From “combinatorial” monoids to bialgebras and Hopf algebras, functorially

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Monoidal categories

Roughly speaking a monoidal category is a usual category equipped with an "associative" multiplication and a two-sided "identity".

Examples include:

1. The category **Set** of sets with the cartesian product as multiplication and any one-point set as unit (one will take $1 := \{ 0 \}$).

2. The category of vector spaces over a field $\mathbb{K}$ with the tensor product $\otimes_{\mathbb{K}}$ as multiplication and $\mathbb{K}$ as unit.

A convenient setting to deal with monoids

A key feature of monoidal categories is their ability to talk about monoids in a quite general setting. Indeed, relative to any monoidal category one can define algebraic structures that behave like usual monoids.

E.g., a monoid in the category of sets is a usual monoid, a monoid in the category of $\mathbb{K}$-vector spaces (with the tensor product) is a unital $\mathbb{K}$-algebra, while a monoid in the category of commutative $\mathbb{K}$-algebras (again with the tensor product) is a commutative bialgebra over $\mathbb{K}$ (hence an affine monoid scheme or even an algebraic monoid if one restricts to finitely generated algebras).

Moreover using (monoidal) functors one can put in relations monoids from different categories.

Another link: the set of isomorphism classes in a monoidal category inherits a structure of monoid from the monoidal structure (called the classifying monoid of the monoidal category under consideration).

Purpose of the talk

• Give a category-theoretic interpretation of some "combinatorial" monoids as monoids in monoidal categories.

• Use monoidal functors to explain functorially some properties of their (completed) monoid algebras.

• Prove that some of these monoid algebras give rise to monoid and even ring schemes (i.e., bialgebras and Hopf rings).

Purpose of the talk

Let $\mathbf{M}$ be a category of "combinatorial" monoids, and let $\mathbf{A}$ be a category of algebras.

$$\begin{array}{ccc} \mathbf{Mon}(\mathbb{C}) \cong \mathbf{M} & \xrightarrow{\tilde{\mathbb{F}}} & \mathbf{Mon}(\mathbb{D}) \cong \mathbf{A} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle V} \\ \mathbb{C} & \xrightarrow[\mathbb{F}]{} & \mathbb{D} \end{array}$$

where $\tilde{\mathbb{F}}$ is the "(large) monoid algebra" functor.

"Combinatorial" monoids

- Finite decomposition monoid: For each $x \in M$, there are only finitely many $y, z \in M$ such that $x = y * z$.

- Filtered monoid: A monoid together with a decreasing filtration $\ldots \subseteq M_2 \subseteq M_1 \subseteq M_0 \subseteq M$ such that $x_m * x_n \in M_{m+n}$ and $1 \in M_0$.

- Locally finite monoid: For each $x \in M$, there are only finitely many $x_1, \cdots, x_n \in M \setminus \{ 1 \}$ such that $x = x_1 * \cdots * x_n$.

- Clifford monoid: An inverse monoid in which every idempotent lies in the center.

The class of finite decomposition monoids is the larger class for which convolution of functions is possible.

Let R be a commutative ring with a unit. Let M be a finite decomposition monoid. Then one can define the R-coalgebra $R^{(M)}$ (free module with basis M)

$$\Delta(x) = \sum_{x=y*z} y \otimes z$$

and

$$\epsilon(x) = 1 \ .$$

It follows that one can consider its dual R-algebra $R[[M]]$, called the large algebra of M, of all functions from M to R. Its multiplication is given by convolution

$$(f * g)(x) = \sum_{x=y*z} f(y)g(z) \ .$$

When M is a locally finite monoid, then $R[[M]]$ admits a structure of a (complete) filtered algebra.

That makes it possible to consider a star operation. Given $f \in R[[M]]$ such that $f(1) = 0$, then one defines $f^\star = \sum_{n\geq 0} f^n$.

It follows that $\{ f \in R[[M]] \colon f(1) = 1 \}$ is a subgroup of invertible elements of $R[[M]]$. The inverse of f is given by $(f - \delta_1)^\star$.

Möbius inversion formula: let $\zeta = \sum_{x\in M} x$ (called the zêta function of M), and let $\mu = \zeta^{-1}$ (called the Möbius function of M). Then for all $f, g \in R[[M]]$,

$$g(x) = \sum_{x=y*z} f(y) \Leftrightarrow f(x) = \sum_{x=y*z} g(y)\mu(z) .$$

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Monoidal category

"Category with a multiplication"

A monoidal category is given as a 6-tuple $\mathbb{C} = (\mathbf{C}, \otimes, I, \alpha, \lambda, \rho)$, where $\mathbf{C}$ is a category, $\otimes\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is a functor (called the (monoidal) tensor) and I is a particular object of $\mathbf{C}$ (the unit) with natural isomorphisms

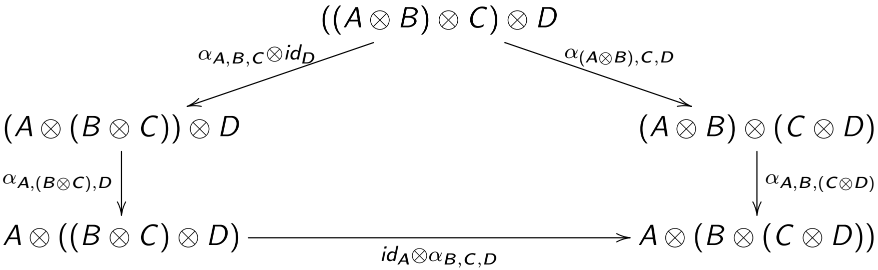
$$
\begin{aligned}
&\text{(Ass)} \quad (A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C), \\
&\text{(lUnit)} \quad I \otimes C \xrightarrow{\lambda_C} C, \\
&\text{(rUnit)} \quad C \otimes I \xrightarrow{\rho_C} C.
\end{aligned}
$$

A symmetric monoidal category $\mathbb{C}$ requires another natural isomorphism $\sigma_{C,D}\colon C \otimes D \to D \otimes C$ such that $\sigma_{D,C} \circ \sigma_{C,D} = id_{C \otimes D}$.

Moreover, certain technical conditions, called coherence conditions, have to be satisfied in order that the multiplication be "well-behaved".

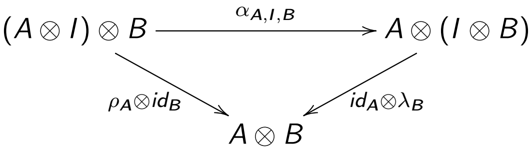
Coherence: Associativity, Mac Lane - Stasheff's Pentagon

How to move brackets ?



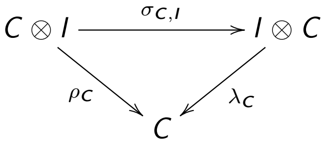
Commutativity of this diagram ensures that $\otimes$ is "associative" and that the bracketing is irrelevant.

Coherence: Left and right unit



Coherence: Symmetry

Commutative multiplication



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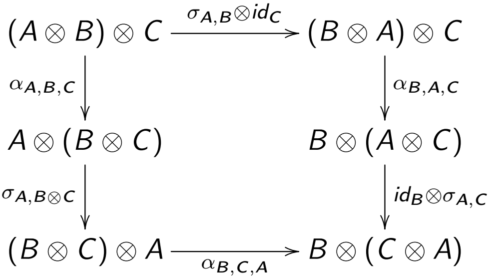


Examples

1. $(\mathbf{Set}, \times, 1)$ where 1 is a one-point set.

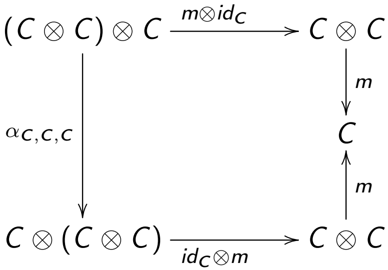
2. $(\mathbf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$, $({}_R\mathbf{Mod}, \otimes_R, R)$, $({}_R\mathbf{Mod}_R, \otimes_R, R)$, $\mathbf{Ban}_1$ (projective tensor product), **Hilb** (Hilbertian tensor product).

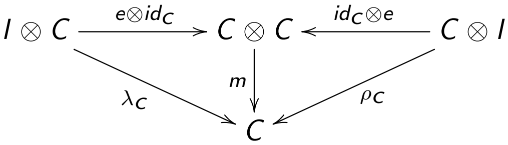
3. $\mathbb{C}^{\mathrm{op}} = (\mathbf{C}^{\mathrm{op}}, \otimes, I)$.

The monoidal category ${}_R\mathbf{Mod}_R$ of R-bimodules fails to be symmetric.

Monoids

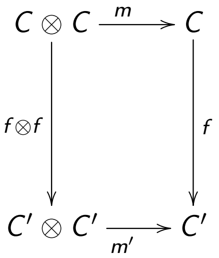
A monoid in $\mathbb{C}$ is a triple $(C, C \otimes C \xrightarrow{m} C, I \xrightarrow{e} C)$ such that the following diagrams commute

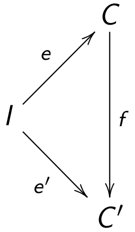




Morphisms

A morphism $f\colon (C, m, e) \to (C', m', e')$ is a **C**-morphism $f\colon C \to C'$ such that the following diagrams commute





This defines a category **Mon**$(\mathbb{C})$.

Comon$(\mathbb{C}) := ($**Mon**$(\mathbb{C}^{\mathrm{op}}))^{\mathrm{op}}$ is called the category of comonoids in $\mathbb{C}$.

Example: **Mon**(**Set**)

A monoid in $(\mathbf{Set}, \times, 1)$ is given as a triple $(C, C \times C \xrightarrow{m} C, 1 \xrightarrow{e} C)$.

Clearly, e picks out an element of C namely $e(0)$ (where 0 is the only member of 1).

It is easily checked that $(C, m, e(0))$ is a (usual) monoid.

Example: **Mon(Ab)**

Let $(C, C \otimes_{\mathbb{Z}} C \xrightarrow{m} C, \mathbb{Z} \xrightarrow{e} C)$ be a monoid in $(\mathbf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$.

Hence C is an abelian group, and, because $\mathbb{Z}$ is the free (abelian) group on one generator, e is uniquely determined by the value $e(1)$.

Moreover there is a unique bi-additive map $m_0 \colon C \times C \to C$ which factors through the tensor product to provide m.

Finally $(C, m_0, e(1))$ is a just a ring.

Examples

1. $\mathbf{Mon}(\mathbf{Mon}) \cong \mathbf{cMon}$.
2. $\mathbf{Mon}({}_R\mathbf{Mod}) \cong {}_R\mathbf{Alg}$ (with R commutative).
3. $\mathbf{Mon}({}_R\mathbf{Mod}_R) \cong {}_R\mathbf{Ring}$.
4. $\mathbf{Mon}(\mathbf{Ban}_1) \cong \mathbf{BanAlg}$.
5. $\mathbf{Comon}({}_R\mathbf{Mod}) \cong {}_R\mathbf{Coalg}$.
6. $\mathbf{Comon}(\mathbf{Set}) \cong \mathbf{Set}$. Indeed, on each set X exists a unique comonoid structure $(X, X \xrightarrow{\Delta_X} X \times X, X \xrightarrow{!} 1)$. (But there might be several cosemigroup structures!)

Monoidal functors

A monoidal functor $\mathbb{C} \to \mathbb{C}'$ is a triple given by

- a functor $F \colon \mathbf{C} \to \mathbf{C}'$,
- a natural transformation $\Phi_{C,D} \colon F(C) \otimes' F(D) \to F(C \otimes D)$,
- a $\mathbf{C}'$-morphism $\phi \colon I' \to F(I)$

subject to certain coherence conditions.

Examples

1. The forgetful functor **Ab** $\xrightarrow{|-|}$ **Set** is monoidal (with $|G| \times |H| \to |G \otimes H|$ the canonical map and $1 \to \mathbb{Z}$ the map that picks out $0 \in \mathbb{Z}$).

2. Let M be a monoid, and let us consider the category ${}_M\mathbf{LAct}$ of left M-actions (i.e., a set X equipped with a homomorphism of monoids $M \to X^X$) with equivariant maps as morphisms. Then, the forgetful functor ${}_M\mathbf{LAct} \to \mathbf{Set}$ is monoidal.

3. The (covariant) powerset functor $\mathfrak{P}\colon \mathbf{Set} \to \mathbf{Set}$ is monoidal (with $\Phi_{C,D}\colon \mathfrak{P}(C) \times \mathfrak{P}(D) \to \mathfrak{P}(C \times D)$ given by $(A, B) \mapsto A \times B$, and $\phi\colon 1 \to \mathfrak{P}(1)$ defined as $\phi(0) = 1$; recall that $1 = \{ 0 \}$).

Monoidal functors transform monoids into monoids

The following folklore result is well-known from category theory.

Proposition

Let $\mathbb{F} := (F, \Phi, \phi) \colon \mathbb{C} \to \mathbb{C}'$ be a monoidal functor, and let (C, m, e) be a monoid in $\mathbb{C}$.

Then, $\tilde{\mathbb{F}}(C, m, e) := (F(C), F(C) \otimes F(C) \xrightarrow{\Phi_{C,C}} F(C \otimes C) \xrightarrow{F(m)} F(C), I' \xrightarrow{\phi} F(I) \xrightarrow{F(e)} F(C))$ is a monoid in $\mathbb{C}'$.

Let $f \colon (C, m, e) \to (C', m', e')$ be a morphism of monoids, then with $\tilde{\mathbb{F}}(f) := F(f)$, $\tilde{\mathbb{F}} \colon \mathbf{Mon}(\mathbb{C}) \to \mathbf{Mon}(\mathbb{C}')$ defines a functor.

Example: the monoid structure on the powerset of a monoid

Given a (set-theoretic) monoid $(M, *, 1)$ (here $1 \in M$ is the identity), it is well-known that $\mathfrak{P}(M)$ admits a structure of monoid given by

$$A * B = \{\, a * b \colon a \in A,\ b \in B \,\}$$

for $A, B \subseteq M$, and with $\{\, 1 \,\}$ as the identity.

This is just the consequence of the fact that $\mathfrak{P}\colon \mathbf{Set} \to \mathbf{Set}$ is a monoidal functor because the monoidal structure on $\mathfrak{P}(M)$ is the image of $(M, *, 1)$ by $\tilde{\mathfrak{P}}$.

Remark

Actually $\tilde{\mathfrak{P}}(M, *, 1)$ is a monoid with a zero (the empty set). This is so because $\mathfrak{P}$ may be shown to be a monoidal functor from $\mathbf{Set}$ to $\mathbf{Set}_\bullet$ the category of pointed sets, base-point preserving maps and with the smash-product as monoidal tensor. The monoids in $\mathbf{Set}_\bullet$ are precisely the monoids with a zero.

Cartesian monoidal categories

Let **C** be a category, and let C, D be objects of **C**.

A product of C, D is an object $C \times D$ together with two morphisms $\pi_C\colon C \times D \to C$ and $\pi_D\colon C \times D \to D$ satisfying the following property.

For all morphisms $f\colon A \to C$ and $g\colon A \to D$, there is a unique morphism $h\colon A \to C \times D$ such that $\pi_C \circ h = f, \pi_D \circ h = g$.

Group objects

The existence of such products for each ordered pair of objects (and also what is called a terminal object) gives rise to a monoidal structure on **C** which is referred to as a cartesian monoidal category. For instance the category of sets with the cartesian product is a cartesian monoidal category, so is the category of $\mathbb{K}$-vector spaces with the direct product.

If $\mathbb{C}$ is such a monoidal category, then in addition to monoids one can consider groups (or even any equational variety of algebras) in $\mathbb{C}$.

E.g., groups in **Set** are the usual groups, groups in the category of topological spaces, with the usual topological product, are topological groups, groups in the category of cocommutative coalgebras are (cocommutative) Hopf algebras.

Coproduct

A coproduct in a category $\mathbf{C}$ is a product in the opposite category $\mathbf{C}^{\text{op}}$.

In more details: let C, D be objects of $\mathbf{C}$. A coproduct of C, D is an object $C \coprod D$ of $\mathbf{C}$ with two morphisms $q_C \colon C \to C \coprod D$ and $q_D \colon D \to C \coprod D$ satisfying the following property.

For all morphisms $f \colon C \to A$ and $g \colon D \to A$, there is a unique morphism $h \colon C \coprod D \to A$ such that $h \circ q_C = f, h \circ q_D = g$.

Examples: the set-theoretic disjoint sum, the free product of monoids or algebras, the tensor product over $\mathbb{Z}$ for commutative rings.

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Finite decomposition monoid

Let M be a monoid. It is said to be a finite decomposition monoid if its product $*$ has finite fibers.

In details this means that for each $x \in M$, there are only finitely many $y, z \in M$ such that $x = y * z$.

Category-theoretic interpretation

Let us consider the category **FinFibSet** of all sets with finite-fiber maps.

It admits a structure of a symmetric monoidal category inherited from the set-theoretic cartesian product.

The category of monoids in $\mathbb{FinFibSet}$ is then (isomorphic to) the category of finite decomposition monoids (homomorphisms of monoids with finite fibers).

Topologically free module

Large algebra

A R-module is said to be a topologically free R-module whenever it is isomorphic to a module of the form R^X for some set X.

Each such module admits a linear topology whose basis of open neighborhoods of zero is given by $R^{(X \setminus A)}$ for finite subsets $A \subseteq X$.

Clearly $\varprojlim_{A \in \mathfrak{P}_{\mathrm{fin}(X)}} R^X / R^{(X \setminus A)} \cong \varprojlim_{A \in \mathfrak{P}_{\mathrm{fin}}(X)} R^A \cong R^X$, hence R^X is complete in the inverse limit topology (where all R^A are discrete), this topology is equivalent to the product topology (with R discrete).

Let us denote by ${}_R\mathbf{TopFreeMod}$ the category of all topologically free modules with continuous linear maps.

Completed tensor product

Let us provide to the algebraic tensor product $R^X \otimes_R R^Y$ a linear topology as follows.

For each $A \in \mathfrak{P}_{\mathrm{fin}}(X)$ and $B \in \mathfrak{P}_{\mathrm{fin}}(Y)$, let us consider the canonical map $R^X \otimes_R R^Y \to R^A \otimes_R R^B \cong R^{A\times B}$ (induced by the projections).

The kernels, say $K_{A,B}$, of these maps form the basis of the topology.

And

$$R^{X\times Y} \cong \varprojlim_{A,B} R^{A\times B} \cong \varprojlim_{A,B} (R^X \otimes_R R^Y)/K_{A,B} \; .$$

Completed tensor product

One thus defines $R^X \hat{\otimes}_R R^Y := R^{X \times Y}$ ($\hat{\otimes}$ is a bifunctor), so that $R^X \hat{\otimes}_R R^Y$ is the completion of $R^X \otimes_R R^Y$ (in the linear topology).

There exists a continuous R-bilinear map $can \colon R^X \times R^Y \to R^X \hat{\otimes} R^Y$.

Theorem (Universal property of $\hat{\otimes}$)

Let $\phi \colon R^X \times R^Y \to R^Z$ be a continuous R-bilinear map. Then, there exists a unique continuous R-linear map $\phi_0 \colon R^X \hat{\otimes}_R R^Y \to R^Z$ such that $\phi_0 \circ can = \phi$.

Monoidal category

$_R\mathbf{TopFreeMod}$ with $\hat{\otimes}$ becomes a symmetric monoidal category $_R\mathbb{TopFreeMod}$, and $\mathbf{Mon}({}_R\mathbb{TopFreeMod})$ consists in complete R-algebras.

Let us define a functor $R^-\colon \mathbf{FinFibSet} \to {}_R\mathbf{TopFreeMod}$ such that

$$X \mapsto R^X$$

and for $\phi\colon X \to Y$, let $R^\phi\colon R^X \to R^Y$ be given by

$$(R^\phi)(f)(y) = \sum_{x \in \phi^{-1}(\{ y \})} f(x)$$

$f \in R^X$, $y \in Y$.

R^- is a monoidal functor, hence it lifts to a functor between categories of monoids. One recovers $M \mapsto R[[M]]$, where M is a finite decomposition monoid, and this corrects the lack of functoriality of the large algebra as defined usually.

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Filtered sets

A (decreasing) filtration on a set X is a decreasing sequence $(X_n)_n$ of finite subsets of X. $(X, (X_n)_n)$ is thus called a filtered set.

A morphism $f\colon (X, (X_n)_n) \to (Y, (Y_n)_n)$ is a set-theoretic map $f\colon X \to Y$ such that for each n, $f(X_n) \subseteq Y_n$. Such a map is said to be a filtration-preserving map.

The category of all filtered sets admits a monoidal tensor $(X, (X_n)_n) \otimes (Y, (Y_n)_n) = (X \times Y, (T^n(X, Y))_n)$ with

$$T^n(X, Y) = \bigcup_{i=0}^{n} X_i \times Y_{n-i} \ .$$

The unit is the one-point set 1 with filtration $1_n = \emptyset$ for all $n > 0$ and $1_0 = 1$.

monoidal sub-categories

A filtered set $(X, (X_n)_n)$ is

- exhausted if $X = X_0$,
- separated if $\bigcap_{n \geq 0} X_n = \emptyset$,
- connected if it is both separated and exhausted, and $X_0 \setminus X_1 = 1$.

A set X with an exhausted and separated filtration is equivalent to a set X with a length function $\ell \colon X \to \mathbb{N}$. ($X_n := \{ x \in X \colon \ell(x) \geq n \}$ and $\ell(x) := \sup\{ n \in \mathbb{N} : x \in X_n \}$.)

A filtered set is connected if, and only if, there is a unique element of length zero.

Locally finite monoid

A monoid M is said to be a locally finite monoid if for each $x \in M$, there are only finitely many $x_1, \cdots, x_n \in M \setminus \{ 1 \}$ such that $x = x_1 * \cdots * x_n$.

Such a monoid is necessarily a finite decomposition monoid.

It may be equipped with a length function $\ell(x) = \sup\{ n \in \mathbb{N} \colon \exists (x_1, \cdots, x_n) \in M \setminus \{ 1 \},\ x = x_1 * \cdots * x_n \}$ that satisfies $\ell(x * y) \geq \ell(x) + \ell(y)$ and $\ell(x) = 0$ if, and only if, $x = 1$. It is called the canonical length function.

Hence a locally finite monoid is also a monoid in the monoidal category of connected filtered sets.

Monoids

One now considers the monoidal category $\mathbb{cSet}$ of all connected filtered sets with finite-fiber and filtration-preserving maps.

Theorem

A monoid in $\mathbb{cSet}$ is precisely a locally finite monoid.

Proof: A monoid object in $\mathbb{cSet}$ is a usual monoid M with a connected filtration $(M_n)_n$ of (two-sided) ideals of M. Let ℓ be its associated length function. It thus satisfies $\ell(x * y) \geq \ell(x) + \ell(y)$. Since it is connected, $\ell^{-1}(\{ 0 \}) = \{ 1 \}$. Let us assume that there exists some $x \in M$ with arbitrary long non-trivial decompositions. Then, for every n, $\ell(x) \geq n$ (since $x = x_1 * \cdots * x_m$, $m \geq n$, $x_i \neq 1$) which is impossible since the filtration is separated. □

Filtered modules

Filtered module: A R-module M endowed with a (decreasing) filtration M_n of submodules.

Filtered maps: Linear maps that respect the filtrations.

Complete filtered module: $M \cong \varprojlim_n M/M_n$. Any filtered module M admits a completion, namely $\hat{M} = \varprojlim_n M/M_n$. Let $\hat{M}_n$ be the kernel of the projection $\hat{M} \to M/M_n$. Then $\hat{M}$ is filtered (with $(\hat{M}_n)_n$) and $\hat{M} \cong \varprojlim_n \hat{M}/\hat{M}_n$.

Filtered tensor product: The algebraic tensor product $M \otimes_R N$ together with the filtration $\sum_{i+j=n} M_i \otimes_R N_j$.

Completed tensor product: $M \hat{\otimes} N = \widehat{M \otimes_R N}$.

Monoid: Filtered (complete) R-algebras.

Large algebra

Let M be a locally finite monoid. Then, its canonical filtration induces (functorially) a structure of an exhausted and separated filtered algebra on $R[[M]]$.

It is given by $\mathfrak{I}_n = \{ f \in R^M : \forall x(\ell(x) < n \Rightarrow f(x) = 0) \}$.

The associated (linear) topology is always stronger than the product topology (i.e., the canonical projections are continuous), and can be even strictly stronger.

$R[[M]]$ is complete in this topology but is not necessarily the completion of $R[M]$ with the induced topology.

Remark

Of course R^- is again a monoidal functor from the category of connected filtered sets (with finite-fiber and filtration-preserving maps) to that of complete filtered modules.

Hence it lifts to a functor $R[[-]]$ from the category of locally finite monoids to that of complete filtered algebras.

Remark

$R[[M]]$ is an augmented algebra with augmentation ideal $\mathfrak{I}_1$ (this is due to the fact that M is connected as a filtered set).

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Representable functors

Let $\mathbf{C}$ be a category and let C be an object of $\mathbf{C}$.

One can define the (covariant) hom functor $\mathbf{C}(C, -) \colon \mathbf{C} \to \mathbf{Set}$. It acts on a morphism $f \colon C_1 \to C_2$ as the set-theoretic map $g \in \mathbf{C}(C, C_1) \mapsto f \circ g \in \mathbf{C}(C, C_2)$.

A functor $F \colon \mathbf{C} \to \mathbf{Set}$ is said to be representable if it is (naturally) isomorphic to some hom functor $\mathbf{C}(C, -)$. The object C is said to be the representing object of F (it is unique up to isomorphism).

Yoneda's lemma

Yoneda's lemma

Let $F\colon \mathbf{C} \to \mathbf{Set}$ be a functor, and let C be an object of $\mathbf{C}$. The set of all natural transformations from $\mathbf{C}(C, -)$ to F is in a bijective correspondence with the set $F(C)$.

In particular if F itself is representable with representing object D, then the set of all natural transformations from $\mathbf{C}(C, -)$ to F is in bijection with the set $\mathbf{C}(D, C)$.

Monoid and group scheme

Let $F\colon {}_c\mathbf{Alg}_R \to \mathbf{Mon}$ be a functor which is representable as a functor ${}_c\mathbf{Alg}_R \to \mathbf{Mon} \to \mathbf{Set}$ with representing (or coordinate) algebra $\mathcal{O}(F)$. Then F is said to be a monoid scheme (i.e., a monoid in the category of representable functors).

One observes that for any functor $F\colon {}_c\mathbf{Alg}_R \to \mathbf{Mon}$ the binary law $*_A\colon F(A) \times F(A) \to F(A)$ and the identity $e_A\colon 1_A \to F(A)$ of the monoid $F(A)$ are natural transformations.

Replacing **Mon** by **Grp**, one gets a group scheme (the inversion map is a natural transformation).

Bialgebra and Hopf algebra

Let us assume that $F\colon {}_c\mathbf{Alg}_R \to \mathbf{Mon}$ is a monoid scheme.

Since $F(-) \times F(-)\colon {}_c\mathbf{Alg}_R \to \mathbf{Mon}$ (product of monoids) is representable with representing algebra $\mathcal{O}(F) \otimes_R \mathcal{O}(F)$ and $1\colon {}_c\mathbf{Alg}_R \to \mathbf{Mon}$, given by $1_A = 1$ (one-point set, hence the trivial monoid), also is representable with representing algebra R, by Yoneda's lemma, the natural transformations $*_-$ and e_- give rise to algebra maps $\Delta\colon \mathcal{O}(F) \to \mathcal{O}(F) \otimes \mathcal{O}(F)$ (the coproduct) and $\epsilon\colon \mathcal{O}(F) \to R$ (the counit).

These maps equip the algebra $\mathcal{O}(F)$ with a structure of a (commutative) R-bialgebra.

If one assumes that F is a group scheme, then one gets in addition another algebra map $S\colon \mathcal{O}(F) \to \mathcal{O}(F)$ (the antipode) corresponding to the inversion in F, and the structure $(\mathcal{O}(F), \Delta, \epsilon, S)$ is called a (commutative) Hopf algebra.

From large algebra to representable functor

Let M be a finite decomposition monoid.

Let us define a functor $(-)^M \colon {}_c\mathbf{Alg}_R \to \mathbf{Set}$ by $A \mapsto A^M$.

It is representable with coordinate ring $R[x_a \colon a \in M]$ (polynomial ring in the indeterminates x_a, $a \in M$).

Indeed, ${}_c\mathbf{Alg}_R(R[x_a \colon a \in M], A) \cong A^M$.

Ring scheme (or Hopf ring)

Both the multiplicative and additive structures of $A[[M]]$ are natural in the commutative algebra A. Hence $(-)[[M]] \colon A \mapsto A[[M]]$ forms a ring in the category of representable functors.

By Yoneda's lemma it induces a structure of a Hopf ring on $R[x_a \colon a \in M]$ (i.e., a ring in the category of cocommutative coalgebras or a monoid in the category of "abelian" Hopf algebras).

The additive part defines the abelian Hopf algebra structure with coalgebra structure maps $\Delta_{\text{prim}}(x_a) = x_a \otimes 1 + 1 \otimes x_a$, $\epsilon_{\text{prim}}(x_a) = 0$ and $S_{\text{prim}}(x_a) = -x_a$, $a \in M$.

The multiplicative part induces a bialgebra with $\Delta(x_a) = \sum_{b*c=a} x_b \otimes x_c$ and $\epsilon(x_a) = 1$.

Of course both structures are related so that ring axioms hold.

Remark

The correspondence $M \mapsto (-)[[M]]$ again comes from the lifting of a monoidal functor from $\mathbb{FinFibSet}$ to the category of abelian group schemes, which is equivalent to the opposite category of abelian Hopf algebras.

Reconstruction theorem

Given a representable functor $F\colon {}_c\mathbf{Alg}_R \to \mathbf{Set}$, its set of R-rational points is $F(R) \cong {}_c\mathbf{Alg}_R(\mathcal{O}(F), R)$.

Theorem

The large algebra $R[[M]]$ is isomorphic to the ring of R-rational points of the ring scheme $(-)[[M]]$.

Proof: This comes from ${}_c\mathbf{Alg}_R(R[x_a\colon a \in M], R) \cong R[[M]]$ (of course as sets but also as rings). $\square$

Locally finite monoids to Hopf algebras

Let M be a locally finite monoid.

Let A be a commutative R-algebra with a unit. Let us define $1 + \mathfrak{I}_1(A) = \{ f : M \to A : f(1) = 1 \}$. It is a subgroup of the group of invertible elements of $A[[M]]$.

It defines a group scheme $A \mapsto 1 + \mathfrak{I}_1(A)$ with representing (or coordinate) Hopf algebra $R[x_a : a \in M \setminus \{ 1 \}]$.

The antipode S is given by $S(x_a) = \mu(a)$ for each $a \in M \setminus \{ 1 \}$, where μ is the Möbius function of M.

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Presheaf of monoids

Let $\mathbf{C}$ be a category.

A presheaf of monoids over $\mathbf{C}$ is a functor $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Mon}$ (or equivalently a monoid in the category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$).

Any poset $(P, \leq)$ may be seen as a category with set of objects P and an arrow between x and y if, and only if, $x \leq y$.

Hence one can consider a presheaf of monoids $F\colon (P, \leq)^{\mathrm{op}} \to \mathbf{Mon}$. This means that $F(x)$ is a monoid for each $x \in P$, and for each $x \leq y$, one has a homomorphism of monoids $F_{y,x}\colon F(y) \to F(x)$ such that $F_{x,x} = id_{F(x)}$ and for $x \leq y \leq z$, $F_{z,x} = F_{y,x} \circ F_{z,y}$.

Remark

In case $(P, \leq)$ is a meet semi-lattice, and F takes its values in $\mathbf{Grp}$ (rather than $\mathbf{Mon}$), one gets a presheaf of groups, and by a "glueing" construction one obtains a Clifford semigroup as a strong semi-lattice of groups.

A structure theorem for finite decomposition monoids

Some preliminaries - 1

For each monoid M let us denote by $M^\times$ its group of units.

If M is a finite decomposition monoid, then $M \setminus M^\times$ is a two-sided ideal of M.

Let M be a finite decomposition monoid. One recursively defines a sequence of semigroups as follows: $I_0 := M \setminus M^\times$ and $I_{n+1} := I_n \setminus I_n^\times$ if I_n admits an identity, and $I_{n+1} = I_n$ otherwise.

A structure theorem for finite decomposition monoids

Some preliminaries - 2

Let $o(M) = \inf\{\, n \in \mathbb{N} \cup \{\, \infty \,\} \colon I_n \text{ has no identity} \,\}$.

- If $o(M) = \infty$, then one defines $M_0 := M^{\times}$ and $M_{n+1} = I_n^{\times}$. It can be shown that $M = \bigcup_{n \geq 0} M_n$ and $M_m \cap M_n = \emptyset$ for all $m \neq n$.
- If $o(M) < \infty$, then there two cases:
  - either $o(M) = 0$, so that I_0 has no identity, and then one defines $M_0 := M$,
  - or $o(M) > 0$, so that $I_{o(M)}$ has no identity, and then one defines $M_0 := M$, $M_k = I_{k-1}^{\times}$ for $1 \leq k < o(M)$, and $M_{o(M)} := I_{o(M)-1}$.

In any case $(M_n)_{0 \leq n}$ is a (finite or not) sequence of pairwise disjoint monoids, $M = \bigcup_n M_n$ and M_n is finite for each $n < o(M)$.

Let e_n be the identity element of M_n (hence e_0 is the identity of M). For each $m \leq n$, and $x \in M_m$, then $xe_n \in M_n$, hence for each $m \leq n$, one can define $F_{m,n} \colon x \in M_m \mapsto xe_n \in M_n$. Actually, F is a presheaf of monoids over $\{\, n \in \mathbb{N} \colon n \leq o(M) \,\}$ (with the opposite order).

A structure theorem for finite decomposition monoids

Theorem (Deneufchâtel, Duchamp, 2013)

Let M be a finite decomposition monoid.

Then, M is (isomorphic to) the strong semi-lattice of monoids determined by the presheaf F.

In more details, $M \cong \bigcup_n M_n \times \{ n \}$, where the right hand-side is a monoid with identity $(e_0, 0)$ and multiplication
$(x, m) \otimes (y, n) := (F_{m,\max\{ m,n \}}(x) *_{\max\{ m,n \}} F_{n,\max\{ m,n \}}(y), \max\{ m, n \})$
(where $*_n$ stands for the product in M_n).

Corollary

If $o(M) = \infty$, then M is a Clifford monoid.

The strong semi-lattice construction is induced by a monoidal functor

One already knows that a monoid in the category of presheaves of sets is just a presheaf of monoids.

Let $(L, \leq)$ be a meet semi-lattice with a top element, and let $\mathbf{C}$ be a category with infinite coproducts.

Then, one can define a functor $E \colon \mathbf{C}^{(L,\leq)^{\mathrm{op}}} \to \mathbf{C}$ by $E(F) := \coprod_{x \in L} F(x)$.

Proposition

Let us assume that $\mathbb{C}$ is a monoidal category with infinite coproducts and that the monoidal tensor preserves all these coproducts in each of its variables.

Then, E is a monoidal functor.

Therefore E lifts to the category of monoids, and one recovers the strong semi-lattice construction when $\mathbb{C} = \mathbf{Set}$ with the cartesian product.

Monoid algebra

The previous proposition also applies in the case where $\mathbb{C}$ is the category of R-modules (with the usual tensor product).

Hence given a presheaf $F\colon (L, \leq)^{\mathrm{op}} \to {}_R\mathbf{Alg}$ (i.e., a monoid in ${}_R\mathbf{Mod}^{(L,\leq)^{\mathrm{op}}}$), then one can consider the strong semi-lattice construction $E(F) = \bigoplus_{x \in L} F(x)$ which becomes an algebra (i.e., a monoid in ${}_R\mathbf{Mod}$).

Let $F\colon (L, \leq)^{\mathrm{op}} \to \mathbf{Mon}$ be a presheaf of monoids. Let $R[F]\colon (L, \leq)^{\mathrm{op}} \to {}_R\mathbf{Alg}$ be the presheaf of algebras given by $R[F](x) := R[F(x)]$.

Then, $\bigoplus_{x \in L} R[F(x)] \cong R[E(F)]$ (as algebras).

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Isomorphism

Let **C** be a category and let $f\colon C \to D$ be a **C**-morphism.

It is said to be an isomorphism if f admits a two-sided inverse, i.e., there exists $g\colon D \to C$ such that $g \circ f = id_C$ and $f \circ g = id_D$.

If a two-sided inverse exists, then it is unique.

Opposite category

Let **C** be a category. Then, $\mathbf{C}^{\mathsf{op}}$ is the opposite category of **C**.

It has the same objects and morphisms as **C** but with the reverse composition.

Hence $f \circ g$ in $\mathbf{C}^{\mathsf{op}}$ corresponds to $g \circ f$ in **C**.

Of course, $\mathbf{C} = (\mathbf{C}^{\mathsf{op}})^{\mathsf{op}}$.

Functors

Let **C** and **D** be categories.

A functor $F\colon \mathbf{C} \to \mathbf{D}$ maps objects of **C** to objects of **D**, morphisms of **C** to morphisms of **D** and

- $F(id_C) = id_{F(C)}$,
- $F(g \circ f) = F(g) \circ F(f)$ whenever (g, f) is a pair of composable morphisms.

A functor $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$ is referred to as a contravariant functor (usual functors are also said to be covariant functors).

Some examples: $\mathfrak{P}\colon \mathbf{Set} \to \mathbf{Set}$, $\mathbf{C}_0\colon \mathbf{Top}^{\mathrm{op}} \to {}_{\mathbb{R}}\mathbf{Alg}$, $\mathbf{GL}_n\colon \mathbf{Ring} \to \mathbf{Grp}$, *etc.*

Natural transformations

Let $F, G\colon \mathbf{C} \to \mathbf{D}$ be two functors.

A natural transformation $\tau\colon F \Rightarrow G$ is a family $(\tau_C)_C$ of $\mathbf{D}$-morphisms indexed by the objects of $\mathbf{C}$ such that for each $\mathbf{C}$-morphism $f\colon C \to C'$ the following diagram commutes.

$$\begin{array}{ccc} F(C) & \xrightarrow{\tau_C} & G(C) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(C') & \xrightarrow[\tau_{C'}]{} & G(C') \end{array}$$

If τ_C is an isomorphism for each object C, then τ is said to be a natural isomorphism.

Some examples: $j_M\colon M \to (M^\star)^\star$ where M is a R-module (R is a commutative ring with a unit), $\det\colon \mathbf{GL}_n \Rightarrow (-)^\times$, *etc.*