On the b-coloring of cographs and P_4-sparse graphs

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Abstract. A *b-coloring* of a graph is a coloring such that every color class admits a vertex adjacent to at least one vertex receiving each of the colors not assigned to it. The *b-chromatic number* of a graph G, denoted by $\chi_b(G)$, is the maximum number t such that G admits a b-coloring with t colors. A graph G is *b-continuous* if it admits a b-coloring with t colors, for every $t = \chi(G), \ldots, \chi_b(G)$. We define a graph G to be *b-monotonic* if $\chi_b(H_1) \geq \chi_b(H_2)$ for every induced subgraph H_1 of G, and every induced subgraph H_2 of H_1. In this work, we prove that P_4-sparse graphs (and, in particular, cographs) are b-continuous and b-monotonic. Besides, we describe a dynamic programming algorithm to compute the b-chromatic number in polynomial time within these graph classes.

Key words. b-coloring, b-continuity, b-monotonicity, cographs, P_4-sparse graphs.

1. Introduction

We consider finite undirected graphs without loops or multiple edges. A graph with only one vertex will be called *trivial*. A *coloring* (i.e. *proper coloring*) of a graph G is an assignment of colors (or natural numbers) to the vertices of G such that any two adjacent

* Partially supported by ANPCyT PICT-2007-00533 and PICT-2007-00518, and UBACyT Grants X069 and X606 (Argentina).

† Partially supported by FONDECyT Grant 1080286 and Millennium Science Institute "Complex Engineering Systems" (Chile), and ANPCyT PICT-2007-00518 and UBACyT Grant X069 (Argentina).

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vertices are assigned different colors. The smallest number t such that G admits a coloring with t colors is called the *chromatic number* of G and is denoted by $\chi(G)$.

Given a coloring of a graph G with t colors, a vertex v is said to be *dominant* or *t-dominant* if v is adjacent to at least one vertex receiving each of the $t-1$ colors not assigned to v. A *b-coloring* of a graph is a coloring such that every color class admits a dominant vertex. Note that every coloring of G with $\chi(G)$ colors is a b-coloring. The *b-chromatic number* of a graph G, denoted by $\chi_b(G)$, is the maximum number t such that G admits a b-coloring with t colors. This parameter has been introduced by R. W. Irving and D. F. Manlove [12], by considering proper colorings that are minimal with respect to a partial order defined on the set of all the partitions of the vertex set of G. They proved that determining $\chi_b(G)$ is NP-hard for general graphs, but polynomial-time solvable for trees. In [20], Kratochvil, Tuza and Voigt show that determining $\chi_b(G)$ is NP-hard even if G is a connected bipartite graph. More results on algorithmic aspects and bounds for some graph classes can be found in [6,7,19].

Recently, several related concepts concerning b-colorings of graphs have been studied in [1,8–10,17,18,21]. A graph G is defined to be *b-continuous* [8] if it admits a b-coloring with t colors, for every $t = \chi(G), \ldots, \chi_b(G)$. For example, the graph in Figure 1 is not b-continuous since it admits b-colorings with 2 colors and 4 colors, but no b-coloring with 3 colors. In [17] (see also [8]) it is proved that chordal graphs and some planar graphs are b-continuous.

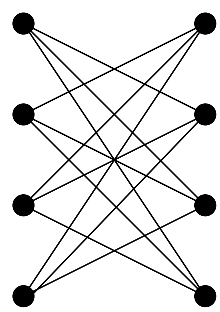

Fig. 1. A non-b-continuous graph, admitting b-colorings with 2 and 4 colors but no b-coloring with 3 colors.

Hoàng and Kouider [9] defined the concept of *b-perfectness* of a graph. A graph G is *b-perfect* if $\chi_b(H) = \chi(H)$ for every induced subgraph H of G. The property $\chi_b(G) = \chi(G)$ is not hereditary: the graph in Figure 2 has $\chi_b(G) = \chi(G) = 3$ but it contains an induced subgraph H with $\chi_b(H) = 4$ and $\chi(H) = 3$. Also a graph G is *b-imperfect* if it is not b-perfect, and *minimally b-imperfect* if it is b-imperfect and every proper induced subgraph of G is b-perfect (see [18,10,21] and references therein).

We define a graph G to be *b-monotonic* if $\chi_b(H_1) \geq \chi_b(H_2)$ for every induced subgraph H_1 of G, and every induced subgraph H_2 of H_1. This property does not hold in general, see Figure 2. Notice that, by the monotonicity of the chromatic number, both b-perfect and minimally non b-perfect graphs are b-monotonic.

A *cograph* is a P_4-free graph, i.e. a graph that does not contain a path with four vertices P_4 as an induced subgraph. A graph is *P_4-sparse* if every 5-vertex subset contains at most one P_4. Cographs and P_4-sparse graphs have been much studied. In this paper, we prove that P_4-sparse graphs (and, in particular, cographs) are b-continuous and b-monotonic. Besides, we describe a dynamic programming algorithm to compute the b-

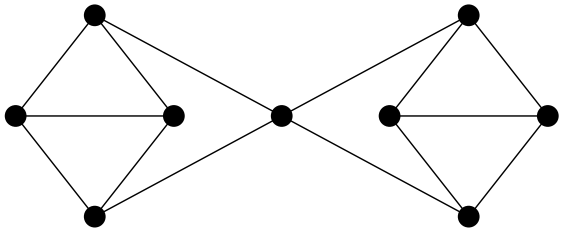

Fig. 2. A non-b-monotonic graph G. We have $\chi_b(G) = 3$, but the subgraph H obtained from G by deleting the central vertex has $\chi_b(H) = 4$.

chromatic number in polynomial time within these graph classes. These algorithms rely on the structural properties of the corresponding classes, and are based on the notion of dominance vector that we will introduce below. Before proving the results mentioned here, we shall need to introduce a few definitions and preliminary results.

1.1. Definitions and preliminary results

Given a graph G, the *dominance sequence* $\mathrm{dom}_G \in \mathbb{Z}^{\mathbb{N}_{\geq \chi(G)}}$, is defined such that $\mathrm{dom}_G[t]$ is the maximum number of distinct color classes admitting dominant vertices in any coloring of G with t colors, for every $t \geq \chi(G)$. Note that it suffices to consider this sequence until $t = |V(G)|$, since $\mathrm{dom}_G[t] = 0$ for $t > |V(G)|$. The algorithmic treatment of this sequence will be based on this observation, i.e., we shall consider the *dominance vector* $(\mathrm{dom}_G[\chi(G)], \ldots, \mathrm{dom}_G[|V(G)|])$ instead of the whole sequence. For example, the dominance vector of the graph in Figure 3 is $(3, 3, 2, 0)$.

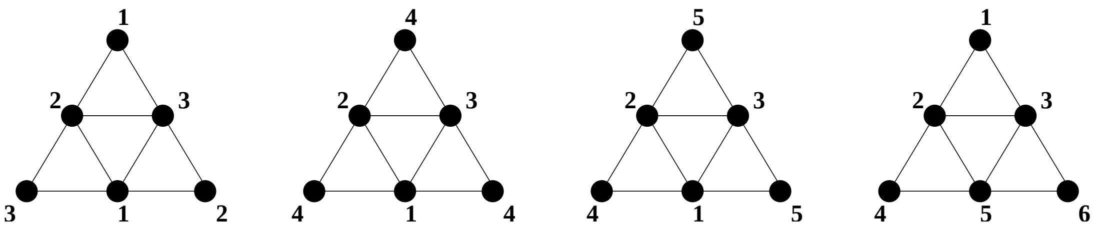

Fig. 3. A pyramid and its coloring with 3, 4, 5 and 6 colors admitting 3, 3, 2 and 0 distinct color classes with dominant vertices, respectively.

Notice that a graph G admits a b-coloring with t colors if and only if $\mathrm{dom}_G[t] = t$. Moreover, it is clear that $\mathrm{dom}_G[\chi(G)] = \chi(G)$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \emptyset$. The *union* of G_1 and G_2 is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$, and the *join* of G_1 and G_2 is the graph $G_1 \vee G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup V_1 \times V_2)$ (i.e., $\overline{G_1 \vee G_2} = \overline{G_1} \cup \overline{G_2}$).

Cographs were defined in [4]. Many NP-complete problems are polynomial time solvable on cographs; but there are some exceptions, e.g., achromatic number [3], list coloring [16], etc. The b-coloring problem on cographs was studied in [18], where b-perfect cographs have been characterized. Nevertheless, the complexity of computing the b-chromatic number of a cograph was not known. Cographs have a really nice structure, since they admit a full decomposition theorem.

Proposition 1. [4] *Every non-trivial cograph is either union or join of two smaller cographs.*

To each cograph G one can associate a corresponding decomposition rooted tree T, called the *cotree* of G, in the following way. Each non-leaf node in the tree is labeled with either "$\cup$" (union-nodes) or "$\vee$" (join-nodes) and each leaf is labeled with a vertex of G. Each non-leaf node has two or more children. Let T_x be the subtree of T rooted at node x and let V_x be the set of vertices corresponding to the leaves in T_x. Then, each node x of the cotree corresponds to the graph $G_x = (V_x, E_x)$. An union-node (join-node) corresponds to the disjoint union (join) of the cographs associated with the children of the node. Finally, the cograph that is associated with the root of the cotree is just G, the cograph represented by this cotree. The cotree associated to a cograph can be computed in linear time [5].

The chromatic number of a cograph can be recursively calculated from its cotree by applying the following result.

Theorem 1. [5] *If G is the trivial graph, then $\chi(G) = 1$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. Then,*

(i) $\chi(G_1 \cup G_2) = \max\{\chi(G_1), \chi(G_2)\}$.
(ii) $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$.

For b-coloring there is a similar result, but the relation between the b-chromatic number of two graphs and the b-chromatic number of their union is weaker.

Theorem 2. [18] *If G is the trivial graph, then $\chi_b(G) = 1$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. Then,*

(i) $\chi_b(G_1 \cup G_2) \geq \max\{\chi_b(G_1), \chi_b(G_2)\}$.
(ii) $\chi_b(G_1 \vee G_2) = \chi_b(G_1) + \chi_b(G_2)$.

The graph H in Figure 2 is an example of a graph verifying the strict inequality in Theorem 2: $\chi_b(H_1) = \chi_b(H_2) = 3$, but $\chi_b(H) = 4$.

A *spider* is a graph whose vertex set can be partitioned into S, C and R, where $S = \{s_1, \ldots, s_k\}$ ($k \geq 2$) is a stable set; $C = \{c_1, \ldots, c_k\}$ is a complete set; s_i is adjacent to c_j if and only if $i = j$ (a *thin spider*), or s_i is adjacent to c_j if and only if $i \neq j$ (a *thick spider*); R is allowed to be empty and if it is not, then all the vertices in R are adjacent to all the vertices in C and non-adjacent to all the vertices in S. Clearly, the complement of a thin spider is a thick spider, and viceversa. The triple (S, C, R) is called the *spider partition*, and can be found in linear time [13].

P_4-sparse graphs were introduced in [11]. They generalize cographs, can be recognized in linear time [13], and are a subclass of perfect graphs [11]. Besides, b-perfect P_4-sparse graphs have been characterized in [9]. P_4-sparse graphs have also a nice decomposition theorem.

Theorem 3. [11,14] *If G is a non-trivial P_4-sparse graph, then either G or $\overline{G}$ is not connected, or G is a spider.*

In fact, Theorem 3 says that if G is a non-trivial P_4-sparse graph, then either (i) $G_1, \ldots, G_p$ ($p > 1$) are the connected components of G ($\overline{G}$) and G is the disjoint union (join) of G_i's ($\overline{G_i}$'s), or (ii) G and $\overline{G}$ are connected and G is a spider.

Let G be a graph and $A \subset V(G)$. Denote by $G[A]$ the subgraph of G induced by A. In [14] it is observed that if G is a spider with vertex partition (S, C, R), then G is P_4-sparse if and only if $G[R]$ is P_4-sparse.

In [15], the following lemma is implicitly stated, which allows to recursively compute the chromatic number of a P_4-sparse graph in linear time.

Lemma 1. [15] *Let G be a spider with spider partition (S, C, R). If R is empty, then $\chi(G) = |C|$. Otherwise, $\chi(G) = |C| + \chi(G[R])$.*

The algorithm is based on the decomposition theorem and the recognition algorithm, which finds the decomposition tree in linear time.

2. b-continuity in cographs

Minimally b-imperfect cographs, i.e., graphs G such that $\chi_b(G) > \chi(G)$ but $\chi_b(H) = \chi(H)$ for every proper induced subgraph H of G, are characterized in [18]. Such graphs are the disjoint union of two diamonds and the disjoint union of three P_3. In both cases $\chi_b(G) = \chi(G) + 1$ holds. It is natural to ask whether there exist cographs with a bigger difference between their chromatic number and their b-chromatic number.

Let B_n be the graph composed by $n+1$ copies of the star $K_{1,n}$. We have that $\chi(B_n) = 2$ and $\chi_b(B_n) = n + 1$. A b-coloring with $n + 1$ colors is obtained by coloring each of the $n + 1$ central vertices with a different color and, for every star, coloring each of the n non-central vertices with a different color (such that this color does not coincide with the color assigned to the corresponding central vertex). In such a coloring, all the central vertices are $(n + 1)$-dominant, and each color class admits a dominant vertex.

Since there are cographs with arbitrarily large difference between their b-chromatic number and their chromatic number, it makes sense to analyze b-continuity in cographs.

Lemma 2. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. If G_1 and G_2 are b-continuous and $G = G_1 \cup G_2$, then G is b-continuous.*

Proof. Assume G admits a b-coloring with $t + 1$ colors such that $t + 1 > \chi(G)$. We shall show that there exists a b-coloring of G with t colors. Since $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$ by Theorem 1, then $t + 1 > \chi(G_i)$ for $i = 1, 2$. We are going to eliminate color $t + 1$ and obtain a b-coloring of G with t colors. To this end, consider the following cases for $i = 1, 2$:

(a) If G_i does not admit dominant vertices assigned the color $t + 1$, recolor every vertex $v \in G_i$ receiving color $t+1$ with some color between 1 and t not used by any neighbor of v.
(b) If G_i admits dominant vertices assigned the color $t+1$ but no dominant vertex assigned color j for some $j \neq t + 1$, then swap the colors $t + 1$ and j, and then resort to Case (a), since now there are no dominant vertices with color $t + 1$ in G_i.
(c) If G_i admits dominant vertices for every color, then we have a b-coloring of G_i with $t + 1$ colors, where $t + 1 > \chi(G_i)$. Since G_i is b-continuous, there exists a b-coloring of G_i with t colors.

After these operations, it is clear that the resulting coloring is a b-coloring of G with t colors (since we have yet dominant vertices with every color from 1 to t). □

Lemma 3. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. If G_1 and G_2 are b-continuous and $G = G_1 \vee G_2$, then G is b-continuous.*

Proof. Assume G admits a b-coloring with $t+1$ colors such that $t+1 > \chi(G)$. We shall show that there exists a b-coloring of G with t colors. We have that $\chi(G) = \chi(G_1)+\chi(G_2)$ and $\chi_b(G) = \chi_b(G_1) + \chi_b(G_2)$ by Theorems 1 and 2. Furthermore, any b-coloring of G_1 and G_2 generates a b-coloring of G by renaming the colors assigned to G_2 starting by the largest color assigned to G_1 plus one, and any b-coloring of G restricted to G_1 (resp. G_2) is also a b-coloring. Therefore, in the b-coloring of G with $t+1$ colors, either G_1 or G_2 (perhaps both) is colored with more colors than its chromatic number. Suppose without loss of generality that this is the case for G_1. By restricting the coloring of G to G_1, we obtain a b-coloring of G_1 with $k+1$ colors such that $k+1 > \chi(G_1)$. Since G_1 is b-continuous, there exists a b-coloring of G_1 with k colors. Combine this coloring with the original b-coloring of G restricted to G_2, thus constructing a b-coloring of G with t colors. □

Theorem 4. *Cographs are b-continuous.*

Proof. We proceed by induction, using Proposition 1, Lemma 2, and Lemma 3, since the trivial graph is b-continuous. □

3. A polynomial time algorithm for b-coloring cographs

Theorem 2 does not lead to an algorithm to compute the b-chromatic number of a cograph. In fact, it is not difficult to build examples showing that the b-chromatic number of the graph $G_1 \cup G_2$ does not depend only on the b-chromatic numbers of G_1 and G_2. For this reason, the notion of dominance vector is introduced. Our goal is to recursively compute this vector using the decomposition theorem for cographs, hence obtaining the b-chromatic number of the graph as the maximum t such that $\mathrm{dom}_G[t] = t$.

Theorem 5. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. If $G = G_1 \cup G_2$ and $t \geq \chi(G)$, then*

$$\mathrm{dom}_G[t] = \min\{t, \mathrm{dom}_{G_1}[t] + \mathrm{dom}_{G_2}[t]\}.$$

Proof. Let $t \geq \chi(G)$. If $t > |V(G)|$, then $t > |V(G_1)|$ and $t > |V(G_2)|$, hence $\mathrm{dom}_G[t] = 0 = \min\{t, \mathrm{dom}_{G_1}[t] + \mathrm{dom}_{G_2}[t]\}$. If $t \leq |V(G)|$, we take a coloring of G with t colors and $\mathrm{dom}_G[t]$ color classes with dominant vertices. Let a_1 be the number of color classes with dominant vertices in G_1, and let a_2 be the number of color classes with dominant vertices in G_2 not having dominant vertices in G_1. Then $\mathrm{dom}_G[t] = a_1+a_2$. Notice that, for $i = 1, 2$, if $a_i > 0$ then the t colors are used in G_i. Therefore, by restricting the coloring to G_1 (resp. G_2) we obtain $\mathrm{dom}_{G_1}[t] \geq a_1$ and $\mathrm{dom}_{G_2}[t] \geq a_2$, so $\mathrm{dom}_G[t] \leq \mathrm{dom}_{G_1}[t]+\mathrm{dom}_{G_2}[t]$. Since clearly $\mathrm{dom}_G[t] \leq t$, we conclude $\mathrm{dom}_G[t] \leq \min\{t, \mathrm{dom}_{G_1}[t] + \mathrm{dom}_{G_2}[t]\}$.

On the other hand, since $t \geq \chi(G)$ then $t \geq \chi(G_1)$ and $t \geq \chi(G_2)$. If $t > |V(G_1)|$ and $t > |V(G_2)|$, then $\mathrm{dom}_G[t] = 0 = \min\{t, \mathrm{dom}_{G_1}[t] + \mathrm{dom}_{G_2}[t]\}$. If $t > |V(G_1)|$ but $t \leq |V(G_2)|$, then $\mathrm{dom}_{G_1}[t] = 0$ and $\mathrm{dom}_G[t] = \mathrm{dom}_{G_2}[t] = \min\{t, \mathrm{dom}_{G_1}[t] + \mathrm{dom}_{G_2}[t]\}$ holds. If $t \leq |V(G_1)|$ and $t \leq |V(G_2)|$, take a coloring of G_1 (resp. a coloring of G_2) with t colors and $\mathrm{dom}_{G_1}[t]$ (resp. $\mathrm{dom}_{G_2}[t]$) color classes with dominant vertices. If $\mathrm{dom}_{G_1}[t] + \mathrm{dom}_{G_2}[t] \leq t$, then we can rename the colors in G_2 in such a way that the dominant vertices use $\mathrm{dom}_{G_2}[t]$ color classes differing from the $\mathrm{dom}_{G_1}[t]$ color classes with dominant vertices in G_1. This implies $\mathrm{dom}_G[t] \geq \mathrm{dom}_{G_1}[t]+\mathrm{dom}_{G_2}[t] = \min\{t, \mathrm{dom}_{G_1}[t]+\mathrm{dom}_{G_2}[t]\}$. If $\mathrm{dom}_{G_1}[t] + \mathrm{dom}_{G_2}[t] > t$, then we can rename the colors in G_2 in such a way that the dominant vertices use the $t - \mathrm{dom}_{G_1}[t]$ color classes differing from the $\mathrm{dom}_{G_1}[t]$ color classes with dominant vertices in G_1 plus some additional colors. We conclude, therefore, that $\mathrm{dom}_G[t] \geq t = \min\{t, \mathrm{dom}_{G_1}[t] + \mathrm{dom}_{G_2}[t]\}$. □

Theorem 6. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. Let $G = G_1 \vee G_2$ and $\chi(G) \leq t \leq |V(G)|$. Let $a = \max\{\chi(G_1), t - |V(G_2)|\}$ and $b = \min\{|V(G_1)|, t - \chi(G_2)\}$. Then $a \leq b$ and*

$$\mathrm{dom}_G[t] = \max_{a \leq j \leq b} \{\mathrm{dom}_{G_1}[j] + \mathrm{dom}_{G_2}[t-j]\}.$$

Proof. We will show first four inequalities that imply $a \leq b$. By Theorem 1, $\chi(G_1) + \chi(G_2) = \chi(G) \leq t$, so $\chi(G_1) \leq t - \chi(G_2)$; on the other hand, $\chi(G_1) \leq |V(G_1)|$ and $\chi(G_2) \leq |V(G_2)|$, so $t - |V(G_2)| \leq t - \chi(G_2)$; finally, $t \leq |V(G)| = |V(G_1)| + |V(G_2)|$, so $t - |V(G_2)| \leq |V(G_1)|$.

Consider any t-coloring of G with $\mathrm{dom}_G[t]$ color classes with dominant vertices. Let t_1 (resp. t_2) be the number of colors used by this coloring in G_1 (resp. G_2), and let a_1 (resp. a_2) be the number of color classes with dominant vertices in G_1 (resp. G_2). Notice that any coloring of G assigns disjoint color sets to G_1 and G_2, so $t = t_1 + t_2$ and $\mathrm{dom}_G[t] = a_1 + a_2$. Since the t_1 colors from G_1 are not used in G_2, the t-dominant vertices in G which are in G_1 are t_1-dominant vertices in the coloring restricted to G_1, hence $a_1 \leq \mathrm{dom}_{G_1}[t_1]$. Similarly, the t-dominant vertices in G which are in G_2 are t_2-dominant vertices in the coloring restricted to G_2. Since $t_2 = t - t_1$, we obtain $a_2 \leq \mathrm{dom}_{G_2}[t - t_1]$, implying $\mathrm{dom}_G[t] = a_1 + a_2 \leq \mathrm{dom}_{G_1}[t_1] + \mathrm{dom}_{G_2}[t - t_1]$. As $t_1 \geq \chi(G_1)$, $t_1 \geq t - |V(G_2)|$, $t_1 \leq |V(G_1)|$ and $t_1 \leq t - \chi(G_2)$, then $a \leq t_1 \leq b$. Therefore, we have $\mathrm{dom}_G[t] \leq \max_{a \leq j \leq b}\{\mathrm{dom}_{G_1}[j] + \mathrm{dom}_{G_2}[t-j]\}$.

Consider any t_1 such that $a \leq t_1 \leq b$. Since $\chi(G_1) \leq t_1 \leq |V(G_1)|$, there exists some coloring of G_1 with exactly t_1 colors. Take any such coloring having $\mathrm{dom}_{G_1}[t_1]$ color classes with dominant vertices. Let $t_2 = t - t_1$. Since $\chi(G_2) \leq t_2 \leq |V(G_2)|$, there exists some coloring of G_2 with exactly t_2 colors. Take any such coloring having $\mathrm{dom}_{G_2}[t_2]$ color classes with dominant vertices, and rename these t_2 colors in such a way that only colors in $\{t_1 + 1, \ldots, t\}$ are used in the new coloring. By combining these two colorings for G_1 and G_2, we obtain a coloring of G with exactly t colors. Each t_1-dominant vertex in G_1 has in G all the vertices in G_2 as neighbors, hence it admits a neighbor with every color in $\{t_1 + 1, \ldots, t\}$ and, therefore, it is a t-dominant vertex in G. Similarly, each t_2-dominant vertex in G_2 can be shown to be t-dominant in G. Conversely, every t-dominant vertex in G is either t_1-dominant in G_1 or t_2-dominant in G_2. Moreover, since the color sets corresponding to G_1 and G_2 are disjoint, the number of t-dominant vertices in such a coloring of G is $\mathrm{dom}_{G_1}[t_1] + \mathrm{dom}_{G_2}[t_2]$. Therefore, $\mathrm{dom}_G[t] \geq \max_{a \leq j \leq b}\{\mathrm{dom}_{G_1}[j] + \mathrm{dom}_{G_2}[t-j]\}$. □

Theorem 7. *The dominance vector and the b-chromatic number of a cograph can be computed in $O(n^3)$ time.*

Proof. The previous results give a dynamic programming algorithm to compute the dominance vector of a cograph from its cotree. If $G = G_1 \cup G_2$ (as in Theorem 5) the value of $\mathrm{dom}_G[t]$ is obtained directly from $\mathrm{dom}_{G_1}[t]$ and $\mathrm{dom}_{G_2}[t]$. If $G = G_1 \vee G_2$ (as in Theorem 6), then at most n values of j must be examined. Moreover, each of these two theorems reduces the computation of $\mathrm{dom}_G[t]$ to the computation on two disjoint subgraphs. Thus there are at most n occurrences of such reduction steps. In total, the computation time is

$O(n^2)$ for every value of t, and so $O(n^3)$ for all possible values of t. From the dominance vector of a graph G, the b-chromatic number can be computed easily as the maximum t such that $\mathrm{dom}_G[t] = t$. □

4. b-monotonicity in cographs

The monotonicity on induced subgraphs is a desirable property that holds for many known optimization parameters of a graph, like chromatic number, maximum clique, maximum degree. This is not the case for the b-chromatic number in general, so it is interesting to analyze the monotonicity of the b-chromatic number within different classes of graphs. In this section we study the b-monotonicity in cographs. We first state some preliminary properties of the dominance vector of a graph, and then use the decomposition theorem to analyze the b-monotonicity in this class of graphs.

Lemma 4. *If G is a graph and $t \geq \chi(G)$, then either $\mathrm{dom}_G[t+1] = t+1$ or $\mathrm{dom}_G[t+1] \leq \mathrm{dom}_G[t]$.*

Proof. If $t+1 > |V(G)|$, then $0 = \mathrm{dom}_G[t+1] \leq \mathrm{dom}_G[t]$. Assume, therefore, $t+1 \leq |V(G)|$ and $\mathrm{dom}_G[t+1] < t+1$. Take any coloring of G with $t+1$ colors having $\mathrm{dom}_G[t+1]$ color classes with dominant vertices. Since $\mathrm{dom}_G[t+1] < t+1$, there exists some color class with no dominant vertices, say the color $t+1$. For every vertex v with color $t+1$, change the color of v to any color in $\{1,\ldots,t\}$ not used by any of the neighbors of v. The resulting coloring is a coloring of G with t colors. Note that every dominant vertex in the original coloring is dominant in the new coloring, and that the number of color classes with dominant vertices in the new coloring is at least the same. Therefore, $\mathrm{dom}_G[t] \geq \mathrm{dom}_G[t+1]$. □

A direct consequence of this lemma is the following.

Corollary 1. *Let G be a graph. The maximum value of $\mathrm{dom}_G[t]$ is attained in $t = \chi_b(G)$.*

Lemma 5. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$, and let $G = G_1 \cup G_2$. Assume that for every $t \geq \chi(G_i)$ and every induced subgraph H of G_i we have $\mathrm{dom}_H[t] \leq \mathrm{dom}_{G_i}[t]$, for $i = 1, 2$. Then, for every $t \geq \chi(G)$ and every induced subgraph H of G, $\mathrm{dom}_H[t] \leq \mathrm{dom}_G[t]$ holds.*

Proof. Let H be an induced subgraph of G and let $t \geq \chi(G)$. By Theorem 5, we have $\mathrm{dom}_G[t] = \min\{t, \mathrm{dom}_{G_1}[t] + \mathrm{dom}_{G_2}[t]\}$. If $\mathrm{dom}_G[t] = t$, then $\mathrm{dom}_H[t] \leq \mathrm{dom}_G[t]$ clearly holds. Assume, therefore, $\mathrm{dom}_G[t] = \mathrm{dom}_{G_1}[t] + \mathrm{dom}_{G_2}[t]$. If H is completely contained in G_i, for $i = 1$ or $i = 2$, then $\mathrm{dom}_H[t] \leq \mathrm{dom}_{G_i}[t] \leq \mathrm{dom}_G[t]$. Otherwise, $H = H_1 \cup H_2$, where H_i is an induced subgraph of G_i, for $i = 1, 2$. By the hypothesis, $\mathrm{dom}_{H_i}[t] \leq \mathrm{dom}_{G_i}[t]$, hence $\mathrm{dom}_{H_1}[t] + \mathrm{dom}_{H_2}[t] \leq \mathrm{dom}_{G_1}[t] + \mathrm{dom}_{G_2}[t]$. Therefore, we conclude $\mathrm{dom}_H[t] \leq \mathrm{dom}_G[t]$. □

Lemma 6. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two b-continuous graphs such that $V_1 \cap V_2 = \emptyset$, and let $G = G_1 \vee G_2$. Assume that for every $t \geq \chi(G_i)$ and for every induced subgraph H of G_i we have $\mathrm{dom}_H[t] \leq \mathrm{dom}_{G_i}[t]$, for $i = 1, 2$. Then, for every $t \geq \chi(G)$ and for every induced subgraph H of G, $\mathrm{dom}_H[t] \leq \mathrm{dom}_G[t]$ holds.*

Proof. Let H be an induced subgraph of G, and let $t \geq \chi(G)$. By hypothesis, G_1 and G_2 are b-continuous. So, by Theorem 3, we have that G is b-continuous. Hence it suffices to consider $t > \chi_b(G)$, otherwise $t = \mathrm{dom}_G[t] \geq \mathrm{dom}_H[t]$. Recall that, by Theorem 2, $\chi_b(G) = \chi_b(G_1) + \chi_b(G_2)$, so $t > \chi_b(G_1) + \chi_b(G_2)$. By Theorem 6, we have

$\mathrm{dom}_G[t] = \max_{a \le j \le b}\{\mathrm{dom}_{G_1}[j] + \mathrm{dom}_{G_2}[t-j]\}$, where $a = \max\{\chi(G_1), t - |V(G_2)|\}$ and $b = \min\{|V(G_1)|, t - \chi(G_2)\}$, and $a \le b$ holds.

If H is completely contained in G_1 or G_2, say G_1, by the hypothesis we have $\mathrm{dom}_H[t] \le \mathrm{dom}_{G_1}[t]$. Let $j' = \max\{a, \chi_b(G_1)\}$. We know that $a \le b$, and $\chi_b(G_1) \le |V(G_1)|$. Furthermore, $\chi_b(G_1) < t - \chi_b(G_2) \le t - \chi(G_2)$, hence $a \le j' \le b$. Finally, $t > \chi_b(G_1) \ge \chi(G_1)$ and clearly $t \ge t - |V(G_2)|$, hence $t \ge j'$. Since $t \ge j' \ge \chi_b(G_1)$ and $\mathrm{dom}_{G_1}[t] < t$, by Lemma 4, $\mathrm{dom}_{G_1}[t] \le \mathrm{dom}_{G_1}[j']$, and since $\mathrm{dom}_{G_2}[t-j'] \ge 0$, we have $\mathrm{dom}_G[t] = \max_{a \le j \le b}\{\mathrm{dom}_{G_1}[j] + \mathrm{dom}_{G_2}[t-j]\} \ge \mathrm{dom}_{G_1}[j'] + \mathrm{dom}_{G_2}[t-j'] \ge \mathrm{dom}_{G_1}[j'] \ge \mathrm{dom}_{G_1}[t] \ge \mathrm{dom}_H[t]$. Therefore, $\mathrm{dom}_H[t] \le \mathrm{dom}_G[t]$.

If H is not completely contained in G_1 or G_2, then $H = H_1 \vee H_2$, where H_i is an induced subgraph of G_i, for $i = 1, 2$. By the hypothesis, $\mathrm{dom}_{H_i}[j] \le \mathrm{dom}_{G_i}[j]$ for each $j \ge \chi(G_i)$. By Theorem 6, we have $\mathrm{dom}_H[t] = \max_{a' \le j \le b'}\{\mathrm{dom}_{H_1}[j] + \mathrm{dom}_{H_2}[t-j]\}$, where $a' = \max\{\chi(H_1), t - |V(H_2)|\}$ and $b' = \min\{|V(H_1)|, t - \chi(H_2)\}$, and $a' \le b'$ holds. Let $j' \in \{a', \dots, b'\}$ be the color realizing such maximum, and consider the following three possible cases:

(a) If $a \le j' \le b$, then $\mathrm{dom}_{H_1}[j'] \le \mathrm{dom}_{G_1}[j']$ and $\mathrm{dom}_{H_2}[t-j'] \le \mathrm{dom}_{G_2}[t-j']$, hence $\mathrm{dom}_G[t] = \max_{a \le j \le b}\{\mathrm{dom}_{G_1}[j] + \mathrm{dom}_{G_2}[t-j]\} \ge \mathrm{dom}_{G_1}[j'] + \mathrm{dom}_{G_2}[t-j'] \ge \mathrm{dom}_{H_1}[j'] + \mathrm{dom}_{H_2}[t-j'] = \mathrm{dom}_H[t]$.

(b) If $j' < a$, then in particular $a' < a$ and, since $t - |V(H_2)| \ge t - |V(G_2)|$, we have $a = \chi(G_1)$. Therefore, $\mathrm{dom}_{H_1}[j'] \le j' < a = \mathrm{dom}_{G_1}[a]$. Since $t > \chi_b(G_1) + \chi_b(G_2)$ and $a \le \chi_b(G_1)$, it holds $t - a > \chi_b(G_2)$. Since $t - j' > t - a > \chi_b(G_2)$, then Lemma 4 implies $\mathrm{dom}_{G_2}[t-j'] \le \mathrm{dom}_{G_2}[t-a]$. Finally, as $\mathrm{dom}_{H_2}[t-j'] \le \mathrm{dom}_{G_2}[t-j']$, we obtain $\mathrm{dom}_G[t] = \max_{a \le j \le b}\{\mathrm{dom}_{G_1}[j] + \mathrm{dom}_{G_2}[t-j]\} \ge \mathrm{dom}_{G_1}[a] + \mathrm{dom}_{G_2}[t-a] \ge \mathrm{dom}_{H_1}[j'] + \mathrm{dom}_{H_2}[t-j'] = \mathrm{dom}_H[t]$.

(c) If $j' > b$ the argumentation is similar. We have $b' > b$ and, since $|V(H_1)| \le |V(G_1)|$, we have $b = t - \chi(G_2)$. Therefore, $\mathrm{dom}_{H_2}[t-j'] \le t - j' < t - b = \chi(G_2) = \mathrm{dom}_{G_2}[t-b]$. Following the same argument as in Case (b), we conclude that $\mathrm{dom}_{H_1}[j'] \le \mathrm{dom}_{G_1}[b]$, hence $\mathrm{dom}_G[t] = \max_{a \le j \le b}\{\mathrm{dom}_{G_1}[j] + \mathrm{dom}_{G_2}[t-j]\} \ge \mathrm{dom}_{G_1}[b] + \mathrm{dom}_{G_2}[t-b] \ge \mathrm{dom}_{H_1}[j'] + \mathrm{dom}_{H_2}[t-j'] = \mathrm{dom}_H[t]$.

In the three cases we obtain $\mathrm{dom}_H[t] \le \mathrm{dom}_G[t]$. □

Theorem 8. *Cographs are b-monotonic.*

Proof. As cographs are hereditary, it is enough to prove that given a cograph G, $\chi_b(G) \ge \chi_b(H)$, for every induced subgraph H of G. By applying Proposition 1, Theorem 4, Lemma 5, and Lemma 6, an induction argument shows that for every cograph G, every $t \ge \chi(G)$, and every induced subgraph H of G, $\mathrm{dom}_H[t] \le \mathrm{dom}_G[t]$ holds. Let G be a cograph, and let H be an induced subgraph of G. If $\chi_b(H) < \chi(G)$, then $\chi_b(H) < \chi_b(G)$. Otherwise, $\chi_b(H) = \mathrm{dom}_H[\chi_b(H)] \le \mathrm{dom}_G[\chi_b(H)]$, and by Corollary 1 $\mathrm{dom}_G[\chi_b(H)] \le \mathrm{dom}_G[\chi_b(G)] = \chi_b(G)$. Hence $\chi_b(G) \ge \chi_b(H)$. □

5. P_4-sparse graphs

In this section we extend the results about cographs to a superclass of them: P_4-sparse graphs. For the b-chromatic number of a spider, a result similar to Lemma 1 can be proved.

Lemma 7. *Let G be a spider with spider partition (S, C, R). If R is empty then $\chi_b(G) = |C|$. Otherwise, $\chi_b(G) = |C| + \chi_b(G[R])$.*

Proof. Let G be a spider with spider partition (S, C, R), where $|S| = |C| = k \geq 2$. If R is empty then $\chi(G) = k$, and the vertices in S have degree at most $k-1$, thus they cannot be dominant in a coloring with more than k colors. So, $\chi_b(G) = k = |C|$. Assume now that R is non-empty. Then, by Lemma 1, $\chi_b(G) \geq \chi(G) = k + \chi(G[R]) \geq k+1$. Any b-coloring of $G[R]$ with p colors generates a b-coloring of G with $p+k$ colors, by using k new colors on C and coloring each vertex in S with a color used by a non-neighbor of it in C, thus $\chi_b(G) \geq k + \chi_b(G[R])$; conversely, any b-coloring of G with t colors, when restricted to $G[R]$ is also a b-coloring with $t-k$ colors, since the color sets used in C and R are disjoint and vertices in S cannot be dominant in a coloring with more than k colors, so $\chi_b(G) \leq k + \chi_b(G[R])$. Hence, the lemma holds. □

Nevertheless, in order to compute recursively the b-chromatic number of a P_4-sparse graph, we will need to calculate the dominance vector of a spider instead.

Theorem 9. *Let G be a spider with spider partition (S, C, R), and $k = |S| = |C| \geq 2$.*

(a) *If R is empty and G is a thin spider, then $\mathrm{dom}_G[k] = \mathrm{dom}_G[k+1] = k$, and $\mathrm{dom}_G[j] = 0$ for $j > k+1$.*
(b) *If R is non-empty and G is a thin spider, then $\mathrm{dom}_G[k+r] = k + dom_{G[R]}[r]$ for $\chi(G[R]) \leq r \leq |R|$, $\mathrm{dom}_G[k+|R|+1] = k$, and $\mathrm{dom}_G[j] = 0$ for $j > k+|R|+1$.*
(c) *If R is empty and G is a thick spider, then $\mathrm{dom}_G[k+s] = \min\{k, 2k-2s\}$ for $0 \leq s \leq k$, and $\mathrm{dom}_G[j] = 0$ for $j > 2k$.*
(d) *If R is non-empty and G is a thick spider, then $\mathrm{dom}_G[k+r] = k + dom_{G[R]}[r]$ for $\chi(G[R]) \leq r \leq |R|$, $\mathrm{dom}_G[k+|R|+s] = \min\{k, 2k-2s\}$ for $1 \leq s \leq k$, and $\mathrm{dom}_G[j] = 0$ for $j > 2k+|R|$.*

Proof. Let G be a spider with spider partition (S, C, R), and $k = |S| = |C| \geq 2$. Let $C = \{c_1, \ldots, c_k\}$ and $S = \{s_1, \ldots, s_k\}$.

(a) If R is empty and G is a thin spider, then $\chi(G) = k$, implying $\mathrm{dom}_G[k] = k$. The k vertices in C have degree k and the vertices in S have degree 1, hence $\mathrm{dom}_G[k+1] \leq k$ and $\mathrm{dom}_G[j] = 0$ for $j > k+1$, since G does not admit any vertex with degree at least $k+1$. Finally, a coloring of G with $k+1$ colors and k colors with dominant vertices can be obtained by assigning colors 1 to k to the vertices in C, and color $k+1$ to the vertices in S.
(b) If R is non-empty and G is a thin spider, then, by Lemma 1, $\chi(G) = k + \chi(G[R])$, implying $\mathrm{dom}_G[k+\chi(G[R])] = k + \chi(G[R])$. The k vertices $c_1, \ldots, c_k$ in C have degree $k+|R|$, the vertices in R have degree at most $k+|R|-1$ and the vertices in S have degree 1, hence $\mathrm{dom}_G[k+|R|+1] \leq k$ and $\mathrm{dom}_G[j] = 0$ for $j > k+|R|+1$. On the other hand, a coloring of G with $k+|R|+1$ colors and k color classes with dominant vertices can be obtained by assigning the colors $1, \ldots, k$ to the vertices in C, the colors $k+1, \ldots, k+|R|$ to the vertices of R, and the color $k+|R|+1$ to the vertices of S. For $\chi(G[R]) < r \leq |R|$, in a coloring with $k+r$ colors, the vertices of S cannot be dominant. Moreover, if they use a color non present in $C \cup R$, then at most the k vertices in C can be dominant. Suppose now that all the colors used in S are also present in $C \cup R$. Then all the vertices in C are dominant, and they have pairwise different colors. In fact, they are dominant also in the coloring restricted to $G[C \cup R]$ as well as the dominant vertices

in R, since there are no edges between R and S. Besides, any coloring of $G[C \cup R]$ can be extended to G without introducing new colors. So, $\mathrm{dom}_G[k+r] = \mathrm{dom}_{G[C\cup R]}[k+r]$, and, by Theorem 6, $\mathrm{dom}_G[k+r] = k + dom_{G[R]}[r]$.

(c) If R is empty and G is a thick spider, then $\chi(G) = k$, implying $\mathrm{dom}_G[k] = k$. Furthermore, the vertices in S have degree $k-1$ and the vertices in C have degree $2k-2$, hence $\mathrm{dom}_G[j] = 0$ for $j \geq 2k$. Finally, for $s = 1,\dots,k$, the vertices in S cannot be dominant in a coloring of G with $k+s$ colors, thus $\mathrm{dom}_G[k+s] \leq k$. In any coloring of G the vertices in C are assigned pairwise different colors, say the colors $1,\dots,k$. Moreover, the vertex c_i is dominant if and only if the color assigned to s_i is also assigned to some other vertex in G. In a coloring of G with $k+s$ colors, at least s vertices from S must be assigned the s colors between $k+1$ and $k+s$. By symmetry, without loss of generality we may assume that $s_1,\dots,s_s$ are assigned the colors $k+1,\dots,k+s$. If $k < 2s$ then at least $s-(k-s)$ of them get a color not used by any other vertex in G. This implies $\mathrm{dom}_G[k+s] \leq k - (s-(k-s)) = 2k-2s$. As in the case $k \geq 2s$ we have $2k-2s \geq k$ and this already is an upper bound for $\mathrm{dom}_G[k+s]$, we obtain $\mathrm{dom}_G[k+s] \leq \min\{k, 2k-2s\}$. A coloring attaining this bound is obtained by assigning the colors $1,\dots,k$ to the vertices in C, and the colors $k+1,\dots,k+s$ to the vertices $s_1,\dots,s_s$. If $k \geq 2s$, the vertices $s_{s+1},\dots,s_{2s}$ receive the colors $k+1,\dots,k+s$, and for $i > 2s$, s_i gets the same color as c_i. In this case, we have k color classes with dominant vertices, since every vertex from C is dominant. If $k < 2s$, the vertices $s_{s+1},\dots,s_k$ are assigned the colors $k+1,\dots,2k-s$. Here, we get $2k-2s$ color classes with dominant vertices ($\{c_1,\dots,c_{k-s}\} \cup \{c_{s+1},\dots,c_k\}$ are dominant vertices). Therefore, $\mathrm{dom}_G[k+s] = \min\{k, 2k-2s\}$.

(d) If R is non-empty and G is a thick spider, then, by Lemma 1, $\chi(G) = k + \chi(G[R])$, implying $\mathrm{dom}_G[k+\chi(G[R])] = k+\chi(G[R])$. Furthermore, the vertices in S have degree $k-1$, the vertices in C have degree $2k+|R|-2$, and the vertices in R have degree at most $k+|R|-1$, hence $\mathrm{dom}_G[j] = 0$ for $j \geq 2k+|R|$. For $s = 1,\dots,k$, neither the vertices in S nor the vertices in R can be dominant in a coloring of G with $k+|R|+s$ colors, hence $\mathrm{dom}_G[k+|R|+s] \leq k$. In any coloring of G, the vertices from C are assigned pairwise different colors, say the colors $1,\dots,k$. Moreover, the vertex c_i is dominant if and only if the color assigned to s_i is also assigned to some other vertex in G. In a coloring of G with $k+|R|+s$ colors, since the vertices in R can use at most $|R|$ colors, say $k+1,\dots,k+|R|$, then at least s vertices from S must be assigned the s colors between $k+|R|+1$ and $k+|R|+s$. By symmetry, without loss of generality we may assume that $s_1,\dots,s_s$ are assigned the colors $k+|R|+1,\dots,k+|R|+s$. If $k < 2s$, at least $s-(k-s)$ of them are assigned a color not used by any other vertex in G. Therefore, $\mathrm{dom}_G[k+|R|+s] \leq k-(s-(k-s)) = 2k-2s$. As in the case $k \geq 2s$ we have $2k-2s \geq k$ and this already is an upper bound $\mathrm{dom}_G[k+|R|+s]$, we obtain $\mathrm{dom}_G[k+|R|+s] \leq \min\{k, 2k-2s\}$. A coloring attaining this bound can be constructed by assigning the colors $1,\dots,k$ to the vertices in C, the colors $k+1,\dots,k+|R|$ to the vertices in R, and the colors $k+|R|+1,\dots,k+|R|+s$ to the vertices $s_1,\dots,s_s$. If $k \geq 2s$, the vertices $s_{s+1},\dots,s_{2s}$ are assigned the colors $k+|R|+1,\dots,k+|R|+s$, and s_i gets the same color as c_i, for $i > 2s$. In this case, we obtain k color classes with dominant vertices, since all the vertices in C are dominant. If $k < 2s$, the vertices $s_{s+1},\dots,s_k$ are assigned the colors $k+|R|+1,\dots,2k+|R|-s$. In this case, we have $2k-2s$ color classes with dominant vertices ($\{c_1,\dots,c_{k-s}\} \cup \{c_{s+1},\dots,c_k\}$ are dominant vertices). Therefore, $\mathrm{dom}_G[k+1+s] = \min\{k, 2k-2s\}$. For $\mathrm{dom}_G[k+r]$ with $\chi(G[R]) < r \leq |R|$, we can use the same argumentation as in case (b), so $\text{dom}_G[k+r] = \text{dom}_{G[C\cup R]}[k+r] = k + dom_{G[R]}[r]$.

□

Theorem 10. *The dominance vector and the b-chromatic number of a P_4-sparse graph can be computed in $O(n^3)$ time.*

Proof. By combining Theorem 5, Theorem 6, Theorem 3, and Theorem 9, and since P_4-sparse graphs are a hereditary class, we can recursively calculate the dominance vector and, consequently, the b-chromatic number of a P_4-sparse graph in $O(n^3)$ time. The complexity analysis is the same as for Theorem 7, noting that for the base cases (spiders) the computation of $\text{dom}_G[t]$ for each value of t is given directly in the proof Theorem 9. □

Now, we study the b-continuity on P_4-sparse graphs.

Theorem 11. *P_4-sparse graphs are b-continuous.*

Proof. We proceed by induction, using Theorem 3 and Lemmas 2 and 3. So, it remains to analyze the case of spiders. Suppose that G is a spider P_4-sparse graph, with spider partition (S, C, R), where $|S| = |C| = k \geq 2$. Assume G admits a b-coloring with $t + 1$ colors such that $t + 1 > \chi(G)$. We shall show that there exists a b-coloring of G with t colors. We have that $\chi(G) = k + \chi(G[R])$ and $\chi_b(G) = k + \chi_b(G[R])$ by Lemmas 1 and 7. So, R must be non-empty and $\chi_b(G[R]) > \chi(G[R])$. As observed in the proof of Lemma 7, any b-coloring of $G[R]$ with p colors generates a b-coloring of G with $p+k$ colors and, conversely, any b-coloring of G restricted to $G[R]$ is also a b-coloring. Therefore, by restricting the b-coloring of G with $t + 1$ colors to $G[R]$, we obtain a b-coloring of $G[R]$ with $t+1-k$ colors, and $t+1-k > \chi(G[R])$. Since P_4-sparse is a hereditary graph class, $G[R]$ is a P_4-sparse graph, and by inductive hypothesis, there exists a b-coloring of $G[R]$ with $t - k$ colors. As observed before, this b-coloring of $G[R]$ generates a b-coloring of G with t colors. □

Finally, we analyze the b-monotonicity on P_4-sparse graphs.

Lemma 8. *Let G be a spider with spider partition (S, C, R). Assume that for every $t \geq \chi(G[R])$ and for every induced subgraph H of $G[R]$ we have $\text{dom}_H[t] \leq \text{dom}_{G[R]}[t]$. Then, for every $t \geq \chi(G)$ and for every induced subgraph H of G, $\text{dom}_H[t] \leq \text{dom}_G[t]$ holds.*

Proof. Let G be a spider with spider partition (S, C, R), where $|S| = |C| = k \geq 2$, and let H be an induced subgraph of G. For convenience, if a graph is empty define its dominance sequence as the zero sequence, beginning at zero. Let R_H be $V(H) \cap R$. By hypothesis, $dom_{G[R_H]}[r] \leq dom_{G[R]}[r]$, for each $r \geq \chi(G[R])$. Following the arguments used in the proof of Theorem 9, it can be seen that:

- If G is a thin spider, then $\text{dom}_H[k+r] \leq k+dom_{G[R_H]}[r] \leq k+dom_{G[R]}[r] = \text{dom}_G[k+r]$ for $\chi(G[R]) \leq r \leq |R|$, $\text{dom}_H[k + |R| + 1] \leq k = \text{dom}_G[k + |R| + 1]$, and $\text{dom}_H[j] = 0 = \text{dom}_G[j]$ for $j > k + |R| + 1$.
- If G is a thick spider, then $\text{dom}_H[k+r] \leq k+dom_{G[R_H]}[r] \leq k+dom_{G[R]}[r] = \text{dom}_G[k+r]$ for $\chi(G[R]) \leq r \leq |R|$, $\text{dom}_H[k + |R| + s] \leq \min\{k, 2k - 2s\} = \text{dom}_G[k + |R| + s]$ for $1 \leq s \leq k$, and $\text{dom}_H[j] = 0 = \text{dom}_G[j]$ for $j > 2k + |R|$.

We conclude $\text{dom}_G[t] \geq \text{dom}_H[t]$, for every $t \geq \chi(G)$. □

Theorem 12. *P_4-sparse graphs are b-monotonic.*

Proof. As P_4-sparse graphs are hereditary, it is enough to prove that given a P_4-sparse graph G, $\chi_b(G) \geq \chi_b(H)$, for every induced subgraph H of G. By applying Lemma 5, Theorem 11, Lemma 6, Lemma 8, and Theorem 3, since P_4-sparse graphs is a hereditary class, we can inductively show that for every P_4-sparse graph G, every induced subgraph H of G, and every $t \geq \chi(G)$, $\mathrm{dom}_H[t] \leq \mathrm{dom}_G[t]$ holds. Let G be a P_4-sparse graph, and let H be an induced subgraph of G. If $\chi_b(H) < \chi(G)$, then $\chi_b(H) < \chi_b(G)$. Otherwise, $\chi_b(H) = \mathrm{dom}_H[\chi_b(H)] \leq \mathrm{dom}_G[\chi_b(H)]$ and by Corollary 1, $\mathrm{dom}_G[\chi_b(H)] \leq \mathrm{dom}_G[\chi_b(G)] = \chi_b(G)$ implying that $\chi_b(G) \geq \chi_b(H)$. □

6. Concluding remarks

In this paper, we have proved that cographs and P_4-sparse graphs are b-continuous and b-monotonic. Besides, we have designed a dynamic programming algorithm to compute the b-chromatic number in polynomial time within these graph classes. One interesting problem is to extend our results to superclasses of these graph families, as for example, the class of distance-hereditary graphs. Finally, it would be an interesting problem to characterize b-monotonic graphs by forbidden induced subgraphs.

Note added in proof. Results similar to Theorem 5, Theorem 6, and Theorem 7 have been obtained independently in [1,2], by resorting to similar proof techniques.

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