1. Limit laws for basic parameters of lattice paths with unbounded jumps

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ABSTRACT: *This paper establishes the asymptotics of a class of random walks on $\mathbb{N}$ with regular but unbounded jumps and studies several basic parameters (returns to zero for meanders, bridges, excursions, final altitude for meanders). All these results are generic (obtained by the kernel method for the combinatorial part and by singularity analysis for the asymptotic part).*

This paper completes the article [3] which was only dealing with the combinatorics (enumeration and bijections) of walks with unbounded jumps (the so-called "factorial walks"), which play an important rôle for uniform random generation of some combinatorial objects. We fully parallelize the analytical approach from [4] which was dealing with walks with bounded jumps only.

1 Introduction

Our main motivation for analyzing a class of walks with unbounded jumps comes from the fact that several classes of combinatorial objects can be generated via the so-called "generating trees". Enumerating these trees (and predicting the number of nodes at a given depth) allows uniform random generation. The concept of generating trees has been used from various points of view and has been introduced in the literature by Chung, Graham, Hoggatt and Kleiman [11] to examine the reduced Baxter permutations. This technique has been successively applied to other classes of permutations and the main references on the subject are due to West [14, 25, 26], then followed by the Florentine school [6, 7, 16, 19, 20, 22, 23] and other authors [3, 12, 18]. A generating tree is a rooted labeled tree (labels are integers) with the property that if v_1 and v_2 are any two nodes with the same label then, for each label ℓ, v_1 and v_2 have exactly the same number of children with label ℓ. To specify a generating tree it therefore suffices to specify: 1) the label of the root; 2) a set of rules explaining how to derive from the label of a parent the labels of all of its children. Points 1) and 2) define what we call a *rewriting rule*. Any random walk in the generating tree can also be seen as a lattice path (random walk on the integers, with an infinite number of possible jumps). The regularity of the rewriting rules determines the "solvability" (combinatorially speaking) of the corresponding random walk process.

Few years ago, Pinzani and al. [6] exhibited several cases of factorial-like rewriting rules for which the generating functions were algebraic. This was calling for a general solution of the factorial-like rewriting rules case. This problem was solved in [3], by establishing a link between the generating trees and families of lattice paths with unbounded jumps (with respect to a given rewriting rule, the number of nodes with label k at depth n in the tree is the number of walks of length n ending at altitude k); then, the corresponding generating functions for walks are always algebraic and are made explicit via the kernel method (we give more details in Section 3). The asymptotic properties of such walks were remaining open.

The article [4] and an important part of the PhD thesis [2] are dedicated to the analysis of several parameters of discrete random walks on $\mathbb{Z}$ or $\mathbb{N}$ with *bounded* jumps. For this case (but not for the case of unbounded jumps), a context-free grammar approach is also possible (as the jumps are bounded and thus can be encoded by a finite alphabet). However this language theory approach (which was previously the main one considered in combinatorics) reveals almost nothing about the shape of the generating function and is even less talkative about the asymptotics. An orthogonal approach (the kernel method) has the merit of giving a direct access to the generating functions and their asymptotics.

A natural question is: can the same approach be the winning one for the study of walks with *unbounded* jumps? We show here that the answer is clearly: yes, for a quite general family of walks! What follows is a slightly modified copy/paste of [4] which gives however some new original results for generating functions and asymptotics of walks with unbounded jumps.

2 Lattice paths and generating functions

This section presents the varieties of lattice paths to be studied as well as their companion generating functions (in the same terms as in [4]).

Definition 2.1. *Fix a set of vectors of $\mathbb{Z}\times\mathbb{Z}$, $\mathcal{S}=\{(x_1,y_1),\ldots\}$. ($\mathcal{S}$ can be finite or not). A* lattice path *or* walk *relative to $\mathcal{S}$ is a sequence $v=(v_1,\ldots,v_n)$ such that each v_j is in $\mathcal{S}$. The geometric realization of a lattice path $v=(v_1,\ldots,v_n)$ is the sequence of points $(P_0,P_1,\ldots,P_n)$ such that $P_0=(0,0)$ and $\overrightarrow{P_{j-1}P_j}=v_j$. The quantity n is referred to as the* size *of the path.*

In what follows, we focus our attention to a class of infinite sets $\mathcal{S}$ and we shall identify a lattice path with the polygonal line admitting $P_0,\ldots,P_n$ as vertices. The elements of $\mathcal{S}$ are called *steps* or *jumps*, and we also refer to the vectors $\overrightarrow{P_{j-1}P_j}=v_j$ as the steps of a particular path.

Various constraints will be imposed on paths. In particular we restrict attention throughout this paper to *directed paths* defined by the fact that if (i,j) lies in $\mathcal{S}$, then necessarily one should have $i>0$. In other words, a step always entails progress along the horizontal axis and the geometric realization of the path naturally lives in the half plane $\mathbb{N}\times\mathbb{Z}$. (This constraint implies that the paths studied can be treated essentially as 1-dimensional objects.) The following conditionings are to be considered (Figure 1).

Definition 2.2. *A* bridge *is a path whose end-point P_n lies on the x-axis. A* meander *is a path that lies in the quarter plane $\mathbb{N}\times\mathbb{N}$. An* excursion *is a path that is at the same time a meander and a bridge; it thus connects the origin to a point lying on the x-axis and involves no point with negative y-coordinate.*

A family of paths is said to be factorial *if each allowed step in $\mathcal{S}$ (Definition 2.1) is of the form $(1,-y)$ for any $y\geq 1$ or of the form $(1,j)$ with $j\in\mathcal{J}$ a given finite subset of $\mathbb{Z}$. We thus simply note $\mathcal{S}=\{\mathbb{Z}_{<0},\mathcal{J}\}$.*

In the factorial case the size of a path coincides with its span along the horizontal direction, that is, its *length.* The terminology of bridges, meanders, and excursions is chosen to be consistent with the standard one adopted in Brownian motion theory; see, e.g., [24]. A factorial walk is simply a walk for which there is, at each step, not only a finite amount of "bounded" jumps below or above the actual position but also the possibility to go anywhere below the actual position.

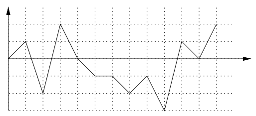
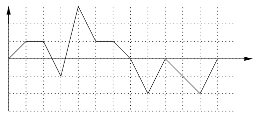
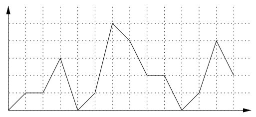
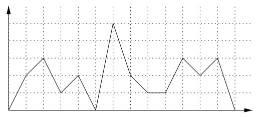
walks	ending anywhere	ending in 0
unconstrained (on $\mathbb{Z}$)	 walk ($\mathcal{W}$) $W(z,u) \equiv \sum_{k\in\mathbb{Z}} W_k(z)\, u^k = \frac{1}{1 - z\,Q(u)}$ $W_n = +\infty$	 bridge ($\mathcal{B}$) $B(z) \equiv W_0(z) = z \sum_{i=0}^{b} \frac{u_i'(z)}{u_i(z)}$ $B_n \sim \beta_0 \frac{Q(\tau)^n}{\sqrt{2\pi n}}$
constrained (on $\mathbb{N}$)	 meander ($\mathcal{M}$) $M(z) \equiv \sum_{k\geq 0} F_k(z) = -\frac{1}{z} \prod_{i=0}^{b} (1 - u_i(z))$ $M_n \sim \mu_0 \frac{Q(\tau)^n}{2\sqrt{\pi n^3}}$	 excursion ($\mathcal{E}$) $E(z) \equiv F_0(z) = \frac{(-1)^{b+1}}{z p_{-b}} \prod_{i=0}^{b} u_i(z)$ $E_n \sim \epsilon_0 \frac{Q(\tau)^n}{2\sqrt{\pi n^3}}$

Figure 1: The four types of paths with unbounded jumps: walks, bridges, meanders, and excursions. We give the corresponding generating functions and the asymptotics of their coefficients. (*N.B.*: there is an infinite number of unconstrained walks as jumps are unbounded.)

The main objective of this paper is to enumerate exactly as well as asymptotically paths, bridges, and meanders, this with special attention to factorial families. Once the set of steps is fixed, we let $\mathcal{W}$ and $\mathcal{B}$ denote the set of paths and bridges respectively ($\mathcal{W}$ being reminiscent of "walk"); we denote by $\mathcal{M}$ and $\mathcal{E}$ the set of meanders and excursions.

Given a class $\mathcal{C}$ of paths, we let $\mathcal{C}_n$ denote the subclass of paths that have size n, and, whenever appropriate, $\mathcal{C}_{n,k} \subset \mathcal{C}_n$ those that have final vertical abscissa (also known as "final altitude") equal to k. With the convention of using standard fonts to denote cardinalities of the corresponding sets (themselves in calligraphic style), $C_n = \mathrm{card}(\mathcal{C}_n)$ and $C_{n,k} = \mathrm{card}(\mathcal{C}_{n,k})$, the corresponding (ordinary) *generating functions* are then

$$C(z) := \sum_{n\in\mathbb{N}} C_n z^n, \qquad C(z,u) = \sum_{k\in\mathbb{Z}} C_k(z)\, u^k = \sum_{n\in\mathbb{N}, k\in\mathbb{Z}} C_{n,k}\, u^k z^n = \sum_{n\in\mathbb{N}} c_n(u)\, z^n.$$

This paper is entirely devoted to characterizing these generating functions: they are either rational functions (W) or algebraic functions (B, M, E)[1]. As we shall see, a strong algebraic decomposition prevails which, as opposed to other approaches, renders the calculation of the generating functions effective. Even more importantly, the decomposability of generating functions makes it possible to extract their singular structure, and in turn solve the corresponding asymptotic enumeration problems in a wholly satisfactory fashion.

Weighted paths. For several applications, it is useful to associate *weights* to single steps. In this case, the set of steps $\mathcal{S}$ is coupled with a system of weights $\Pi = \{w_i\}_{i\in\mathbb{Z}}$, with $w_i > 0$ the weight associated to $(1,i) \in \mathcal{S}$; the weight of a path is then defined as the *product* of the weights of its individual steps. Then the quantity C_n, still referred to as *number of paths* (of size n), represents the sum of the weights of all paths of size n. Such weighted paths cover several situations of interest: (i) combinatorial paths in the standard sense above when each $w_i = 1$; (ii) paths with coloured steps, e.g., $w_i = 2$ means that the corresponding step $(1,i)$ has two possible coloured incarnations (say blue and red); (iii) $\sum w_i = 1$ corresponds to a probabilistic model of paths where, at each stage, step $(1,i)$ is chosen with probability w_i.

3 Functional equation and the kernel method

In this section, we characterize the generating functions of the four types of directed paths (unconstrained, bridges, meanders, and excursions). It will be seen that a specific algebraic curve, the "characteristic curve" plays a central rôle.

Definition 3.1. *Let $\mathcal{S} = \{\mathbb{Z}_{<0}, \mathcal{J}\}$ be a factorial set of jumps, with $\Pi = \{w_i\}_{i\in\mathbb{Z}}$ the corresponding system of weights ($w_i \equiv 1$ in the unweighted case). The* characteristic series *of $\mathcal{S}$ is defined as the Laurent series*[2]

$$Q(u) := \sum_{i\in\mathbb{Z}_{<0}\cup\mathcal{J}} w_i\, u^i\,.$$

Let $b = -\min \mathcal{J} \cup \{0\}$ and $a = \max \mathcal{J}$ be the two extreme vertical amplitudes of any jump of $\mathcal{J}$, and assume throughout $a > 0, b \geq 0$. We restrict now attention to the unweighted case (but with possibly coloured jumps in $\mathcal{J}$, see the paragraph "weighted paths" in Section 2). The characteristic series can be then rewritten as

$$Q(u) = \sum_{i=-\infty}^{-1} u^i + P(u) \equiv P(u) - \frac{1}{1-u}\,,\ \text{where } P(u) := \sum_{j=-b}^{a} p_j u^j \quad (p_j \in \mathbb{N}). \quad (1)$$

So p_j can be seen as the multiplicity of the jump $(1,j)$. The kernel *is defined by*

$$K(z,u) := (1-u)u^b - z(u^b(1-u)P(u) - u^b)\,. \quad (2)$$

[1]The attentive reader should have understood that this does not stand for the acronym of a well-known Belgian theorem (Brownian Motion Everywhere)!

[2]By Laurent series, we mean objects like $\sum_{k=m}^{+\infty} g_k u^k$ ($m \in \mathbb{Z}$) or $\sum_{k=-\infty}^{m} g_k u^k$. The reader can check that our generating functions are holomorphic/meromorphic functions; they can be expanded at 0 or at infinity, and so they can be seen as belonging either to $\mathbb{C}[[\frac{1}{u}]][u]$ or $\mathbb{C}[\frac{1}{u}][[u]]$.

The characteristic curve *of the lattice paths determined by $\mathcal{S}$ is the plane algebraic curve defined by the* kernel equation

$$1 - z\,Q(u) = 0, \qquad \text{or equivalently} \quad K(z,u) = 0. \tag{3}$$

As we shall see the characteristic equation plays a central rôle, the second form being the entire version (that is, a form without negative powers).

Proposition 3.2. *The kernel equation* (3) *admits $a+b+1$ roots in u : $b+1$ roots $u_0(z), \ldots, u_b(z)$ finite for $z \sim 0$ and a large roots $v_1(z), \ldots, v_a(z)$ infinite for $z \sim 0$.*

Proof : This polynomial has degree $a+b+1$ in u, and hence, admits $a+b+1$ solutions, which are algebraic functions of z. The classical theory of algebraic functions and the Newton polygon construction enable us to expand the solutions near any point as Puiseux series (that is, series involving fractional exponents; see [13]). The $a+b+1$ solutions, expanded around 0, can be classified as follows:

- the "unit" branch, denoted by u_0, is a power series in z with constant term 1;
- b "small" branches, denoted by $u_1, \ldots, u_b$, are power series in $z^{1/b}$ whose first nonzero term is $\zeta z^{1/b}$, with $\zeta^b + 1 = 0$;
- a "large" branches, denoted by $v_1, \ldots, v_a$, are Laurent series in $z^{1/a}$ whose first nonzero term is $\zeta z^{-1/a}$, with $\zeta^a + 1 = 0$.

In particular, all the roots are distinct. □

Formulae (4) and (5) in the following theorem were first derived in [3]:

Theorem 3.3 (Excursions and meanders.). *The generating function $F(z,u)$ for factorial walks starting from 0 is algebraic; it is given by* (8)*, where $u_0, \ldots, u_b$ (resp. $v_1, \ldots, v_a$) are the finite (resp. infinite) solutions at $z = 0$ of the equation $K(z,u) = 0$ and the kernel K is defined by* (2)*. In particular, the generating function for all walks, irrespective of their endpoint, is*

$$M(z) = F(z,1) = -\frac{1}{z}\prod_{i=0}^{b}(1-u_i), \tag{4}$$

and the generating function for excursions*, i.e., walks ending at 0, is, for $b < 0$:*

$$E(z) = F(z,0) = \frac{(-1)^{b+1}}{zp_{-b}}\prod_{i=0}^{b} u_i\,. \tag{5}$$

(For $b = 0$, the relation becomes $F(z,0) = \frac{u_0}{1+(1-p_0)z}$.)

More generally, the generating function for meanders ending at altitude k is

$$F_k(z) = \frac{1}{zp_a}\sum_{i=1}^{a}\frac{v_i^{-k-1}}{\prod_{j\neq i} v_j - v_i}\,. \tag{6}$$

Proof : The allowed jumps imply that from position k (encoded by u^k), one can go to the position encoded[3] by $u^0+u^1+\ldots+u^{k-1}+\{u^{\geq 0}\}P(u)u^k = \frac{u^k-1}{u-1}+\{u^{\geq 0}\}P(u)u^k$, as this is a linear mapping, this leads to the recurrence on the $f_n(u)$'s (the polynomials encoding the possible walk positions at time n):

$$f_{n+1}(u)=\frac{f_n(u)-f_n(1)}{u-1}+\{u^{\geq 0}\}P(u)f_n(u)$$

and equivalently to the following equality

$$\begin{aligned} F(z,u) &= \sum_{n\geq 0} f_n(u)z^n \\ &= 1+z\left(\frac{F(z,1)-F(z,u)}{1-u}+P(u)F(z,u)-\{u^{<0}\}[P(u)F(z,u)]\right). \end{aligned}$$

Thus, $F(z,u)$ satisfies the following functional equation:

$$F(z,u)\left(1+\frac{z}{1-u}-zP(u)\right)=1+\frac{zF(z,1)}{1-u}-z\sum_{k=0}^{b-1} r_k(u)F_k(z)\,, \qquad (7)$$

where $r_k(u)$ is a Laurent polynomials whose exponents belong to $[k-b,-1]$:

$$r_k(u) := \{u^{<0}\}\left(P(u)u^k\right) \equiv \sum_{j=-b}^{-k-1} p_j u^{j+k}.$$

Now comes the second ingredient of the proof, the so called "kernel method". The right-hand side of (7), once multiplied by $u^b(1-u)$, is

$$R(z,u)=u^b(1-u)\left(1+\frac{z}{1-u}F(z,1)-z\sum_{k=0}^{b-1} r_k(u)F_k(z)\right).$$

By construction, it is a *polynomial* in u of degree $b+1$ and leading coefficient -1. Hence, it admits $b+1$ roots, which depend on z. Replacing u by the series $u_0,u_1,\ldots,u_b$ in Eq. (7) shows that these series are exactly the $b+1$ roots of R, so that

$$R(z,u)=-\prod_{i=0}^{b}(u-u_i).$$

Let $p_a := [u^a]P(u)$ be the multiplicity of the largest forward jump.

Then the coefficient of u^{a+b+1} in $K(z,u)$ is $p_a z$, and we can write

$$K(z,u)=p_a z\prod_{i=0}^{b}(u-u_i)\prod_{i=1}^{a}(u-v_i).$$

[3] We make use of the conventional notations for coefficients of entire and Laurent series: $[z^n]\sum_n f_n z^n := f_n$ and $\{u^{\geq 0}\}g(u)$ is the sum of the monomials of $g(u)$ with a nonnegative exponent.

Finally, as $K(z,u)F(z,u) = R(z,u)$, we obtain

$$F(z,u) = \frac{-\prod_{i=0}^{b}(u-u_i)}{u^b(1-u)+zu^b-zu^b(1-u)P(u)} = -\frac{1}{p_a z \prod_{i=1}^{a}(u-v_i)}. \tag{8}$$

Setting $u = 1$ and $u = 0$ gives formulae (4) and (5) and a partial fraction decomposition of the rightmost part of (8) gives (6). □

The "kernel method" has been part of the folklore of combinatorialists for some time and is related to the what is known as "the quadratic method" in enumeration of planar maps [10]. Earlier references (see [17] Ex. 2.2.1.11 for Dyck paths, [21, Sec. 15.4] for a pebbling game) were dealing with the case of a single unknown in the right part of (7). The kernel method in its more general version was developed by Banderier, Bousquet-Mélou, Flajolet, Petkovšek [1, 2, 3, 4, 9]. A somewhat similar idea (involving a reduction to a Riemann–Hilbert problem) was used in [15] for a queuing theory application.

Theorem 3.4 (Bridges). *The bivariate generating function of paths (with z marking size and u marking final altitude) relative to a simple set of steps $\mathcal{S}$ with characteristic series $Q(u)$ is a rational function. It is given by*

$$W(z,u) = \frac{1}{1-zQ(u)}. \tag{9}$$

The generating function of bridges is an algebraic function given by

$$B(z) = z\sum_{j=0}^{b}\frac{u_j'(z)}{u_j(z)} = z\frac{d}{dz}\log\left(u_0(z)\cdots u_b(z)\right), \tag{10}$$

where the expressions involve all the small branches $u_0,\ldots,u_b$ of the characteristic curve (3)*. Generally, the generating function W_k of paths terminating at altitude k is, for $-\infty < k < b$,*

$$W_k(z) = z\sum_{j=0}^{b}\frac{u_j'(z)}{u_j(z)^{k+1}} = -\frac{z}{k}\frac{d}{dz}\left(\sum_{j=0}^{b}u_j(z)^{-k}\right), \tag{11}$$

and for $-a < k < +\infty$,

$$W_k(z) = -z\sum_{j=1}^{a}\frac{v_j'(z)}{v_j(z)^{k+1}} = \frac{z}{k}\frac{d}{dz}\left(\sum_{j=1}^{a}v_j(z)^{-k}\right), \tag{12}$$

where $v_1,\ldots,v_a$ are the large branches.

(For W_0, the second form in (11) and (12) is to be taken in the limit sense $k \to 0$.)
Proof : The proof of an identity similar to (10) for walks with bounded jumps is given in [4] and holds verbatim for walks with unbounded jumps: Consider a bridge and let m (with $m \leq 0$) be the minimal altitude of any vertex. Any nonempty bridge β decomposes uniquely into a walk φ_1 of size ≥ 1 from 0 to m that only reaches level m at its right end, followed by an excursion ε, followed

by a path φ_2 of size ≥ 0 from m to 0 that only touches level m at its beginning. By rearrangement, one can write $\beta = \varepsilon \cdot (\varphi_2|\varphi_1)$, where the gluing of $\varphi_2\varphi_1$ is an arch (that is, an excursion which reaches 0 only at its beginning and its end) and the bar keeps track of where the splitting should occur. This links bridges and excursions:

$$\overbrace{B(z)-1}^{\text{bridges}} = \overbrace{E(z)}^{\text{excursions}} \cdot \overbrace{\left(z\frac{d}{dz}A(z)\right)}^{\text{split arches}}, \tag{13}$$

as $E(z) = 1/(1-A(z))$ ($A(z)$ stands for the generating function of arches), this is equivalent to

$$B(z)-1 = E(z)\cdot z\frac{d}{dz}\left(1-\frac{1}{E(z)}\right) = z\frac{E'(z)}{E(z)},$$

using Formula (5) for $E(z)$ gives the identity (10).

This reinforces the discussion of [4] about ubiquitous Spitzer, Andersen-like relations and here also, this gives the possibility of analysing the number of times a bridge attains its minimum or maximum value by adapting the decomposition (13).

Set $w_n(u) = [z^n]W(z,u)$, the Laurent series that describes the possible altitudes and the number of ways to reach them in n steps. We have $w_0(u) = 1$, $w_1(u) = Q(u)$, and $w_{n+1}(u) = Q(u)w_n(u)$, so that $w_n(u) = Q(u)^n$ for all n. The determination of $W(z,u)$ in (9) follows from

$$\sum_{n\geq 0} Q(u)^n z^n = \frac{1}{1-zQ(u)}.$$

Observe that the resulting series is entire in z but of the Laurent type in u (it involves arbitrary negative powers of u).

For positive $Q(u)$, the radius of convergence of $W(z,u)$ viewed as a function of z is exactly $1/Q(u)$. Also, by the link between $E(z)$ and $B(z)$ (see above), the radius of convergence of $B(z)$ as a function of z is $\rho = 1/Q(\tau)$, the radius of convergence of $E(z)$ ($\tau > 1$, as it is proven in the next section). Consider now $|z| < r$, where $r := \frac{\rho}{2}$ and then follow the scheme of the proof from [4]. □

4 Asymptotics

Lemma 4.1. *Let $Q(u) = P(u) - 1/(1-u)$ be the rational series associated to the jumps a factorial walk. Then, there exists a unique number τ, called the* structural constant*, such that $Q'(\tau) = 0$, $\tau > 1$. The* structural radius *is by definition the quantity*

$$\rho := \frac{1}{Q(\tau)}.$$

The following domination amongst the roots holds

$$|u_i(z)| < u_0(z) \leq v_1(z) < |v_j(z)| \quad \forall |z| \leq \rho \text{ for } i = 1,\ldots,b \text{ and } j = 2,\ldots,a. \tag{14}$$

Proof : Differentiating twice Q as given in (1), we see that $Q''(x) > 0$ for all $x > 1$. Thus, the real function $x \mapsto Q(x)$ is strictly convex on $[1, +\infty]$. Since it satisfies $Q(1^+) = Q(+\infty) = +\infty$, it must have a unique positive minimum attained at some τ, and $Q'(\tau) = 0$.

As Q is aperiodic, a strong version of the triangular inequality gives

$$Q(v_1) = \frac{1}{z} = |Q(v_i)| < Q(|v_i|)$$

since Q is strictly increasing on the interval $[1, +\infty]$ and since $|v_i| > \tau > 1$ belongs to this interval for $z \in [0, \rho]$, one has the three last inequalities of (14); a duality argument gives the first inequality of (14). □

As one of the referee pointed out, the structural constant τ is such that the jumps with law $\frac{w_j \tau^j}{Q(\tau)}$ are centered. Similarly, the factoriality assumption results in steps which can be seen as a mixture of a geometric probability law and a finitely supported one.

Theorem 4.2. *The asymptotics for the number of bridges, meanders, excursions is given by*

$$B_n \sim \beta_0 \frac{Q(\tau)^n}{\sqrt{2\pi n}} (1 + \frac{\beta_1}{n} + \frac{\beta_2}{n^2} + \ldots), \quad \beta_0 = \frac{1}{\tau} \sqrt{\frac{Q(\tau)}{Q''(\tau)}},$$

$$M_n \sim \mu_0 \frac{Q(\tau)^n}{2\sqrt{\pi n^3}} (1 + \frac{\mu_1}{n} + \frac{\mu_2}{n^2} + \ldots), \quad \mu_0 = \widetilde{U}(\rho) \sqrt{\frac{2Q^3(\tau)}{Q''(\tau)}},$$

$$E_n \sim \epsilon_0 \frac{Q(\tau)^n}{2\sqrt{\pi n^3}} (1 + \frac{\epsilon_1}{n} + \frac{\epsilon_2}{n^2} + \ldots), \quad \epsilon_0 = U(\rho)(-1)^b \sqrt{\frac{2Q^3(\tau)}{Q''(\tau)}},$$

where $U(\rho) = u_1(\rho) \ldots u_b(\rho)$ *and* $\widetilde{U}(\rho) = (1 - u_1(\rho)) \ldots (1 - u_b(\rho))$.

Proof : Here again, the approach used in [4] is the winning one. A saddle point method gives

$$\begin{aligned} B_n &= \frac{1}{2i\pi} \int_{|u|=\tau} Q(u)^n \frac{du}{u} \\ &\sim \frac{1}{2i\pi} \int_{\tau e^{-i\epsilon}}^{\tau e^{+i\epsilon}} \exp\left(n \left(\log Q(\tau) + \frac{1}{2} \frac{Q''(\tau)}{Q(\tau)} (u - \tau)^2 + O((u - \tau)^3)\right)\right) \frac{du}{u} \\ &\sim \frac{Q(\tau)^n}{2\pi\tau} \int_{-\infty}^{+\infty} e^{-nht^2/2}\, dt = \frac{Q(\tau)^n}{\tau\sqrt{2\pi n h}}, \qquad h = \frac{Q''(\tau)}{Q(\tau)}. \end{aligned}$$

The approximation is valid as $Q(\tau)$ dominates on the circle of integration (this can be seen by the Laurent series expression of $Q(u)$).

Contrary to what is observed for the bounded jumps case, it may happen that the small roots cross for $|z| < \rho$ (but their product remains analytic). We follow the scheme of proof from [4] which uses the link between $B(z)$ and $E(z)$. One has, by local inversion of the kernel equation,

$$u_0(z) = \tau - \sqrt{2 \frac{Q(\tau)}{Q''(\tau)}} \sqrt{1 - z/\rho} + \cdots \qquad (z \to \rho^-). \tag{15}$$

Then the only possible behaviour compatible with the above asymptotics for B_n is that $U(z) := u_1(z) \dots u_b(z)$ is analytical for $|z| < \rho$; the same hold for $\widetilde{U}(z) := (1 - u_1(z)) \dots (1 - u_b(z))$.

Singularity analysis on the following expressions then gives the asymptotic expansions from the theorem

$$E_n \sim [z^n] U(\rho) \frac{(-1)^{b+1}}{z} \sqrt{2 \frac{Q(\tau)}{Q''(\tau)}} \sqrt{1 - z/\rho},$$

$$M_n \sim [z^n] \widetilde{U}(\rho) \frac{-1}{z} \sqrt{2 \frac{Q(\tau)}{Q''(\tau)}} \sqrt{1 - z/\rho}.$$

□

5 Returns to zero

Theorem 5.1 (Excursions). *The number of returns to zero of an excursion with unbounded jumps is asymptotically the sum of two independent geometric laws. The average is $2E(\rho) - 1 + O\left(\frac{1}{n}\right)$ returns to zero, with a variance $2E(\rho)(E(\rho) - 1) + O\left(\frac{1}{n}\right)$.*

Proof : An excursion is a sequence of arches, so $E(z) = \frac{1}{1 - A(z)}$ and $A(z) = 1 - \frac{1}{E(z)}$ for $E(z)$ and $A(z)$ generating functions of excursions and arches respectively. We note $F(z, u, t)$ the generating functions with respect to their length, final altitude, number of returns to zero. Thus, one has

$$F(z, 0, t) = \sum_{n,j} f_{nj}(0) t^j z^n = \frac{1}{1 - tA} = \frac{1}{1 - t\left(1 - \frac{1}{E}\right)},$$

where $f_{nj}(0)$ stands for the number of excursions of length n with j returns to 0. Then, all the moments can be made explicit as the m-th derivatives in t of $F(z, 0, t)$ are computable $\left(\partial_t^m F(z, 0, t) = m! \frac{(1 - E^{-1})^m}{(1 - t(1 - E^{-1}))^{m+1}}\right)$ and simplify when $t = 1 : \partial_t^m F(z, 0, 1) = m! E(z)(E(z) - 1)^m$.

Thus, the average number of returns to zero is

$$\mu_n = \frac{[z^n] \partial_t F(z, 0, 1)}{[z^n] F(z, 0, 1)} = \frac{[z^n] E(z)^2}{[z^n] E(z)} - 1 = 2e_0 - 1 + O\left(\frac{1}{n}\right)$$

as $E(z) = e_0 - e_1 \sqrt{\rho - z} + \cdots$ and the variance is given by

$$\sigma_n^2 = \frac{f_n''(1)}{f_n(1)} + \mu_n - \mu_n^2 = \frac{[z^n] 2E(z) - 4E(z)^2 + 2E(z)^3}{[z^n] E(z)} + \mu_n - \mu_n^2$$

$$= 6e_0^2 - 8e_0 + 2 + \mu_n - \mu_n^2 = 2e_0(e_0 - 1) + O\left(\frac{1}{n}\right).$$

The number of excursions of length n with j returns to zero is given by

$$f_{nj}(0) = [z^n]\left(1 - \frac{1}{E(z)}\right)^j = [z^n](1 - \frac{1}{e_0})^j - \frac{e_1 j (1-e_0^{-1})^{j-1}\sqrt{r-z}}{e_0^2} + O(r-z)\,.$$

Consequently, the probability to get asymptotically j returns to zero is $\pi_j = f_{nj}(0)/f_n \to \frac{j(1-e_0^{-1})^{j-1}}{e_0^2}$ for $n \to +\infty$, and $\sum_{j\geq 0}\pi_j = 1$ for any e_0. The probability generating function is $x\,(\frac{1}{e_0}\frac{1}{1-x(1-e_0^{-1})})^2$ and one has so a discrete limit law which is asymptotically the sum of two independent geometric laws of parameters $1 - 1/e_0$. □

Perhaps it can seem strange than a walk with a infinite negative drift has such a small average number of returns to zero[4], the explanation of this "paradox" is that most of the walks have much more returns, but their probabilities are very low, decreasing exponentially (so, like for Zeno's paradox, the sum is finite).

Theorem 5.2 (Meanders). *The average number of returns to zero of a meander with unbounded jumps follows a discrete limit law of a geometrical type.*

Proof : Equation (8) gives $F(z,u)$, the bivariate generating function for meanders (length, final altitude). Taking into account the number of returns to zero (via another variable t) leads to

$$F(z,u,t) = \sum_{n,j\geq 0} f_{nj}(u)t^j z^n = \frac{1}{1-t(1-1/E(z))}\frac{F(z,u)}{E(z)}\,.$$

This reflects the fact that a meander is a sequence of arches, followed by a prefix (i.e. a left part) of an arch, so $M(z) = \frac{1}{1-A(z)}M^+(z)$ and that a prefix of arch (note $M^+(z,u)$ their generating function) times an excursion gives a meander, so $M^+(z,u) = F(z,u)/E(z)$. The number $f_{nj}(1)$ of meanders of length n with j returns to zero is then given by

$$f_{nj}(1) = [z^n](1-1/E(z))^j\frac{M(z)}{E(z)}\,.$$

Notice that

$$\frac{(1-1/E(z))^j}{E(z)} \sim \frac{(1-\frac{1}{e_0})^j}{e_0} + \left(-\frac{j}{e_0} + (1-\frac{1}{e_0})\right)\left(1-\frac{1}{e_0}\right)^{j-1}\frac{e_1}{e_0^2}\sqrt{\rho - z} + \dots$$

Multiplying by the behaviour of $M(z) = F(z,1) = m_0 + m_1\sqrt{\rho - z}$ around $z = \rho$ gives

$$f_{nj}(1) \sim \left(m_1\frac{(1-\frac{1}{e_0})^j}{e_0} + m_0\left(-\frac{j}{e_0} + (1-\frac{1}{e_0})\right)\left(1-\frac{1}{e_0}\right)^{j-1}\frac{e_1}{e_0^2}\right)[z^n]\sqrt{\rho - z}.$$

So $f_{nj}(1)/f_n(1) \to (\frac{1}{e_0} + \frac{m_0}{m_1}\frac{e_1}{e_0^2})(1-\frac{1}{e_0})^j - \frac{m_0 e_1 j}{m_1 e_0^3}\left(1-\frac{1}{e_0}\right)^{j-1}$ for $n \to +\infty$. Asymptotics of moments is also easily computable from

$$\partial_t F(z,u,1) = F(z,u)(E(z)-1) \text{ and } \partial_t^2 F(z,u,1) = 2(E(z)-1)^2 F(z,u)\,.$$

Average and variance are $O(1)$. □

[4]One referee pointed out that a similar result was known in a special case of bridge, cf. Proposition 2.2 page 101 of [8].

Theorem 5.3 (Bridges). *The number of returns to zero of a bridge with unbounded jumps is asymptotically the sum of two independent geometric laws. The average is $2B(\rho)-1+O\left(\frac{1}{n}\right)$ returns to zero, with a variance $2B(\rho)(B(\rho)-1)+O\left(\frac{1}{n}\right)$.*

Proof : We can play the same game as above:

$$W_k(z,t)=\frac{1}{1-t\left(1-\frac{1}{B(z)}\right)}\frac{W_k(z)}{B(z)},$$

The number of walks w_{nj} of length n ending at altitude k with j returns to zero is then given by

$$w_{nj}(1)=[z^n](1-1/B(z))^j\frac{W_k(z)}{B(z)}.$$

□

6 Final altitude of a meander.

The *final altitude* of a path is the abscissa of its end point. The random variable associated to finite altitude when taken over the set of all meanders of length n is denoted by X_n, and it satisfies

$$\Pr(X_n=k)=\frac{[z^nu^k]F(z,u)}{[z^n]F(z,1)}.$$

We state:

Theorem 6.1 (Meanders). *The final altitude of a random meander of size n admits a discrete limit distribution characterized in terms of the large branches:*

$$\lim_{n\to\infty}\Pr(X_n=k)=[u^k]\,\varpi(u),\quad \textit{where}\quad \varpi(u)=\frac{(1-\tau)^2}{(u-\tau)^2}\prod_{\ell\geq 2}\frac{1-v_\ell(\rho)}{u-v_\ell(\rho)}.$$

The limiting distribution admits an explicit form

$$[u^k]\,\varpi(u)=\tau^{-k}(c_0+c_1k)+\sum_{\ell\geq 2}c_\ell v_\ell(\rho)^{-k},$$

for a set of constants c_j that can be made explicit by a partial fraction expansion of $\varpi(u)$.

Proof : Similarly to [4], one directly shows that the probability generating function of X_n at u converges pointwise to a limit that precisely equals $\varpi(u)$, the convergence holding for $u\in(0,1)$. By the fundamental continuity theorem for probability generating functions, this entails convergence in law of the corresponding discrete distributions.

We now fix a value of u taken arbitrarily in $(0,1)$ and treated as a parameter. The probability generating function of X_n is

$$\frac{[z^n]F(z,u)}{[z^n]F(z,1)},$$

where $F(z,u)$ is given by Theorem 3.3. We know from the proof of Theorem 4.2 that $\tau = v_1(\rho)$ satisfies $\tau > 1$ while the radius of convergence of $F(z,1)$ coincides with the structural radius ρ. Then, the quantity

$$\widetilde{V}(z,u) = \prod_{\ell \geq 2}^{a} \frac{1}{u - v_\ell(z)}$$

is analytic in the closed disk $|z| \leq \rho$: being a symmetric function of the nonprincipal large branches, it has no algebraic singularity there; given the already known domination relations between the large branches (Lemma 4.1), the denominators cannot vanish.

It then suffices to analyse the factor containing the principal large branch v_1. This factor has a branch point at ρ, where

$$\frac{1}{u - v_1(z)} \sim \frac{1}{u-\tau} + \frac{1}{(u-\tau)^2}\sqrt{2\frac{Q(\tau)}{Q''(\tau)}}\sqrt{1 - z/\rho},$$

as follows directly from (15) and the fact that v_1 is conjugate to u_0 at $z = \rho$. Singularity analysis then gives instantly the fact that, for some nonzero constant C,

$$[z^n]F(z,u) \sim C\rho^{-n}n^{-3/2}\Omega(u), \quad \text{where} \quad \Omega(u) = \frac{1}{(u-\tau)^2}\widetilde{V}(\rho,u),$$

and the result follows after normalization by $[z^n]F(z,1)$.

□

7 Variations...

All the above theorems hold with a slightly more general model of walks, for which all the backward unbounded jumps are coloured (say, there is m colors). The only modification is that the roots are then the roots of the kernel $K(z,u) = (1-u)u^b - z(u^b(1-u)P(u) - mu^b)$. The analysis for the $F_k's$ and $W_k's$ is more delicate as it involves a better "individual" knowledge of the small and large roots.

Some more general models of walks were considered in [5], there is still some algebraic generating functions but their asymptotic properties remain to be established, this seems quite difficult as there is no clear simple closed form formula (in terms of the roots of the kernel) in the general case.

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References

[1] Cyril Banderier. Combinatoire analytique : application aux marches aléatoires. *Mémoire de DEA, Université Paris VII*, 1998.

[2] Cyril Banderier. *Combinatoire analytique des chemins et des cartes*. PhD thesis, Université de Paris 6, 2001.

[3] Cyril Banderier, Mireille Bousquet-Mélou, Alain Denise, Philippe Flajolet, Danièle Gardy, and Dominique Gouyou-Beauchamps. Generating functions for generating trees. *Discrete Mathematics*, 246(1-3):29–55, 2002.

[4] Cyril Banderier and Philippe Flajolet. Basic analytic combinatorics of directed lattice paths. *Theoretical Computer Science*, 281(1-2):37–80, 2002.

[5] Cyril Banderier and Donatella Merlini. Lattice paths with an infinite set of jumps. *Proceedings of Formal power series and algebraic combinatorics (Melbourne, 2002)*, 2001.

[6] Elena Barcucci, Alberto Del Lungo, Elisa Pergola, and Renzo Pinzani. A methodology for plane tree enumeration. *Discrete Mathematics*, 180(1-3):45–64, 1998.

[7] Elena Barcucci, Alberto Del Lungo, Elisa Pergola, and Renzo Pinzani. Permutations avoiding an increasing number of length-increasing forbidden subsequences. *Discrete Mathematics & Theoretical Computer Science. DMTCS. An Electronic Journal*, 4(1):31–44 (electronic), 2000.

[8] Philippe Bougerol and Thierry Jeulin. Brownian bridge on hyperbolic spaces and on homogeneous trees. *Probability Theory and Related Fields*, 115(1):95–120, 1999.

[9] Mireille Bousquet-Mélou and Marko Petkovšek. Linear recurrences with constant coefficients: the multivariate case. *Discrete Mathematics*, 225(1-3):51–75, 2000.

[10] W. G. Brown and W. T. Tutte. On the enumeration of rooted non-separable planar maps. *Canad. J. Math.*, 16:572–577, 1964.

[11] F. R. K. Chung, R. L. Graham, V. E. Hoggatt, and M. Kleiman. The number of Baxter permutations. *Journal of Combinatorial Theory, Series A*, 24:382–394, 1978.

[12] Sylvie Corteel. Séries génératrices exponentielles pour les eco-systèmes signés. In *FPSAC'00*. Springer, June 2000.

[13] J. Dieudonné. *Infinitesimal calculus*. Hermann, Paris, 1971. Appendix 3.

[14] S. Dulucq, S. Gire, and J. West. Permutations with forbidden subsequences and nonseparable planar maps. *Discrete Mathematics*, 153(1-3):85–103, 1996. Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993).

[15] Guy Fayolle and Roudolf Iasnogorodski. Two coupled processors: the reduction to a Riemann-Hilbert problem. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 47(3):325–351, 1979.

[16] O. Guibert and E. Pergola. Enumeration of vexillary involutions which are equal to their mirror/complement. *Discrete Mathematics*, 224(1-3):281–287, 2000.

[17] Donald Ervin Knuth. *The Art of Computer Programming*, volume 1. Addison-Wesley, third edition, 1997.

[18] Darla Kremer. Permutations with forbidden subsequences and a generalized Schröder number. *Discrete Mathematics*, 218(1-3):121–130, 2000.

[19] D. Merlini and M. C. Verri. Generating trees and proper Riordan Arrays. *Discrete Mathematics*, 218:167–183, 2000.

[20] Donatella Merlini, Renzo Sprugnoli, and Maria Cecilia Verri. An algebra for generating trees. In *Mathematics and computer science (Versailles, 2000)*, pages 127–139, Basel, 2000. Birkhäuser.

[21] A. M. Odlyzko. *Asymptotic enumeration methods*, volume II. Elsevier, 1995. In Handbook of Combinatorics, (R. Graham, M. Grötschel, and L. Lovász, eds.).

[22] Elisa Pergola, Renzo Pinzani, and Simone Rinaldi. Towards an algebra of succession rules. In *Mathematics and computer science (Versailles, 2000)*, pages 141–152, Basel, 2000. Birkhäuser.

[23] Elisa Pergola and Robert A. Sulanke. Schröder triangles, paths, and parallelogram polyominoes. *Journal of Integer Sequences*, 1:Article 98.1.7, 1998.

[24] Jim Pitman. Brownian motion, bridge, excursion, and meander characterized by sampling at independent uniform times. *Electronic Journal of Probability*, 4:no. 11, 33 pp. (electronic), 1999.

[25] J. West. Generating trees and the Catalan and Schröder numbers. *Discrete Mathematics*, 146:247–262, 1995.

[26] J. West. Generating trees and forbidden subsequences. *Discrete Mathematics*, 157:363–374, 1996.

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