Differential (Lie) algebras from a functorial point of view

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Lie algebras

Let R be a commutative ring with a unit.

A Lie algebra $(\mathfrak{g}, [-,-])$ is the data of a R-module $\mathfrak{g}$ and a bilinear binary operation $[-,-]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the Lie bracket, such that

- It is alternating: $[x, x] = 0$ for every $x \in \mathfrak{g}$.
- It satisfies the Jacobi identity

$$[x,[y,z]] + [y,[x,z]] + [z,[x,y]] = 0$$

  for each $x, y, z \in \mathfrak{g}$.

A Lie algebra is said to be commutative whenever its bracket is the zero map.

Universal enveloping algebra

Any (say unital and associative) algebra $(A, \cdot)$ may be turned into a Lie algebra when equipped with the commutator bracket

$$[x, y] = x \cdot y - y \cdot x \ .$$

Actually this defines a functor from the category **Ass** to the category **Lie**.

This functor admits a left adjoint namely the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$.

One has

$$\mathcal{U}(\mathfrak{g}) \cong \mathrm{T}(\mathfrak{g})/\langle xy - yx - [x, y] \colon x, y \in \mathfrak{g} \rangle$$

where $\mathrm{T}(M)$ is the tensor algebra of a R-module M.

Poincaré-Birkhoff-Witt theorem

Let $\mathfrak{g}$ be a Lie algebra (over R).

Let $j\colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ be the Lie map defined as the composition $\mathfrak{g} \xrightarrow{incl} \mathrm{T}(\mathfrak{g}) \xrightarrow{\pi} \mathcal{U}(\mathfrak{g})$ (where π is the canonical projection, and $\mathcal{U}(\mathfrak{g})$ is seen as a Lie algebra under its commutator bracket).

PBW Theorem

If R is a field, then j is one-to-one.

More generally, P.M. Cohn proved in 1963 that if the underlying R-module of $\mathfrak{g}$ is torsion-free, then j is one-to-one.

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The other one is rather different (since it is not based on the commutator) and is sketched hereafter.

Wronskian bracket

Now, let us assume that $(A, \cdot, d)$ is a differential commutative algebra.

There is another bracket given by the Wronskian

$$W(x, y) = x \cdot d(y) - d(x) \cdot y$$

which turns A into a Lie algebra.

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1. Does it admit a left adjoint ? In other terms, is there a universal enveloping differential (commutative) algebra ? (Call it the Wronskian enveloping algebra.)

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In this talk I will only address the first question.

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Equational varieties

A class **V** of Σ-algebras is said to be an equational variety when each member of the class satisfies some given axioms or identities.

Each variety of Σ-algebras with its homomorphisms (maps preserving the structural operations) forms a category.

One of the key features of equational varieties is the fact that they come equipped with a forgetful functor $U_{\mathbf{V}}\colon \mathbf{V} \to \mathbf{Set}$ (it maps an algebra to its carrier set). So they are concrete categories over **Set** (and even monadic).

Some (counter-)examples

- Semigroups, inverse semigroups, monoids, commutative monoids, groups, abelian groups, rings, R-algebras for a unital commutative ring, Lie algebras, Jordan algebras, etc.

- Fields (the inverse operation is only partially defined) and the category of monoids with invertible elements (groups!), because it is not closed under sub-algebras (e.g., the sub-monoid $\mathbb{N}$ of $\mathbb{Z}$).

Algebraic functors

Let $\mathbf{V}$ and $\mathbf{W}$ be two equational varieties of Σ-algebras.

A functor $F\colon \mathbf{V} \to \mathbf{W}$ is said to be an algebraic functor if it preserves the forgetful functors, i.e., $U_{\mathbf{W}} \circ F = U_{\mathbf{V}}$.

Theorem (Bill Lawvere)

Any algebraic functor admits a left adjoint.

In particular the forgetful functor $U_{\mathbf{V}}$ itself has a left adjoint. Hence for any set X, there exists a free algebra $\mathbf{V}[X]$ in the variety $\mathbf{V}$.

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Generalities about differential algebras

Let R be a commutative ring with a unit.

Let **V** be a variety of (not necessarily associative nor unital) R-algebras (i.e., R-modules M with a binary operation $\cdot\colon M \otimes_R M \to M$ subject to some axioms).

For **V** have in mind **Ass** or **Lie**.

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For **V** have in mind **Ass** or **Lie**.

A derivation $d\colon M \to M$ is a R-linear map that satisfies Leibniz identity

$$d(x \cdot y) = d(x) \cdot y + x \cdot d(y) \ .$$

By considering algebras $(M, \cdot)$ of **V** with a derivation d and homomorphisms of algebras commuting with derivations, one gets a variety, say **DiffV**, of differential algebras (in **V**).

Differential ideals

A (differential) ideal I of a differential algebra $(M, \cdot, d)$ is just an ideal of $(M, \cdot)$ (i.e., $M \cdot I \subseteq I \supseteq I \cdot M$) such that $d(I) \subseteq I$.

It turns out that M/I becomes a differential algebra with derivation $\tilde{d}(x + I) = d(x) + I$ and the natural epimorphism $M \to M/I$ is a homomorphism of differential algebras.

It makes also sense to talk about the least differential ideal generated by a set.

Reflective sub-category (1/2)

The variety **V** embeds into the variety **DiffV** since any algebra in **V** may be seen as a differential algebra with the zero (or trivial) derivation.

Of course this embedding preserves the forgetful functors, hence admits a left adjoint, i.e., **V** is a reflective sub-category of **DiffV**, this means that any differential algebra (in **V**) "freely generates" an algebra in **V**.

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The construction: let $(M, \cdot, d)$ be a member of **DiffV**. Let I_d be the (algebraic) ideal generated $im(d)$. Thus, M/I_d is a member of **V**, and the natural projection $\pi\colon M \to M/I_d$ is a homomorphism of algebras.

Reflective sub-category (2/2)

Universal property

Given an algebra $(N, \cdot)$ and a homomorphism of differential algebras $\phi\colon (M, \cdot, d) \to (N, \cdot, 0)$, because $\phi \circ d = 0$, it passes to the quotient and gives rise to a unique homomorphism of algebras $\hat{\phi}\colon (M/I_d, \cdot) \to (N, \cdot)$ such that $\hat{\phi} \circ \pi = \phi$.

Forgetful functor (1/2)

Conversely, there is an obvious forgetful functor **DiffV** $\rightarrow$ **V** which also admits a left adjoint.

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The construction: let $(M, \cdot)$ be an algebra in **V**. Let $FDiffV(|M|)$ be the free differential algebra generated by the set $|M|$ (carrier set of $(M, \cdot)$), and let $j \colon |M| \to |FDiffV(|M|)|$ be the canonical inclusion. Let I be the differential ideal generated by $j(x+y) - j(x) - j(y)$, $j(x \cdot y) - j(x)j(y)$, $j(rx) - rj(x)$, $x, y \in |M|$, $r \in R$.

Then, $FDiffV(|M|)/I$ is the free differential algebra generated by $(M, \cdot)$.

Forgetful functor (2/2)

Universal property

Let $(N, \cdot, e)$ be a differential algebra, and let $\phi\colon (M, \cdot) \to (N, \cdot)$ be an algebra map.

Let $\hat{\phi}\colon \mathit{FDiffV}(|M|) \to (N, \cdot, e)$ be the unique differential algebra map such that $\hat{\phi} \circ j = \phi$.

Of course $I \subseteq \ker \hat{\phi}$ (since ϕ is an algebra map).

Hence there is a unique differential algebra map $\tilde{\phi}\colon \mathit{FDiffV}(|M|)/I \to (N, \cdot, e)$ such that $\tilde{\phi} \circ \pi \circ j = \phi$.

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Extension of the usual universal enveloping algebra to the differential setting

Let (A, d) be a differential (associative) algebra.

One has $d([x, y]) = d(xy - yx) = d(x)y + xd(y) - d(y)x - yd(x) = [d(x), y] + [x, d(y)]$. Hence, $(A, [-,-], d)$ is a differential Lie algebra.

This gives rise to a functor **DiffAss** $\to$ **DiffLie** which makes commute the following diagram (of forgetful functors).

$$\begin{array}{ccc}
\mathbf{DiffAss} & \xrightarrow{Comm.\ bracket} & \mathbf{DiffLie} \\
{\scriptstyle forgets\ der.}\downarrow & & \downarrow{\scriptstyle forgets\ der.} \\
\mathbf{Ass} & \xrightarrow[Comm.\ bracket]{} & \mathbf{Lie}
\end{array}$$

All functors in this diagram admit a left adjoint.

A construction

Let $(\mathfrak{g}, [-,-], d)$ be a differential Lie algebra.

Let ∂ be the unique derivation on $\mathsf{T}(\mathfrak{g})$ that extends d. It satisfies $\partial(xy - yx - [x,y]) = d(x)y + xd(y) - d(y)x - yd(x) - [d(x), y] - [x, d(y)] = d(x)y - yd(x) - [d(x), y] + xd(y) - d(y)x - [x, d(y)]$, so it factors as a linear map $\tilde{\partial} \colon \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ which is easily seen to be a derivation.

Universal property (1/2)

$(\mathcal{U}(\mathfrak{g}), \tilde{\partial})$ satisfies the following universal property:

Let (A, D) be a differential algebra, and let $\phi\colon (\mathfrak{g}, [-,-], d) \to (A, [-,-], D)$ be a homomorphism of differential Lie algebras.

Then, there is a unique homomorphism of differential algebras $\hat{\phi}\colon (\mathcal{U}(\mathfrak{g}), \tilde{\partial}) \to (A, D)$ such that $\hat{\phi} \circ j = \phi$.

Universal property (2/2)

Indeed, ϕ is of course a homomorphism of Lie algebras from $(\mathfrak{g}, [-,-])$ to $(A, [-,-])$, hence there is a unique algebra map $\hat{\phi}\colon \mathcal{U}(\mathfrak{g}) \to A$ such that $\hat{\phi} \circ j = \phi$.

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The Wronskian bracket

Let (A, d) be a commutative differential (associative and unital) R-algebra.

Let us define the Wronskian bracket

$$W(x, y) := xd(y) - d(x)y \ .$$

Of course it is alternating $W(x, x) = xd(x) - d(x)x = 0$ (since A is commutative).

Moreover it satisfies Jacobi identity.

Hence (A, W) turns to be a Lie algebra.

Moreover $d(W(x, y)) = d(xd(y) - d(x)y) =$
$d(x)d(y) + xd^2(y) - d^2(x)y - d(x)d(y) = xd^2(y) - d^2(x)y.$

While
$W(d(x), y) + W(x, d(y)) = d(x)d(y) - d^2(x)y + xd^2(y) - d(x)d(y).$

Hence (A, W, d) is a differential Lie algebra.

This defines a functor, say the Wronskian, $(A, d) \mapsto (A, W, d)$ from **DiffComAss** to **DiffLie**.

Wronskian enveloping algebra

One observes that the Wronskian functor preserves the obvious forgetful functors,

so it is an algebraic functor,

and it admits a left adjoint $\mathcal{W}$, the Wronskian enveloping algebra.

Construction of the differential enveloping algebra (1/2)

Let $(\mathfrak{g}, [-,-], d)$ be a differential Lie algebra.

Let $\mathsf{S}(\mathfrak{g})$ be the symmetric algebra of the module $\mathfrak{g}$ which becomes a differential algebra with the unique derivation ∂ that extends the map $\partial(x) = d(x)$ on the generators $x \in \mathfrak{g}$

Remark

Actually, one defines the derivation ∂ on the tensor algebra $\mathsf{T}(\mathfrak{g})$, and since it commutes to the permutation of variables, it factors through $\mathsf{S}(\mathfrak{g})$.

Construction of the Wronskian enveloping algebra (2/2)

Let us consider the (algebraic) ideal I generated by $d(x)y - xd(y) - [x, y]$, $x, y \in \mathfrak{g}$.

One observes that $\partial(I) \subseteq I$. Hence I is actually a differential ideal.

Then, the Wronskian enveloping algebra $\mathcal{W}(\mathfrak{g}, [-, -], d)$ is $(\mathsf{S}(\mathfrak{g})/I, \tilde{\partial})$.

Universal property of the Wronskian enveloping algebra

Let (A, δ) be any commutative differential algebra, and let $\phi\colon (g, [-,-], d) \mapsto (A, W, \delta)$ be a homomorphism of differential Lie algebras.

Then, there exists a unique differential algebra map $\tilde{\phi}\colon (\mathsf{S}(\mathfrak{g})/I, \tilde{\partial}) \to (A, \delta)$ such that $\tilde{\phi}(x + I) = \phi(x)$ for each $x \in \mathfrak{g}$.

Proof

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Moreover it satisfies
$\hat{\phi}(d(x)y - xd(y) - [x, y]) = \delta(\phi(x))\phi(y) - \phi(x)\delta(\phi(y)) - [\phi(x), \phi(y)]$

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Hence it factors through I and provides a unique homomorphism of differential algebras $\tilde{\phi}$ from $(\mathrm{S}(\mathfrak{g})/I, \tilde{\partial})$ to (A, δ) such that $\tilde{\phi}(x + I) = \phi(x)$, $x \in \mathfrak{g}$. □

A special case: a Lie algebra with the zero derivation

Let $(\mathfrak{g}, [-,-])$ be a Lie algebra. Then it may be faithfully identified with the differential Lie algebra $(\mathfrak{g}, [-,-], 0)$.

The derivation on $\mathsf{S}(\mathfrak{g})$ that extends the zero derivation is also just the zero derivation.

The differential ideal I is equal to the (algebraic) ideal generated by $[x, y]$, $x, y \in \mathfrak{g}$.

Hence it follows that in case $\mathfrak{g}$ is not commutative (i.e., $[-,-]$ is not identically null), $\mathfrak{g}$ does not embed into its universal enveloping differential (commutative) algebra $\mathcal{W}(\mathfrak{g})$ even if R is a field!

Of course if $\mathfrak{g}$ is a commutative Lie algebra (i.e., with a zero bracket), then it embeds into its Wronskian enveloping algebra which is just $\mathsf{S}(\mathfrak{g})$ (and the same as its universal enveloping algebra).

$\mathfrak{sl}_2(\mathbb{K})$

Let $\mathbb{K}$ be a field of characteristic zero.

The Lie algebra $\mathfrak{sl}_2(\mathbb{K})$ embeds into the algebra of vector fields of $\mathbb{K}[x]$ by the identification of the elements of its Chevalley basis $e = -1$, $h = -2x$, and $f = x^2$ (the familiar commutation rules are satisfied $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$).

It is a differential Lie algebra when equipped with the usual derivation of polynomials.

Hence it embeds into the commutative differential algebra $(\mathbb{K}[x], \partial)$ as a sub-Lie algebra under the Wronskian bracket, therefore it embeds into its Wronskian enveloping algebra.

Conclusion

The problem of embeddability of a differential Lie algebra into its Wronskian enveloping algebra seems to be quite harder than the classical situation (e.g., the case of a non-commutative differential Lie algebra with a zero derivation).

It also seems to be connected to the (faithful) realization of a Lie algebra as a Lie algebra of vector fields. For instance, given two polynomials (seen as vector fields) $P(x)\frac{d}{dx}, Q(x)\frac{d}{dx}$, one has $[P(x)\frac{d}{dx}, Q(x)\frac{d}{dx}] = W(P(x), Q(x))\frac{d}{dx}$.

But Lie algebras of vector fields satisfy some non-trivial identities.

Generalities about Rota-Baxter algebras

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By considering algebras $(M, \cdot)$ of $\mathbf{V}$ with a Rota-Baxter operator R and homomorphisms commuting with the Rota-Baxter operators, one obtains a variety **RBV** of Rota-Baxter algebras. In what follows we are interested in the cases where $\mathbf{V} = \mathbf{Ass}$ and $\mathbf{V} = \mathbf{Lie}$.

The variety $\mathbf{V}$ embeds into the variety **RBV** since any algebra in $\mathbf{V}$ may be seen as a Rota-Baxter algebra with the trivial Rota-Baxter operator.

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Let us once again define the double bracket

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Hence $(\mathfrak{g},[-,-]_B)$ is again a Lie algebra

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Moreover $B([x, y]_B) = B([B(x), y] + [x, B(y)]) = [B(x), B(y)]$ hence B is a Lie map from $(\mathfrak{g}, [-, -]_B)$ to $(\mathfrak{g}, [-, -])$.

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Therefore, one gets a functor **RBAss** $\to$ **RBLie**.

Commutative diagram of “forgetful” functors

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\mathbf{Ass} & \xrightarrow{\textit{Comm. bracket}} & \mathbf{Lie} \\
\uparrow {\scriptstyle \textit{forgets RB op.}} & & \uparrow {\scriptstyle \textit{forgets RB op.}} \\
\mathbf{RBAss} & \xrightarrow[\textit{Comm. bracket}]{} & \mathbf{RBLie} \\
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