Puzzles and... analytic combinatorics... and Maple!

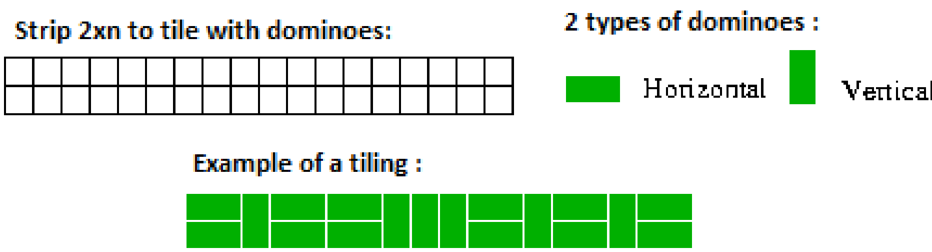
For any questions, don't hesitate to contact me:
`http://lipn.fr/~banderier/` Location of this file:
`http://lipn.fr/~cb/Papers/puzzles.pdf`

Puzzle 1

Compute the number of cubes in $\mathbb{Z}/2017^{2017}\mathbb{Z}$: 1,8,27,64,125...

Puzzle 2

We tile a a $2 \times n$ strip with horizontal and vertical dominoes.
How many tilings is there when $n = 2017^{2017}$? (just give the last 10 digits).



Puzzle 3

We take a coin a make a sequence of flips.
The result of each coin is head (H) or tail (T).
Do you think that the pattern HHH will appear before the pattern TTT?
Do you think that the pattern HTH will appear before the pattern THH?
Prove by a symmetry argument (?) that all the patterns appear on average at the same time!? ;-)

Puzzle 4

There is a table with n people around, and a bottle of water. The guy number 1 takes the bottle, fills his glass with probability p and then gives the bottle to his right or left neighboor (with probability 1/2, 1/2), and so on.
After how many moves everybody should get some water?
What is $\lim E[T_n]$?

Puzzle 5

In each bar of chocolate, there is the same probability to get any football player photo. How many bars of chocolate you need to buy on average to get the full collection of images (100 football players)?

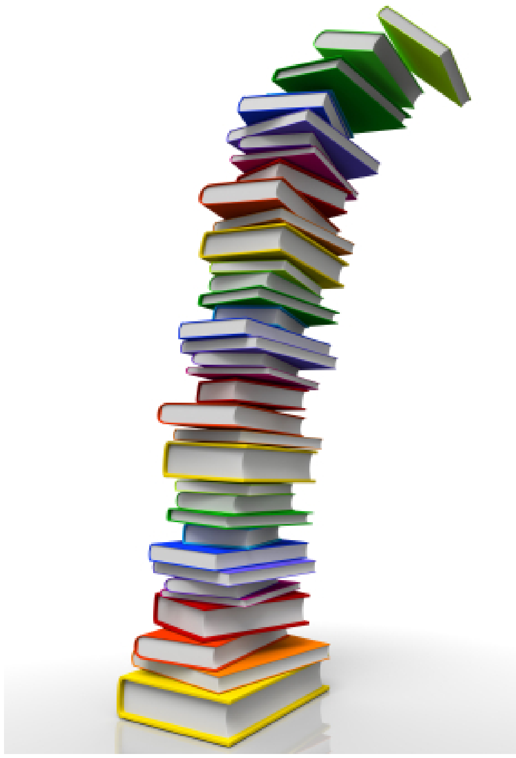
Puzzle 6 (the Alzheimer disease problem)

n people go to some temple. They leave their pair of shoes at the entrance. When they leave, they forgot where were their own shoes, and so they take a pair at random. What is the probability that nobody gets back his own pair of shoes? What is the limit when n goes to infinity?

Puzzle 7

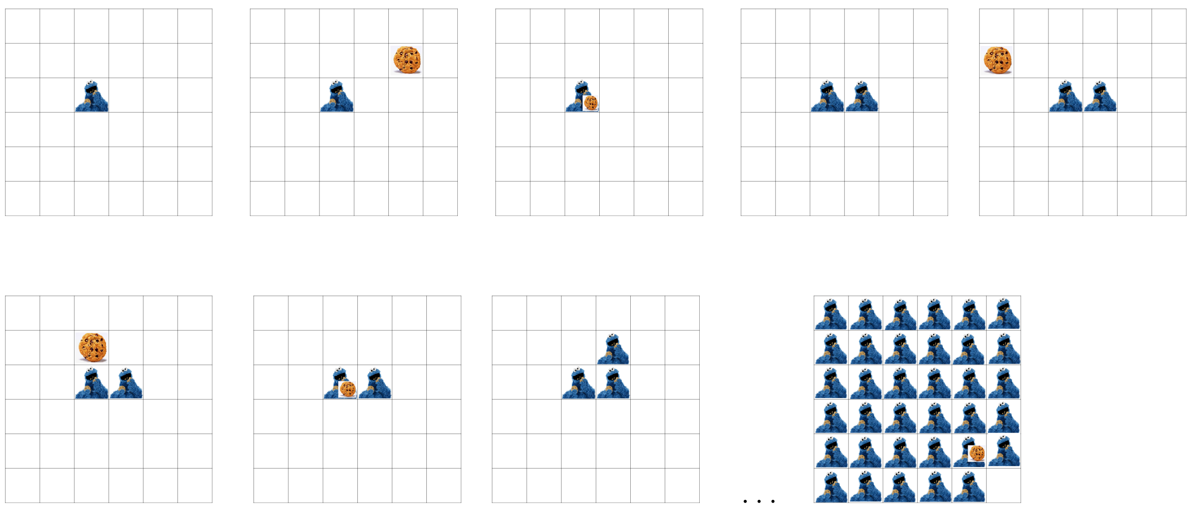
We have 100 prisoners numbered from 1 to 100. There is room with 100 drawers in it. In each drawer, there is a key with a number on it.
Each prisoner will enter to this room, open 50 drawers like he likes. If he finds his key, he goes to the exit (and the guard puts everything like initially). If he fails, ALL the prisoners will be killed.
If all prisoners reach the exit, they are all FREE. If one of them fail, they are all killed. The prisoner can talk all together BEFORE to enter the room, but NOT AFTER. What could be their strategy to improve their chance to survive?
Hints:
1. What is the probability that they all survive if each of them looks in 50 drawers at random?
2. Can you find a better strategy (because you were some good students)?
3. Fight with the Devil!

Puzzle 8: The largest stack of books



You have 1000 books of size 30x22x4 cm.
You can only stack the books, and you can do only one stack.
What is the largest size you get?
Can you get more than 60 cm?

Puzzle 9: The Cookie Monster Problem



On a $n \times n$ grid, there is a cookie monster.

At each step, one throws at random a new cookie on one of the $m = n^2$ cells of the grid.
If the monster occupies this cell, he grows up by one cell (whenever he likes).
If the monster is not occupying this cell, this cookie is lost and nothing happens. How many cookies we have to throw so that the monster will occupy the full grid? What is the asymptotics when n goes to infinity?

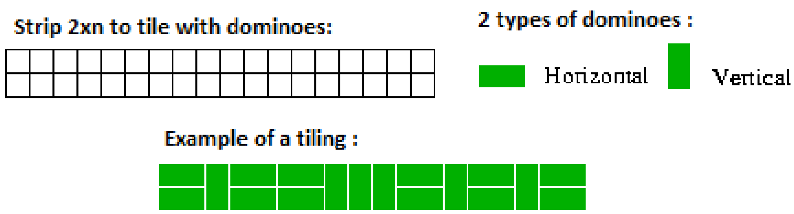
Puzzle 1

Compute the number of cubes in $\mathbb{Z}/2017^{2017}\mathbb{Z}$: 1,8,27,64,125...

Solution: $m = 2017^{2017}$ has 6666 digits, and computing the answer with a naive algorithm would take more than the age of the universe since the Big Bang! Of course, the trick is to apply the formula (which works for any m) which I proved in my other talk. NB: 2017 is prime, this makes our multiplicative formula even faster to use.

Puzzle 2

We tile a a $2 \times n$ strip with horizontal and vertical dominoes.
How many tilings is there when $n = 2017^{2017}$? (just give the last 10 digits).



Solution: $\mathrm{Seq}(z+z^2) = 1/(1-z-z^2)$. Fibonacci numbers ($f_0 = f_1 = 1$). Binet Formula (berk), binary exponention approach with matrix. Taking everything $\mod 10^{10}$ gives the answer.

Puzzle 3

We take a coin a make a sequence of flips.
The result of each coin is head (H) or tail (T).
Do you think that the pattern HHH will appear before the pattern TTT?
Do you think that the pattern HTH will appear before the pattern THH?
Prove by a symmetry argument (?) that all the patterns appear on average at the same time !? ;-)

Solution: An automata with 3 states gives the corresponding probability generating functions. Not all patterns are equal, the automata shows the faster patterns.

Puzzle 4 and 5

Solution: Variant of coupon collector problem, see my article `http://lipn.fr/~cb/Papers/walksgraphs.pdf`.

Puzzle 6 (Solution of the Alzheimer disease problem)


n people go to some temple. They leave their pair of shoes at the entrance. When they leave, they forgot where were their own shoes, and so they take a pair at random. What is the probability that nobody gets back his own pair of shoes? What is the limit when n goes to infinity?

Solution:

Indeed, we have here a permutation without fixed points! Any permutation can be seen as set of cycles. The underlying combinatorial structure behind this problem is thus a set of "cycles of length > 1".

Via the magic dictionary between combinatorial structure and their generating functions, we have:

combinatorial structure	associated generating function
Set(C)	$\exp(C)$
Cycle(z)	$\ln(\frac{1}{1-z})$
Cycle(z) excepted length 1	$\ln(\frac{1}{1-z}) - z$
Cycles of length > 1	$\exp(\ln(\frac{1}{1-z}) - z) = \frac{\exp(-z)}{1-z}$

So this implies $\displaystyle\sum_{n\geq 0} f_n z^n/n! = \sum_{n\geq 0}\left(\sum_{k=0}^{n}(-1)^k/k!\right) z^n$.

This gives the exact formula $f_n = n! \sum_{k=0}^{n}(-1)^k/k!$ and the wanted probability is $f_n/n!$, which converges to $\exp(-1) \approx 0.367$.

It is quite surprising that by taking shoes at random, at least one guy will get his own shows (with proba $1 - \exp(-1) \approx 0.632$ when n gets large).

Puzzle 7 (solution)

100 prisoners are in a jail ; the boss of the jail is a crazy guy who however wants to give them a chance: he creates a room with 100 drawers in it. In each drawer, there is a paper with the name of a prisoner (all names are different, each name is one drawer).
Each prisoner will enter this room, open 50 drawers like he likes. If he finds his name, he goes to the exit (and the guard puts everything like *initially*). If he fails, ALL the prisoners will be killed.
If all prisoners reach the exit, they are all FREE. If one of them fails, they are all killed. The prisoners can talk all together BEFORE to enter the room, but NOT AFTER.
What could be their strategy to improve their chance to survive?

Hints:

- What is the probability that they all survive if each of them looks in 50 drawers at random?
- Can you find a better strategy (because you were some good students)?
- Defeat the Devil!

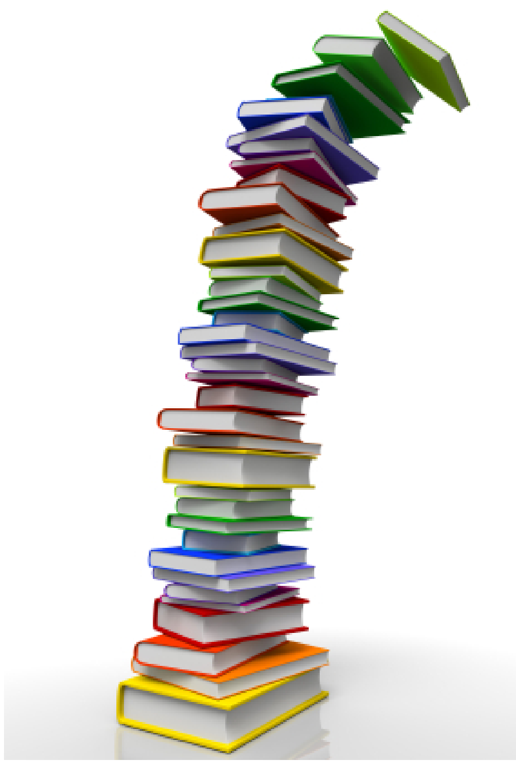
Solution: If each prisoner opens 50 drawers at random, the probabilty that they all succeed is $(1/2)^{100}$. But it is possible to reach better success rate by using the underlying structure: it is a permutation! In which to share information to increase the success rate? The optimal strategy consist in following the cycle structure: prisoner $\#i$ opens drawer $\#i$, and then follows the cycle. Now, the prob have success of each prisonner is no more independant. In fact, this will fail only if (and only if) one cycle is of length more than 50.

And in case the boss of the jail knows that you will use this strategy and if tries to ruin the prisoner's strategy by choosing a permutation with cycles of length more 50, the prisoners can defeat this devil by first relabelling alll the drawers with a random permutation they agreed on initially.

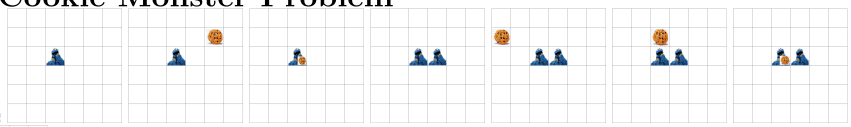
Puzzle 8: The largest stack of books

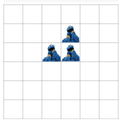
You have 1000 books of size 30x22x4 cm.
You can only stack the books, and you can do only one stack.
What is the largest width you can get?
Can you get more than 60 cm?

Solution: With 1000 such books, you can reach 1,94 meter: Stack the books according to the sequence $1/k$, in reverse order. For each new book that you had, the stack is therefore at its limit of stability. The equilibrium point and the ratio of masses implies that the distance is thus $d_n = 1/2 + 1/3 + \cdots + 1/n$. An upper and a lower bound given by a comparison with the integral of $1/x$ proves $d_n \sim \ln n$. Also, to avoid any physical border effect, you can e.g. take $1/k - 1/k^2$, then $d_n \sim H_n - 1 - \pi^2/6$. This still gives the $\ln n$ asymptotics (assuming uniform gravity in a infinite space).



Puzzle 9: The Cookie Monster Problem[1]

 ...  On a $m \times m$ grid, there is a cookie monster. At each step, one throws at random a new cookie on one of the $n = m^2$ cells of the grid.
If the monster occupies this cell, he grows up by one cell (whenever he likes).
If the monster is not occupying this cell, this cookie is lost and nothing happens.
How many cookies we have to throw so that the monster will occupy the full grid?

What is the asymptotics when n goes to infinity?

Solution This is clearly another variant of the coupon collector problem.
When the monster is of size k, one has proba k/n to hit him.
So the waiting time to hit him is 1/(this proba).
The total sum is $n/1 + n/2 + \cdots + n/(n-1) = nH_{n-1} \sim n \ln n$

Nice variant (Problem 3. from the appendix of Knuth & Greene's book) The Modest Cookie Monster: Consider a monster whose appetite for cookies decreases as its size increases. When we throw a cookie to such a monster with k cookies in its belly, with probability $1 - pk^2$ the monster will grow to size $k + 1$.

Let $n > 0$ be an integer and $p = 1/n^2$. Initially the monster has an empty stomach. Let $X_{n,m}$ be the random variable corresponding to the size of the monster after m cookies have been thrown. Let $\alpha > 0$ be any fixed real number. *Question:* Determine the asymptotic mean and variance of $X_{n,\lceil \alpha n \rceil}$ to within a multiplicative factor of $1 + o(1)$, as $n \to \infty$.

Comment: This question arises in the analysis of a modified version of hashing with separate chaining, in which each key has two random hashing addresses. A key will be added to the list headed by its first hashing address, only if both addresses are nonempty.

[1] There is another problem called the cookie monster problem.