Free-cut elimination in linear logic and an application to a feasible arithmetic

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6th October, 2016

Bologne
ELICA meeting

Outline

Introduction

Normal forms in first-order linear logic

An arithmetic in linear logic

Bellantoni-Cook programs and the WFM for $I\Sigma_1^{N^+}$

Conclusions

Implicit computational complexity (ICC)

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In a nutshell:

ICC studies correspondences between features of logic and complexity classes

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Proof-theoretic approach

For a logic or theory,

'representable' functions = given complexity class

where representability can mean definability, typability etc.

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We distinguish the following two methodologies:

1. Theories whose definable functions = given complexity class.
2. Logics that type terms with normalisation complexity of a given class.

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This work is about the first methodology.

Provably convergent functions

Correspondence between a theory $\mathcal{T}$ and a class $\mathcal{C}$:

$$\mathcal{T} \vdash \forall x.\exists y.A(x, y) \quad \Leftrightarrow \quad \mathbb{N} \models \forall x.A(x, \mathfrak{f}(x)) \text{ for some } \mathfrak{f} \in \mathcal{C}$$

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For example:

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- Via a Dialectica-style functional interpretation.
- **Extracted programs:** higher-order variant of primitive recursive functions.

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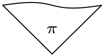
Buss' and Mints' proof.

- Via the witness function method.
- **Extracted programs:** regular primitive recursive functions of ground type.

□

The idea

- A formal witness predicate over $\mathbb{N}$ for each 'tame' formula.
- Arithmetic proofs $\rightsquigarrow$ functions from witnesses to witnesses:



$$\Gamma \vdash \Delta \quad \rightsquigarrow \quad f^{\pi} : \left\{ \begin{array}{c} \text{witnesses} \\ \text{of } \bigwedge \Gamma \end{array} \right\} \to \left\{ \begin{array}{c} \text{witnesses} \\ \text{of } \bigvee \Delta \end{array} \right\}$$

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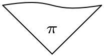

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Crucial points

- π free-cut free: tames the complexity of formulae; no bad $\forall$.

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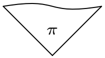

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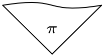

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Crucial points

- π free-cut free: tames the complexity of formulae; no bad $\forall$.
- De Morgan normal form: only functions at ground type, i.e. $\mathbb{N}^k \to \mathbb{N}$.
- Right-contraction: tests the witness predicate (should be decidable).

Free-cut elimination

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- Due to Buss and Mints.
- $\rightsquigarrow$ bounded arithmetic. Theories for $\mathbf{NC}_i, \mathbf{AC}_i, \mathbf{P}, \mathbf{PH}$,...
- The best method available to delineate hierarchies of classical theories.

Context and motivation

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Question

Can WFM be useful for characterising complexity classes via linear logic?

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- De Morgan duality is everywhere!

Free-cut elimination in linear logic

A nonlogical rule has the following format:

$$\frac{\{!\Gamma, \Sigma_i \vdash \Delta_i, ?\Pi\}_{i \in \mathcal{I}}}{!\Gamma, \Sigma \vdash \Delta, ?\Pi}$$

The formulae in Σ and Δ are considered principal.

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Corollary

Every theorem has a proof where all formulae are subformulae of the conclusion or a principal formula of a nonlogical step.

Outline

We consider an axiomatisation inspired by Bellantoni & Hofmann:

$$
\begin{array}{rl}
N_{cntr} & \forall x^N.(N(x) \otimes N(x)) \\
\\
N_\varepsilon & N(\varepsilon) \\
N_0 & \forall x^N.N(s_0 x) \\
N_1 & \forall x^N.N(s_1 x) \\
\varepsilon & \forall x^N.(\varepsilon \neq s_0 x \otimes \varepsilon \neq s_1 x) \\
inj_0 & \forall x^N, y^N.(s_0 x = s_0 y \multimap x = y) \\
inj_1 & \forall x^N, y^N.(s_1 x = s_1 y \multimap x = y) \\
tree & \forall x^N.s_0 x \neq s_1 x \\
surj & \forall x^N.(x = \varepsilon \oplus \exists y^N.x = s_0 y \oplus \exists y^N.x = s_1 y)
\end{array}
$$

$$
PIND \quad \begin{array}{rl}
 & A(\varepsilon) \\
\multimap & !(\forall x^{!N}.(A(x) \multimap A(s_0 x))) \\
\multimap & !(\forall x^{!N}.(A(x) \multimap A(s_1 x))) \\
\multimap & \forall x^{!N}.A(x)
\end{array}
$$

Peano's N predicate: $N(t) :=$ "t is a natural number".

Functions are specified by equational programs. E.g.:

$$\Phi \begin{cases} \texttt{add}(0, x) & = & x \\ \texttt{add}(su, x) & = & s(\texttt{add}(u, y)) \\ & & \\ \texttt{mult}(0, x) & = & 0 \\ \texttt{mult}(su, x) & = & \texttt{add}(x, \texttt{mult}(u, x)) \end{cases}$$

Convergence

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Convergence statement:

$$\forall \mathbf{x}^{!N}.\Phi(\mathbf{x}) \multimap \forall x^N, y^N.N(\texttt{mult}(x, y))$$

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We will consider the theory $I\Sigma_1^{N^+}$, admitting *PIND* only over:

$$E \quad ::= \quad N(t) \mid s = t \mid s \neq t \mid E \parr E \mid E \otimes E \mid \exists x.E$$

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Bellantoni-Cook characterisation of polytime functions

Arguments of a function are separated into normal and safe inputs:

$$f(\mathbf{u}; \mathbf{x})$$

Normal: left of ; so $\mathbf{u}$ above. Safe: right of ; so $\mathbf{x}$ above.

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Predicative recursion on notation
If g, h_0, h_1 are BC then so is f defined by:

$$\begin{aligned} f(\varepsilon, \mathbf{v}; \mathbf{x}) &= g(\mathbf{v}; \mathbf{x}) \\ f(s_0 u, \mathbf{v}; \mathbf{x}) &= h_0(u, \mathbf{v}; \mathbf{x}, f(u, \mathbf{v}; \mathbf{x})) \\ f(s_1 u, \mathbf{v}; \mathbf{x}) &= h_1(u, \mathbf{v}; \mathbf{x}, f(u, \mathbf{v}; \mathbf{x})) \end{aligned}$$

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Safe composition
Can compose functions as long as safe inputs are hereditarily safe.

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Theorem (Bellantoni & Cook '92)

BC programs compute just the polynomial-time functions.

Multiplication again...

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Via the WFM.

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  - No ?-formulae $\rightsquigarrow$ no contraction-right.

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- **!, ?-free** *PIND*: predicative recursion.
- **Anchored cuts:** safe composition of functions.

□

Example case: induction

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- By deduction and invertibility, we can assume *PIND* occurs as:

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- Define f by PRN:

$$\begin{aligned} f\left(0, u^{\Gamma};\right) &:= g\left(u^{\Gamma};\right) \\ f\left(s_i u^{N(t)}, u^{\Gamma};\right) &:= h_i\left(u^{N(t)}, u^{\Gamma}; f\left(u^{N(t)}, u^{\Gamma};\right)\right) \end{aligned}$$

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  - Relationships to BC-versions of equational theory PV?
- Characterise polynomial hierarchy via minimisation principles.
  - Functions conditional on Σ^p_i tests.
  - Relies on evaluation of witness predicate in Δ^p_i.

Thank you.