Idomatic partitions of direct products of complete graphs*

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Abstract

In this paper we give a full characterization of the idomatic partitions of the direct product of three complete graphs. We also show how to use such a characterization in order to construct idomatic partitions of the direct product of finitely many complete graphs.

Keywords: Direct product, independent dominating sets, idomatic partitions, complete graphs, Cayley graphs.

1 Introduction and preliminary results

Let $G = (V, E)$ be an undirected finite simple graph without loops. A set $S \subseteq V$ is called a *dominating set* if for every vertex $v \in V \setminus S$ there exists a vertex $u \in S$ such that u is adjacent to v. The minimum cardinality of a dominating set in G is called the *domination number* of G and is denoted $\gamma(G)$. A set $S \subseteq V$ is called *independent* if no two vertices in S are adjacent. A set $S \subseteq V$ is called an *independent dominating set* of G if it is both independent and dominating set of G. The minimum cardinality of an independent dominating set in G is called the *independent domination number* of G and is denoted $i(G)$. The *domatic number* $d(G)$ is the maximum order of a partition of V into dominating sets. The domatic number of a graph was introduced by Cockayne and Hedetniemi [3]. A partition of the vertex set V into independent dominating sets is called an *idomatic partition* of G [2, 3]. Clearly, an idomatic partition of a graph G represents a proper coloring of the vertices of G. The maximum order of an idomatic partition of G is called the *idomatic number* $id(G)$. An idomatic partition of a graph

*This work was supported by the French Paris-Nord University BQR-2008 Grant.

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G into k parts is called an *idomatic k-partition of* G. Notice that not every graph has an idomatic k-partition, for any k. For example, the cycle graph on five vertices C_5 has no an idomatic k-partition for any k.

The *direct product* $G \times H$ of two graphs G and H is defined by $V(G \times H) = V(G) \times V(H)$, and where two vertices $(u_1, u_2), (v_1, v_2)$ are joined by an edge in $E(G \times H)$ if $\{u_1, v_1\} \in E(G)$ and $\{u_2, v_2\} \in E(H)$. This product is commutative and associative in a natural way (see reference [8] for a detailed description on product graphs).

Let n be a positive integer. We denote by $[n]$ the set $\{0, 1, \ldots, n-1\}$. The complete graph K_n will usually be on the vertex set $[n]$.

Let Γ be a group and C a subset of Γ closed under inverses and identity free. The Cayley graph $\mathrm{Cay}(\Gamma, C)$ is the graph with Γ as its vertex set, two vertices u and v being joined by an edge if and only if $u^{-1}v \in C$. The set C is then called the *connector set* of $\mathrm{Cay}(\Gamma, C)$. Simple examples of Cayley graphs include the cycles, which are Cayley graphs of cyclic groups, and the complete graphs K_n which are Cayley graphs of any group of order n. Cayley graphs constitute a rich class of vertex-transitive graphs (see [5, 6] and references therein).

Let $t \geq 1$ be an integer and let $n_1, n_2, \ldots, n_t$ be positive integers. Notice that the direct product graph $G = K_{n_1} \times K_{n_2} \times \ldots \times K_{n_t}$ can be seen as the Cayley graph of the direct product group $\mathcal{G} = Z_{n_1} \times Z_{n_2} \times \ldots \times Z_{n_t}$ with connector set $[n_1] \setminus \{0\} \times \ldots \times [n_t] \setminus \{0\}$, where Z_{n_i} denotes the additive cyclic group of integers modulo n_i.

Some recent results concerning independence parameters in graphs with connection to direct products graphs and Cayley graphs can be found in [1, 9, 4] (see also references therein).

Idomatic partitions of graphs were studied in [4] as an special coloring problem on graphs defined as *fall colorings*. In this work, the authors show the following result.

Theorem 1 ([4]). *Let $n_1 > 1$ and $n_2 > 1$ be two integers. The direct product graph $K_{n_1} \times K_{n_2}$ admits an idomatic n_1-partition and an idomatic n_2-partition. Furthermore, if $t > 1$ is an integer such that $t \notin \{n_1, n_2\}$, then $K_{n_1} \times K_{n_2}$ has no idomatic t-partition.*

Moreover, in [4] is posed the question of characterizing the idomatic partitions of the direct product of three or more complete graphs. In this note, we give in Section 2, a full characterization of the idomatic partitions of the direct product of three complete graphs by using an standard algebraic approach. In Section 3, we show how to use such a characterization in order to construct idomatic partitions of the direct product of four or more complete graphs.

2 Direct product of three complete graphs

In the following, we characterize the independent dominating sets and the idomatic partitions of the direct product of three complete graphs.

2.1 Independent dominating sets

Lemma 1. *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$ with $n_0, n_1, n_2 \geq 2$ and let I be an independent dominating set in G. If the set I contains at least two vertices agreeing in exactly two coordinates, then $I = pr_i^{-1}(k)$, where $i \in [3]$, pr_i is the projection of G on K_{n_i} and $k \in [n_i]$.*

Proof. As G is vertex-transitive and the direct product is commutative, we can assume w.l.o.g. that the vertices (x, i, j) and (y, i, j) of G belong to I, with i and j fix, and $x \neq y$. First note that for all $z \in [n_0]$, with $z \notin \{x, y\}$, we have that $(z, i, j) \in I$. Otherwise, let $z \notin \{x, y\}$ such that $(z, i, j) \notin I$. As I is a dominating set, then there exists a vertex $(a, b, c) \in I$ such that $a \neq z$, $b \neq i$ and $c \neq j$. If $a \notin \{x, y\}$ then (a, b, c) is adjacent to vertices (x, i, j) and (y, i, j). If $a \in \{x, y\}$, say $a = x$ (the case $a = y$ is analogous), then (a, b, c) is adjacent to vertex (y, i, j). In both cases, we obtain a contradiction to the independence of I. Now, assume that there exists a vertex $(w, q, j) \notin I$, with $q \neq i$. Otherwise, $I = \mathrm{pr}_2^{-1}(j)$ and there is nothing to prove. As I is a dominating set, then there exists a vertex $(a, b, c) \in I$ with $a \neq w$, $b \neq q$ and $c \neq j$. As (z, i, j) belongs to I for any $z \in [n_0]$, then $b = i$, otherwise I is not an independent set. Thus, the vertices (a, i, j) and (a, i, c) belong to I. By using a similar argument as before, we can deduce that $(a, i, h) \in I$ for all $h \in [n_2]$. Therefore, we have that (z, i, j) and (a, i, h) belong to I for all $z \in [n_0]$ and for all $h \in [n_2]$ which implies, by the hypothesis that I is an independent dominating set of G, that $I = \mathrm{pr}_1^{-1}(i)$. □

Lemma 2. *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_0, n_1, n_2 \geq 2$, and let I be an independent set of G such that no two vertices in it agree in exactly two coordinates. Thus, the set I is a dominating set of G if and only if*

$$I = \{(\alpha_0, \alpha_1, \alpha_2), (\alpha_0, \beta_1, \beta_2), (\beta_0, \alpha_1, \beta_2), (\beta_0, \beta_1, \alpha_2)\},$$

for some $\alpha_i, \beta_i \in [n_i]$, with $\alpha_i \neq \beta_i$ and $i \in [3]$.

Proof. Assume first that such independent set I is also a dominating set of G. By hypothesis, I contains at least two vertices, and any pair of such vertices agreeing in exactly one coordinate. As G is vertex-transitive, we can assume w.l.o.g. that vertex $(0,0,0)$ belongs to I. By the commutativity of the direct product, we can assume that I contains also the vertex $(0,\beta_1,\beta_2)$, with $\beta_i \neq 0$ for $i = 1,2$. Furthermore, by hypothesis, I contains no vertex of the form $(0,0,z)$, for any $z \neq 0$. As I is a dominating set, then there exists $(\beta_0,b,c) \in I$ with $\beta_0 \neq 0$, $b \neq 0$ and $c \neq z$. If $c \neq 0$ then vertices $(0,0,0)$ and (β_0,b,c) are adjacent which is a contradiction to the independence of I. So $c = 0$ which implies that $b = \beta_1$, otherwise there is again a contradiction with the independence of I. Therefore, vertices $(0,0,0)$, $(0,\beta_1,\beta_2)$ and $(\beta_0,\beta_1,0)$ belong to I. Similarly, by hypothesis, I contains no vertex of the form $(0,y,0)$ for any $y \neq 0$. As I is a dominating set, there exists a vertex $(u,v,w) \in I$ with $u \neq 0$, $v \neq y$ and $w \neq 0$, which implies that vertex $(\beta_0,0,\beta_2)$ belongs to I. By hypothesis, it is clear that no other vertex different to the previous four vertices can belong to I, otherwise there is a contradiction to the independence of I.
Conversely, let $I = \{(\alpha_0,\alpha_1,\alpha_2),(\alpha_0,\beta_1,\beta_2),(\beta_0,\alpha_1,\beta_2),(\beta_0,\beta_1,\alpha_2)\}$, for some $\alpha_i,\beta_i \in [n_i]$, with $\alpha_i \neq \beta_i$ and $i \in [3]$. Clearly, I is a maximal independent set w.r.t. the property that any pair of vertices in it agree in exactly one coordinate. Suppose that there is a vertex $(x_0,x_1,x_2) \in G \setminus I$ such that it is not adjacent to any vertex in I. Thus, $x_i = \alpha_i$ for some (but not for all) $i \in [3]$. So, assume that $x_2 \neq \alpha_2$ (the other cases can be proved similarly). If $x_0 = \alpha_0$ and $x_1 = \alpha_1$ then $(\beta_0,\beta_1,\alpha_2)$ is adjacent to it. Therefore, assume that $x_1 \neq \alpha_1$. As $x_0 = \alpha_0$, then it implies that $x_1 = \beta_1$, otherwise (x_0,x_1,x_2) is adjacent to $(\beta_0,\beta_1,\alpha_2)$. But, the last implies that $x_2 = \beta_2$, otherwise (x_0,x_1,x_2) is adjacent to $(\beta_0,\alpha_1,\beta_2)$. Thus, $(x_0,x_1,x_2) = (\alpha_0,\beta_1,\beta_2) \in I$ that is a contradiction. Similarly, if we assume that $x_0 \neq \alpha_0$, $x_1 = \alpha_1$, and $x_2 \neq \alpha_2$ we obtain that $(x_0,x_1,x_2) = (\beta_0,\alpha_1,\beta_2) \in I$ that is a contradiction. Therefore, I is an independent dominating set of G. ☐

Definition 1. *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$, and let I be an independent dominating set in G. The set I is said to be of* **Type A** *if it verifies the hypothesis in Lemma 1, and it is said to be of* **Type B** *if it verifies the hypothesis in Lemma 2.*

The following result is a consequence of Lemmas 1 and 2.

Theorem 2. *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$, and let I be an independent set in G. Then, I is also a dominating set in G if and only if* *it is of Type A or Type B.*

2.2 Idomatic partitions

Definition 2. *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$, and let $G_1, G_2, \ldots, G_t$ be an idomatic t-partition of G, with $t > 1$. Such an idomatic partition is called*

- *of* **Type A***: If all independent dominating sets G_i are of Type A.*
- *of* **Type B***: If all independent dominating sets G_i are of Type B.*
- *of* **Type C***: If there is at least one independent dominating set G_i of Type A, and at least one independent dominating set G_j of Type B, with $i \neq j$.*

Theorem 3. *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$. Then, G has an idomatic n_i-partition of Type A for each $i \in [3]$. Moreover, such partitions are the only idomatic partitions of Type A of G.*

Proof. Let pr_i be the projection of G on K_{n_i}, for $i \in [3]$. It is easy to deduce that $\mathrm{pr}_i^{-1}(0), \mathrm{pr}_i^{-1}(1), \ldots, \mathrm{pr}_i^{-1}(n_i - 1)$ is an idomatic n_i-partition of G. In order to proof the second part, assume that G has an idomatic partition of Type A containing two different independent dominating sets I_i and I_j such that $I_i = K_{n_k} \times K_{n_j} \times \{\alpha_i\}$ for some fixed $\alpha_i \in [n_i]$ and $I_j = K_{n_k} \times \{\alpha_j\} \times K_{n_i}$ for some fixed $\alpha_j \in [n_j]$, where $i, j, k \in [3]$ and i, j, k pairwise different. Clearly, $I_i \cap I_j \neq \emptyset$ that is a contradiction. $\square$

Proposition 1. *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$. If G has an idomatic partition of Type B then there exist $j, k \in [3]$, with $j \neq k$, such that n_j and n_k are both even.*

Proof. By Lemma 2, we know that each part in an idomatic partition of Type B has four vertices, and thus 4 is a divisor of $n_0.n_1.n_2$. That is, there is at least one n_j, with $j \in [3]$ such that $2|n_j$. By the commutativity of the direct product, we can assume w.l.o.g. that $j = 2$. Let G_k be a part of the idomatic partition of Type B. By definition, G_k is an independent dominating set of Type B. So, let $G_k = \{(\alpha_0, \alpha_1, \alpha_2), (\alpha_0, \beta_1, \beta_2), (\beta_0, \alpha_1, \beta_2), (\beta_0, \beta_1, \alpha_2)\}$, where $\alpha_i, \beta_i \in [n_i]$ with $\alpha_i \neq \beta_i$. Fix the element $\alpha_2 \in [n_2]$. The number of vertices (x, y, α_2) in G is exactly $n_0.n_1$. Moreover, as $\alpha_i \neq \beta_i$ then, there are exactly $\frac{n_0.n_1}{2}$ parts in any idomatic partition of Type B each one containing exactly two different vertices (x, y, α_2) and (x', y', α_2), with $x \neq x'$ and $y \neq y'$. Therefore, $2|n_0.n_1$, which implies that $2|n_0$ or $2|n_1$. $\square$

Proposition 2. *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$. If there exist $j, k \in [3]$, with $j \neq k$, such that n_j and n_k are both even, then G has an idomatic partition of Type B of order $\frac{n_0.n_1.n_2}{4}$.*

Proof. As mentioned previously, the graph $G = K_{n_0} \times K_{n_1} \times K_{n_2}$ can be seen as the Cayley graph associated with the direct product group $\mathcal{G} = Z_{n_0} \times Z_{n_1} \times Z_{n_2}$ with connector set $[n_0] \setminus \{0\} \times [n_1] \setminus \{0\} \times [n_2] \setminus \{0\}$, where Z_{n_i} denotes the additive cyclic group of the integers modulo n_i. By the commutativity of the direct product, we can assume w.l.o.g. that $2|n_1$ and $2|n_2$. Let a_j be an element of order $\frac{n_j}{2}$ in the group Z_{n_j}, for $j \in \{1, 2\}$. Let $H_0 = <(1, 0, 0)>$ be the cyclic subgroup of $\mathcal{G}$ generated by the element $(1, 0, 0)$. Similarly, let $H_1 = <(0, a_1, 0)>$ and $H_2 = <(0, 0, a_2)>$ be cyclic subgroups of $\mathcal{G}$. It is easy to deduce that $H_i \cap H_j = \{(0, 0, 0)\}$ for all $i, j \in [3]$, with $i \neq j$. As $\mathcal{G}$ is an abelian group then, by using standard group theoretic concepts, it can be deduced that the set $H_0.H_1.H_2 = \{h_0 + h_1 + h_2 : h_i \in H_i \text{ for } i \in [3]\}$ is a subgroup of order $\frac{n_0.n_1.n_2}{4}$ in $\mathcal{G}$. Let P denotes the subgroup $H_0.H_1.H_2$ and let $r = \frac{n_0.n_1.n_2}{4}$. Moreover, let $P = \{p_1, p_2, \ldots, p_r\}$, where $p_1 = (0, 0, 0)$ is the identity element. The following claim can be obtained by using standard arguments in group theory.

Claim 1. *Let P be the subgroup of $\mathcal{G} = Z_{n_0} \times Z_{n_1} \times Z_{n_2}$ defined previously. For $j = 1, 2$, let a_j be the element of order $n_j/2$ in Z_{n_j} chosen in order to construct the subgroup H_j of $\mathcal{G}$. Let β_0 be any element in Z_{n_0}, with $\beta_0 \neq 0$. Moreover, for $j = 1, 2$, let β_j be any element in Z_{n_j} such that $\beta_j \notin <a_j>$. Then, $P, (0, \beta_1, \beta_2) + P, (\beta_0, 0, \beta_2) + P, (\beta_0, \beta_1, 0) + P$ is a partition of $\mathcal{G}$ into left cosets of P.*

In fact, let $D = \{(0, \beta_1, \beta_2), (\beta_0, 0, \beta_2), (\beta_0, \beta_1, 0)\}$. By construction, no element in the set D belongs to the subgroup P. Moreover, let x, y be any two different elements in D. It is easy to show that there exists no element $z \in P$ such that $x + z = y$. Otherwise, $z = (p_0, p_1, p_2) \in P$ is such that $p_1 = \pm\beta_1$ or $p_2 = \pm\beta_2$ that is a contradiction. Therefore, Claim 1 holds.

Now, for each $1 \leq i \leq r$, let $C_i = \{p_i, (0, \beta_1, \beta_2) + p_i, (\beta_0, 0, \beta_2) + p_i, (\beta_0, \beta_1, 0) + p_i : p_i \in P\}$. We want to show that $C_1, C_2, \ldots, C_r$ is an idomatic r-partition of the graph $G = K_{n_0} \times K_{n_1} \times K_{n_2}$. By using the fact that G is the Cayley graph $\mathrm{Cay}(\prod Z_{n_i}, \prod([n_i] \setminus \{0\}))$, we obtain the following claim.

Claim 2. *Let x, y, z be three vertices of G. Then, vertices $x + y$ and $x + z$ are adjacent in G if and only if vertices y and z are adjacent in G.*

Notice that, by Claim 2, each part C_i is an independent set of the graph G. Moreover, by Lemma 2, each set C_i is an independent dominating set of Type B, which completes the proof. $\square$

By Propositions 1 and 2, we obtain the following theorem.

Theorem 4. *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$. Then, G has an idomatic partition of Type B if and only if there exist $j, k \in [3]$, with $j \neq k$, such that n_j and n_k are both even.*

Example 1. *Let $G = K_2 \times K_3 \times K_4$. An idomatic 6-partition of Type B of G can be constructed as follows : let $P =< (0,0,0) > . < (0,1,0) > . < (0,0,2) >= \{p_0, p_1, p_2, p_3, p_4, p_5\}$ be a subgroup of the group $Z_2 \times Z_3 \times Z_4$, where $p_0 = (0,0,0)$, $p_1 = (0,1,0)$, $p_2 = (0,2,0)$, $p_3 = (0,0,2)$, $p_4 = (0,1,2)$, and $p_5 = (0,2,2)$. Let $x_1 = (0,1,1)$, $x_2 = (1,0,1)$, and $x_3 = (1,1,0)$. Then, $C_i = \{p_i, p_i + x_1, p_i + x_2, p_i + x_3\}$, for $i = 0, 1, \ldots, 5$, is an idomatic 6-partition of Type B of G .*

Theorem 5. *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$, and let q_1, q_2 be two positive integers. Then, G has an idomatic $(q_1 + q_2)$-partition of Type C if and only if there exists $i \in [3]$ such that $n_i - q_1 > 1$ and $K_{n_j} \times K_{n_k} \times K_{n_i - q_1}$ admits an idomatic q_2-partition of Type B, with $j, k, i \in [3]$ and j, k, i pairwise different.*

Proof. Assume first that G has an idomatic $(q_1 + q_2)$-partition of Type C, where q_1 (resp. q_2) denotes the number of independent dominating sets of Type A (resp. Type B) in such a partition. By Theorem 3, it can be deduced that the q_1 dominating sets of Type A must be all of the form $K_{n_j} \times K_{n_k} \times \{s\}$ for some $s \in K_{n_i}$ with i fix, where $j, k, i \in [3]$ and j, k, i pairwise different. So, by permuting (if necessarily) the elements in the factor K_{n_i}, we can assume w.l.o.g. that the q_1 independent dominating sets of Type A are the sets $K_{n_j} \times K_{n_k} \times \{s\}$, for $s = n_i - q_1, \ldots, n_i - 1$. Clearly, the remaining q_2 independent dominating sets of Type B induce an idomatic q_2-partition of Type B of the direct product graph $K_{n_j} \times K_{n_k} \times K_{n_i - q_1}$. Finally, note that if $n_i - q_1 = 1$, then all the independent dominating sets in the idomatic partition are of Type A, which is a contradiction, and thus, $n_i - q_1 > 1$. The other direction of the proof is trivial. $\square$

Example 2. *Let $G = K_2 \times K_3 \times K_4$. An idomatic 5-partition of Type C of G can be constructed as follows : consider first the graph $G' = K_2 \times K_2 \times K_4$ and let $P =< (0,0,0) > . < (0,0,0) > . < (0,0,1) >= \{p_0, p_1, p_2, p_3\}$ be a subgroup of the group $Z_2 \times Z_2 \times Z_4$, where $p_0 = (0,0,0)$, $p_1 = (0,0,1)$,*

$p_2 = (0,0,2)$, and $p_3 = (0,0,3)$. Let $x_1 = (0,1,1)$, $x_2 = (1,0,1)$, and $x_3 = (1,1,0)$. Then, $C'_i = \{p_i, p_i + x_1, p_i + x_2, p_i + x_3\}$, for $i = 0,1,2,3$ is an idomatic 4-partition of G' of Type B. Then, $(K_2 \times \{2\} \times K_4) \cup (\cup C'_i)$ is an idomatic 5-partition of Type C for G.

From Theorems 3, 4 and 5, we have a full characterization of the idomatic partitions of the direct product of three complete graphs as follows.

Theorem 6. *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$. If $\mathcal{I}$ is an idomatic partition of G, then $\mathcal{I}$ must be of Type A, B or C.*

By Theorem 1 (see [4]) we know that the idomatic number of the graph $G = K_{n_0} \times K_{n_1}$, with $n_0, n_1 \geq 2$, is equal to $\max\{n_0, n_1\}$. Now, having the characterization of the idomatic partitions of the direct product of three complete graphs then, by using Theorems 3, 4, 5, and Proposition 2, we can easily deduce the following corollary.

Corollary 1. *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_0, n_1, n_2 \geq 2$, and let $id(G)$ denote the idomatic number of graph G. Let $t = \max\{n_0, n_1, n_2\}$. Then,*

1. *If n_i is an odd integer for all $i \in [3]$, then $id(G) = t$.*
2. *If n_i is an even integer and $n_j \leq n_k$ are odd integers, with $i, j, k \in [3]$ and i, j and k pairwise different, then $id(G) = \max\{t, \frac{n_i.n_j.(n_k-1)}{4} + 1\}$.*
3. *If n_i and n_j are even integers, with $i, j \in [3]$ and $i \neq j$, then $id(G) = \frac{n_i.n_j.n_k}{4}$.*

3 The general case

Theorem 7. *Let $G \times H$ be the direct product graph of graphs G and H respectively. If G admits an idomatic r-partition for some $r > 0$, and if H has no isolated vertices, then $G \times H$ admits an idomatic r-partition.*

Proof. Assume that G admits an idomatic r-partition, for some positive integer r. Let $G_1, G_2, \ldots, G_r$ be such an idomatic r-partition of G. Set $S_i = G_i \times H$, for $1 \leq i \leq r$. Clearly, $\bigcup_{i=1}^{r} S_i$ is a vertex partition of the graph $G \times H$. As for each $1 \leq i \leq r$, we have that G_i is an independent dominating set in G, it follows, by the definition of direct product graph and by the hypothesis that H has at least one edge, that S_i is an independent dominating set in $G \times H$, and therefore $\bigcup_{i=1}^{r} S_i$ is an idomatic r-partition of $G \times H$. □

So, by using Theorem 7, we can directly deduce the following result.

Proposition 3. *Let $G = K_{n_0} \times K_{n_1} \times \ldots \times K_{n_t}$, with $t \geq 3$ and $n_i \geq 2$ for any $i \in [t+1]$. Let $\mathcal{J}$ be any subset of $[t+1]$. If $\prod_{i \in \mathcal{J}} K_{n_i}$ has an idomatic partition of size r, then G has an idomatic r-partition.*

Notice that Theorem 3 can be generalized as follows.

Theorem 8. *Let $G = K_{n_0} \times K_{n_1} \times \ldots \times K_{n_t}$, with $t \geq 3$ and $n_i \geq 2$ for any $i \in [t+1]$. Then, G has an idomatic n_i-partition of Type A for each $i \in [t+1]$. Moreover, such partitions are the only idomatic partitions of Type A of G.*

From Theorem 8 and Proposition 3 we are able to construct many idomatic partitions for a direct product of four or more complete graphs. However, we do not know if there exist other different types of idomatic partitions. Therefore, a full characterization of such idomatic partitions for the direct product of finitely many complete graphs remains an open question.

Acknowledgments

The author is grateful to Sandi Klavžar and the anonymous referees for their many suggestions that improved the presentation of this paper.

References

[1] B. Brešar, S. Klavžar and D. F. Rall. Dominating direct products of graphs. *Discrete Math.*, 307:1636–1642, 2007.

[2] E. J. Cockayne and S. T. Hedetniemi. Disjoint independent dominating sets in graphs. *Discrete Math.*, 15:213–222, 1976.

[3] E. J. Cockayne and S. T. Hedetniemi. Towards a theory of domination in graphs. *Networks*, 7:247–261, 1977.

[4] J. E. Dunbar, S. M. Hedetniemi, S. T. Hedetniemi, D. P. Jacobs, J. Knisely, R. C. Laskar and D. F. Rall. Fall colorings of graphs. *J. of Comb. Math. and Comb. Comput.*, 33:257–273, 2000.

[5] C. D. Godsil and G. Royle. *Algebraic graph theory.* Graduate text in mathematics: 207, Springer-Verlag, 2001.

[6] G. Hahn and C. Tardif. Graph homomorphism: structure and symmetry. *Graph Symmetry*, G. Hahn and G. Sabidussi (eds.), Kluwer Academic Publishers, 107–166, 1997.

[7] T. W. Haynes, S. T. Hedetniemi and P. J. Slater. *Fundamentals of Domination in Graphs.* Marcel Dekker, New York, 1998.

[8] W. Imrich and S. Klavžar. *Product Graphs: Structure and Recognition.* Wiley-Interscience Series in Discrete Mathematics and Optimization, 2000.

[9] J. Lee. Independent Perfect Domination Sets in Cayley Graphs. *J. of Graph Theory*, 37:213–219, 2001.