Hilbertian Frobenius algebras
(Extended Abstract)

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A Frobenius algebra may be described in several equivalent ways and the most suitable for the purpose of this Extended Abstract of the note [7] is as follows. A Frobenius algebra [1, Theorem 1, p. 572] is a finite-dimensional unital algebra which at the same time is a counital coalgebra such that both structures interact nicely. More precisely the compatibility condition between the algebra and the coalgebra of a Frobenius algebra – the so-called *Frobenius relation* – asserts that the comultiplication of the latter is a morphism of bimodules over the former.

Because the above compatibility relation is stated entirely using only tensor products and linear maps, the former description has the advantage over others to allow for talking about Frobenius algebras in the realm of monoidal categories. This is precisely the point of view adopted in the recent book [5]. This approach is used with success in [3, 2] where the authors take advantage of the presence of the Hilbertian adjoint to define a comultiplication from a multiplication. More precisely they consider commutative *†-Frobenius monoids*, that is, Frobenius algebras $(H, \mu, \eta, \delta, \epsilon)$ over a finite-dimensional Hilbert space H, where $\delta = \mu^\dagger$ and $\epsilon = \eta^\dagger$.[1]

Most notably the main result in [3] is the statement that orthogonal bases on a given finite-dimensional Hilbert space and its structures of commutative (unital and counital) †-Frobenius monoids are in a one-one correspondence.

In an effort to extend this result to arbitrary Hilbert spaces, commutative †-Frobenius semigroups in the category of Hilbert spaces are studied in [2], obtained by dropping the unital and counital assumptions. More precisely, these are semigroups (H, μ) (in the monoidal category $(\mathbf{Hilb}, \hat{\otimes}_2, \mathbb{C})$ of Hilbert spaces and bounded linear

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[1]In the note [7] "*Hilbertian Frobenius*" stands for "*†-Frobenius*", to recall the fact that the comultiplication is the Hilbert adjoint of the multiplication, because no other kinds of Frobenius semigroups in the category of Hilbert spaces are considered in [7].

maps), that is, $H\hat{\otimes}_2 H \xrightarrow{\mu} H$ is commutative and associative, such that furthermore the adjoint $H \xrightarrow{\mu^\dagger} H\hat{\otimes}_2 H$ of μ satisfies the Frobenius condition.

It is precisely the intention of the note [7] to provide a better understanding of the relations between orthogonal bases or better orthogonal sets, and algebraic structures of commutative †-Frobenius semigroups over not necessarily finite-dimensional Hilbert spaces.

We, hence, provide a structure theorem for commutative †-Frobenius semigroups in **Hilb** ([7, Theorem 30, p. 21]): it is an easy fact that they split as an orthogononal direct sum of their Jacobson radical and the topological closure of the linear span of their *group-like* elements, that is, those non zero elements x sent to $x \otimes x$ by the comultiplication $\mu^\dagger$. But what is less immediate is that in fact, the orthogonal complement of the Jacobson radical is a subalgebra, that is, that the set of group-like elements is closed under multiplication. In fact it is not difficult to notice that the product of two *distinct* group-like elements is equal to zero, but what is not as immediate is that the square of a group-like element belongs to the one-dimensional space spanned by the element (to be fair this observation is free when the comultiplication is assumed isometric as in [2], which is not the case in [7]), which by the way turns this space into a minimal ideal.

A similar result is claimed in [2] for the particular case of commutative †-Frobenius semigroups which are *special*, that is, with an isometric comultiplication, but its proof is not completely correct (see the introduction of [6] for more details). Furthermore our structure theorem is completed by the observation that the radical is precisely the annihilator of the algebra, which is a direct consequence unnoticed elsewhere, of the Frobenius condition (see [7, Proposition 29, p. 21]) which forces the multiplication operators to be normal, that is, that they commute with their own adjoint.

Thus, clearly, a commutative †-Frobenius semigroup (H, μ) in **Hilb** not only splits into an orthogonal direct sum of a subalgebra and an ideal, but actually of two (closed) ideals, one being radical while the other is semisimple. As becomes clear the question of semisimplicity of such an algebra is completely governed by this structure theorem: a necessary and sufficient condition for a commutative †-Frobenius algebra to be semisimple is that its regular representation is faithful, or equivalently that its multiplication $H\hat{\otimes}_2 H \xrightarrow{\mu} H$ has a dense range or using the Hilbert adjoint, that its comultiplication is one-to-one ([7, Theorem 33, p. 22]). (In particular every commutative *special* †-Frobenius algebra in **Hilb** is semisimple.) It is a remarkable fact that in the finite-dimensional situation this may be interpreted as the existence of a unit ([7, Corollary 35, p. 23]). Consequently is recovered (with a proof free of a C^*-argument) the result of [3] that finite-dimensional commutative †-Frobenius *monoids* are semisimple.

By the way, because group-like elements of a commutative †-Frobenius semigroup, are non-zero and pairwise orthogonal (even orthonormal when furthermore the algebra is special, that is, when the comultiplication is an isometry) several bijective correspondences ([7, Theorem 37, p. 24]) between structures of †-Frobenius semigroups of certain kinds available on a given Hilbert space, and some of its orthogonal (or orthonormal) sets are obtained, which extend the main result of [3]. It is worth mentioning that contrary to the finite-dimensional situation, not all orthogonal sets correspond to a structure of a commutative †-Frobenius semigroup but only those which are bounded above (or below by a strictly positive constant), including the empty set; in fact one cannot expect unbounded orthogonal families to be in the range of the above bijections as boundedness of the norm of the group-like elements, is a direct consequence of the fact that in a Banach algebra, the multiplication as a bilinear map, is jointly continuous.

Besides the above structure theorem also has some important consequences at the level of the category ${}_c\mathbf{FrobSem}(\mathbb{Hilb})$ of commutative †-Frobenius semigroups in **Hilb** and (bounded) semigroup morphisms. Most notably it is shown that every semigroup morphism between commutative †-Frobenius semigroups arises from a unique set-theoretic base-point preserving map (of some specific kind), from the set of minimal ideals of its codomain to the set of minimal ideals of its domain, both with zero added as base-point. Among others one proves that

1. ${}_c\mathbf{FrobSem}(\mathbb{Hilb})$ is equivalent to the product ${}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{Hilb})\times\mathbf{Hilb}$ ([7, Proposition 46, p. 31]) where the first factor is the full subcategory of ${}_c\mathbf{FrobSem}(\mathbb{Hilb})$ spanned by the semisimple algebras. The splitting into a semisimple †-Frobenius semigroup and the radical (which is essentially a Hilbert space as its multiplication is trivial) provides the equivalence.

2. ${}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{Hilb})$ is dually equivalent to a category of pointed weighted sets ([7, Theorem 56, p. 37]). The functor whose object component sends a semisimple commutative †-Frobenius semigroup to its set of minimal ideals (with the trivial ideal added), implements the equivalence. Its equivalence inverse is similar to the $\ell^2_\bullet$-functor introduced in [6] which is itself up to the adjoint, an extension of the ℓ^2-functor from [4].

In [7] is also discussed an example of a non-commutative non-semisimple Hilbertian Frobenius semigroup whose merit is to show that the Frobenius condition does not govern anymore semisimplicity in the non-commutative situation.

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