup \{\{y_i, y_j\} : 1 \le i < j \le p\} \\
&\cup \{\{x_{i,j}, \overline{x_{i,k}}\} : 1 \le i \le n,\ 1 \le j, k \le p\} \\
&\cup \{\{y_i, x_{j,k}\} : 1 \le i \le p,\ 1 \le j \le n,\ 1 \le k \le p,\ x_j \in c_i\} \\
&\cup \{\{y_i, \overline{x_{j,k}}\} : 1 \le i \le p,\ 1 \le j \le n,\ 1 \le k \le p,\ \overline{x_j} \in c_i\}.
\end{aligned}
$$

Notice that $|V| = 2np + 4p$.

The resulting graph $\mathbf{G}(X,C) = (V,E)$ is shown in Figure 1.

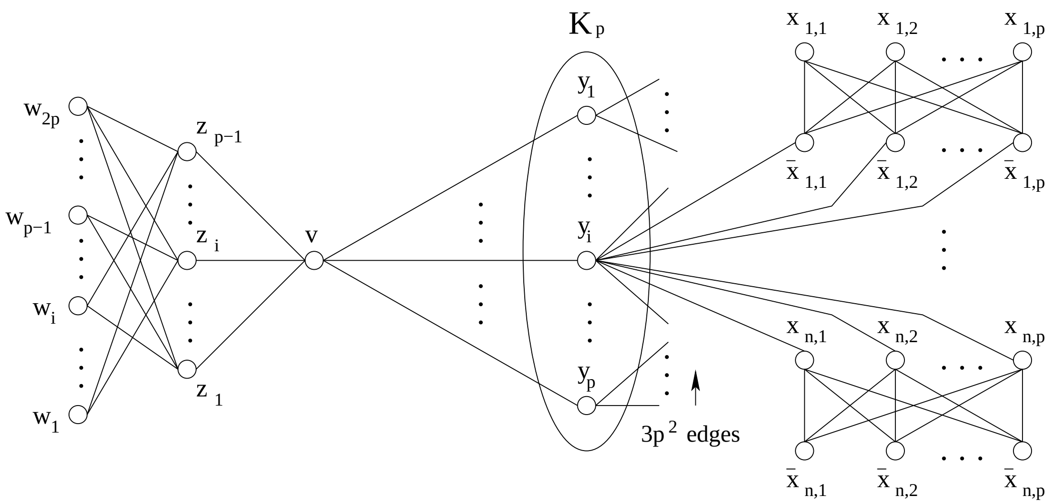

Figure 1: Partial construction of G from (X,C), where the clause $c_i \in C$ contains the literals $\overline{x_1}$ and x_n.

Theorem 3 *Let (X,C) be a non-trivial instance of the 3-ESAT problem, where $|X| = n$ and $|C| = p$. Then, $\varphi(\mathbf{G}(X,C)) = p+t$ where t is the maximum number of clauses that can be satisfied in C.*

The proof of Theorem 3 requires Propositions 1 and 2 below.

Proposition 1 *Let (X,C) be a non-trivial instance of the 3-ESAT problem, where $|X| = n$ and $|C| = p$. Let t be the maximum number of clauses that can be satisfied in C. Then there is a b-coloring of $\mathbf{G}(X,C)$ with $p+t$ colors.*

Proof : Fix a truth assignment of the variables that satisfies exactly t clauses. W.l.o.g. assume that the clauses satisfied in C are $c_1, c_2, \ldots, c_t$.

Color the vertices of $\mathbf{G}(X,C)$ with $p+t$ colors as follows:

- for $1 \le i \le p-1$, assign color i to vertex z_i,
- assign color p to vertex v,
- for $1 \le i \le t$, assign color $p+i$ to vertex y_i.

The previous vertices will be the dominating vertices of each one of the $p+t$ color classes.

For $1 \le j \le p+t$, assign color j to vertex w_j, and for $1 \le j \le p-t$, assign color $p+t$ to vertex w_{p+t+j}. In this way, the vertex z_i is dominating for the color class i.

Vertex v is already a dominating vertex for the color class p.

For $t+1 \le i \le p$, assign to vertex y_i the color $i-t$.

For every $1 \le i \le n$, do the following. If x_i is true, choose $1 \le s \le p$ such that $x_i \in c_s$. Notice that c_s is satisfied and therefore $s \le t$. Assign to each $x_{i,j}$ color j and to $\overline{x_{i,j}}$ color $p+s$, for $1 \le j \le p$. If x_i is false then $\overline{x}_i$ is true, and proceed in the analogous way.

Now, we just need to check that the coloring is proper and that for $1 \le i \le t$, y_i is a dominating vertex for its color class.

The coloring is not proper only if there are $1 \le i \le p$, $1 \le j \le n$ and $1 \le k \le p$ such that there is an edge between y_i and $l_{j,k}$, where $l_{j,k} = x_{j,k}$ or $l_{j,k} = \overline{x_{j,k}}$, with y_i and $l_{j,k}$ of the same color (all the other edges are taken care of directly by the construction). Without loss of generality we assume $l_{j,k} = x_{j,k}$, because the other case is analogous. By construction of $\mathbf{G}(X,C)$, we know that $x_j \in c_i$. There are two cases. If $1 \le i \le t$, as the color of $x_{j,k}$ is the same as the color of y_i, and this is $p+i > p$, then x_j is false, so $\overline{x_j}$ is true. Therefore by the construction of the coloring $\overline{x_j} \in c_i$, but then $x_j, \overline{x_j} \in c_i$ contradicting the non-triviality of the instance. If $t < i \le p$, as the color of $x_{j,k}$ is the same as the color of y_i, and this is $i-t < p$, x_j is true. Therefore c_i is satisfied, but this contradicts our assumption that the truth assignment satisfies exactly the first t clauses.

Now, consider $1 \le i \le t$, and let l_i be a literal in clause c_i such that the truth assignment satisfies l_i. Notice that y_i is adjacent to the p vertices that correspond to this literal, and they received colors $1, \ldots, p$. Since vertex y_i is also adjacent to every other vertex y_j, for $1 \le j \ne i \le t$, vertex y_i is a dominating vertex. □

Proposition 2 *Let (X,C) be a non-trivial instance of MAX 3-ESAT and let $1 < t$. If there is a b-coloring of $\mathbf{G}(X,C)$ with $p+t$ colors, Then there exists a truth assignment for X such that at least t clauses are satisfied in C.*

Proof : Fix a b-coloring of $\mathbf{G}(X,C)$ with $p+t$ colors. There are three possible cases:

- There exist $1 \le j \le n$ and $1 \le k \le p$ such that $x_{j,k}$ is a dominating vertex. In this case, vertex $x_{j,k}$ is adjacent at least to $p+t-1$ other vertices and therefore $x_{j,k}$ is adjacent to at least $t-1$ of the vertices $y_i' s$. This implies x_j belongs to at least $t-1$ of the $c_i' s$. If x_j belongs to at least t of the $c_i' s$, any truth assignment where x_j is true will satisfy t clauses in C. If x_j belongs to exactly $t-1$ $y_i' s$, take $c \in C$ such that $x_j \notin c$, and let $j' \ne j$, $1 \le j' \le n$, be such that $x_{j'} \in c$ (or $\overline{x_{j'}} \in c$). Then any truth assignment where x_j is true and $x_{j'}$ is true (resp. $x_{j'}$ is false) will satisfy at least t clauses in C.

- There are $1 \le j \le n$ and $1 \le k \le p$ such that $\overline{x_{j,k}}$ is a dominating vertex. This case is completely analogous to the first one.

- For every $1 \le j \le n$ and $1 \le k \le p$ neither $x_{j,k}$ nor $\overline{x_{j,k}}$ is a dominating vertex. In this case the dominating vertices are among the set $\{v\} \cup \{z_i : 1 \le i \le p-1\} \cup \{y_i : 1 \le i \le p\}$. Now let B the set of dominating vertices belonging to $\{y_i : 1 \le i \le p\}$. Then $|B| \ge t$. Without loss of generality assume that for $1 \le i \le p$ the color of each y_i is i and that the color assigned to v is $p+1$. Now define the following truth assignment for the boolean variables:

$\sigma(x_j)$ is True if and only if for all $1 \leq k \leq p$ the color of $\overline{x_{j,k}}$ is not $p+2$.

Now, let $1 \leq i \leq p$ be such that $y_i \in B$. As y_i is a dominating vertex, it has to be connected to some vertex of color $p+2$, and this one has to be one of the $x_{j,k}$ or $\overline{x_{j,k}}$ for some $1 \leq j \leq n$ and $1 \leq k \leq p$. Notice that if $x_{j,k}$ has color $p+2$ then for all $1 \leq l \leq p$, the color of $\overline{x_{j,l}}$ is not $p+2$ and thus $\sigma(x_j)$ is True. On the other hand if $\overline{x_{j,k}}$ has color $p+2$ then $\sigma(x_j)$ is False. In either case σ satisfies c_i. □

Proof of the Theorem 3. From Theorem 1, $t \geq 7p/8 > 1$, and the result follows from Propositions 1 and 2. □

By Corollary 1 and Theorem 3, the hardness approximation result for the b-chromatic number problem now follows.

Theorem 4 *The b-chromatic number problem is not approximable within* $120/113 - \varepsilon$ *for any* $\varepsilon > 0$, *unless* $P = NP$.

Proof : Suppose that the b-chromatic number problem can be approximated within a factor of $120/113 - \varepsilon$, for some $\varepsilon > 0$. Let (X, C) be a non-trivial instance of 3-ESAT, as defined in Section 2. Let p be the number of clauses in C, and let t be the maximum number of clauses of C that can be satisfied by a truth assignment to X. By Theorem 3, we can construct in polynomial time a graph G, namely $\mathbf{G}(X, C)$, such that $\varphi(G) = p + t$. By the assumption, we can compute in polynomial time a b-coloring for G with l colors such that

$$\frac{\varphi(G)}{120/113 - \varepsilon} \leq l \leq \varphi(G),$$

and by Proposition 2, we can derive a truth assignment of (X, C) which satisfies at least $l - p$ clauses. Then

$$\frac{p+t}{120/113 - \varepsilon} - p \leq l - p \leq t.$$

$$\frac{113t - 7p + 113p\varepsilon}{120 - 113\varepsilon} \leq l - p \leq t$$

But, from Theorem 1, $p \leq 8t/7$, therefore

$$\frac{t}{8/7 - \varepsilon} = \frac{105t}{120 - 105\varepsilon} \leq \frac{105t + 113p\varepsilon}{120 - 113\varepsilon} \leq \frac{113t - 7p + 113p\varepsilon}{120 - 113\varepsilon} \leq l - p \leq t$$

Thus, we can get a $8/7 - \varepsilon$ approximation to t which contradicts Corollary 1. □

4 Conclusion

We have shown that the b-chromatic number of a graph is hard to approximate in polynomial time within a factor of $120/113 - \varepsilon$, for any $\varepsilon > 0$, unless P = NP. This is the first hardness result for approximating the b-chromatic number. An interesting open problem is the existence of a constant-factor approximation algorithm for the b-chromatic number in general graphs.

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