# Differential (Lie) algebras from a functorial point of view

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## Lie algebras

Let R be a commutative ring with a unit.

A Lie algebra  $(\mathfrak{g}, [-, -])$  is the data of a *R*-module  $\mathfrak{g}$  and a bilinear binary operation  $[-, -]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ , called the Lie bracket, such that

- It is alternating: [x, x] = 0 for every  $x \in \mathfrak{g}$ .
- It satisfies the Jacobi identity

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = 0$$

for each  $x, y, z \in \mathfrak{g}$ .

A Lie algebra is said to be commutative whenever its bracket is the zero map.

# Universal enveloping algebra

Any (say unital and associative) algebra  $(A, \cdot)$  may be turned into a Lie algebra when equipped with the commutator bracket

 $[x,y] = x \cdot y - y \cdot x \; .$ 

Actually this defines a functor from the category Ass to the category Lie.

This functor admits a left adjoint namely the universal enveloping algebra  $\mathcal{U}(\mathfrak{g})$  of a Lie algebra  $\mathfrak{g}$ .

One has

$$\mathcal{U}(\mathfrak{g}) \cong \mathsf{T}(\mathfrak{g})/\langle xy - yx - [x, y] \colon x, y \in \mathfrak{g} 
angle$$

where T(M) is the tensor algebra of a *R*-module *M*.

## Poincaré-Birkhoff-Witt theorem

Let  $\mathfrak{g}$  be a Lie algebra (over R).

Let  $j: \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$  be the Lie map defined as the composition  $\mathfrak{g} \xrightarrow{incl} \mathsf{T}(\mathfrak{g}) \xrightarrow{\pi} \mathcal{U}(\mathfrak{g})$  (where  $\pi$  is the canonical projection, and  $\mathcal{U}(\mathfrak{g})$  is seen as a Lie algebra under its commutator bracket).

#### **PBW** Theorem

If R is a field, then j is one-to-one.

More generally, P.M. Cohn proved in 1963 that if the underlying R-module of  $\mathfrak{g}$  is torsion-free, then j is one-to-one.

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The first one is a somewhat "trivial" extension. Indeed, a derivation on an algebra is also a derivation for its commutator bracket. Moreover the universal enveloping algebra may be equipped with a derivation that extends the derivation of the Lie algebra, and the Poincaré-Birkhoff-Witt theorem remains unchanged.

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The other one is rather different (since it is not based on the commutator) and is sketched hereafter.

Now, let us assume that  $(A, \cdot, d)$  is a differential commutative algebra.

There is another bracket given by the Wronskian

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 Does it admit a left adjoint ? In other terms, is there a universal enveloping differential (commutative) algebra ? (Call it the Wronskian enveloping algebra.)

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In this talk I will only address the first question.

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## Equational varieties

A class V of  $\Sigma$ -algebras is said to be an equational variety when each member of the class satisfies some given axioms or identities.

Each variety of  $\Sigma$ -algebras with its homomorphisms (maps preserving the structural operations) forms a category.

One of the key features of equational varieties is the fact that they come equipped with a forgetful functor  $U_{\mathbf{V}}: \mathbf{V} \to \mathbf{Set}$  (it maps an algebra to its carrier set). So they are concrete categories over **Set** (and even monadic).

#### Some (counter-)examples

- Semigroups, inverse semigroups, monoids, commutative monoids, groups, abelian groups, rings, *R*-algebras for a unital commutative ring, Lie algebras, Jordan algebras, etc.
- Fields (the inverse operation is only partially defined) and the category of monoids with invertible elements (groups!), because it is not closed under sub-algebras (e.g., the sub-monoid N of Z).

# Algebraic functors

Let  $\boldsymbol{V}$  and  $\boldsymbol{W}$  be two equational varieties of  $\boldsymbol{\Sigma}\text{-algebras}.$ 

A functor  $F: \mathbf{V} \to \mathbf{W}$  is said to be an algebraic functor if it preserves the forgetful functors, i.e.,  $U_{\mathbf{W}} \circ F = U_{\mathbf{V}}$ .

#### Theorem (Bill Lawvere)

Any algebraic functor admits a left adjoint.

In particular the forgetful functor  $U_V$  itself has a left adjoint. Hence for any set X, there exists a free algebra V[X] in the variety V.

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## Generalities about differential algebras

Let R be a commutative ring with a unit.

Let **V** be a variety of (not necessarily associative nor unital) *R*-algebras (i.e., *R*-modules *M* with a binary operation  $\cdot: M \otimes_R M \to M$  subject to some axioms).

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A derivation  $d: M \rightarrow M$  is a *R*-linear map that satisfies Leibniz identity

 $d(x \cdot y) = d(x) \cdot y + x \cdot d(y) .$ 

By considering algebras  $(M, \cdot)$  of **V** with a derivation *d* and homomorphisms of algebras commuting with derivations, one gets a variety, say **DiffV**, of differential algebras (in **V**).

# Differential ideals

A (differential) ideal I of a differential algebra  $(M, \cdot, d)$  is just an ideal of  $(M, \cdot)$  (i.e.,  $M \cdot I \subseteq I \supseteq I \cdot M$ ) such that  $d(I) \subseteq I$ .

It turns out that M/I becomes a differential algebra with derivation  $\tilde{d}(x + I) = d(x) + I$  and the natural epimorphism  $M \to M/I$  is a homomorphism of differential algebras.

It makes also sense to talk about the least differential ideal generated by a set.

# Reflective sub-category (1/2)

The variety V embeds into the variety **DiffV** since any algebra in V may be seen as a differential algebra with the zero (or trivial) derivation.

Of course this embedding preserves the forgetful functors, hence admits a left adjoint, i.e., V is a reflective sub-category of DiffV, this means that any differential algebra (in V) "freely generates" an algebra in V.

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The construction: let  $(M, \cdot, d)$  be a member of **DiffV**. Let  $I_d$  be the (algebraic) ideal generated im(d). Thus,  $M/I_d$  is a member of **V**, and the natural projection  $\pi: M \to M/I_d$  is a homomorphism of algebras.

#### Reflective sub-category (2/2) Universal property

Given an algebra  $(N, \cdot)$  and a homomorphism of differential algebras  $\phi: (M, \cdot, d) \to (N, \cdot, 0)$ , because  $\phi \circ d = 0$ , it passes to the quotient and gives rise to a unique homomorphism of algebras  $\hat{\phi}: (M/I_d, \cdot) \to (N, \cdot)$  such that  $\hat{\phi} \circ \pi = \phi$ .

# Forgetful functor (1/2)

Conversely, there is an obvious forgetful functor  $\text{DiffV} \to \text{V}$  which also admits a left adjoint.

Hence any algebra in V "freely generates" a differential algebra (in V).

# Forgetful functor (1/2)

Conversely, there is an obvious forgetful functor  $\text{DiffV} \to \text{V}$  which also admits a left adjoint.

Hence any algebra in V "freely generates" a differential algebra (in V).

The construction: let  $(M, \cdot)$  be an algebra in **V**. Let FDiffV(|M|) be the free differential algebra generated by the set |M| (carrier set of  $(M, \cdot)$ ), and let  $j: |M| \rightarrow |FDiffV(|M|)|$  be the canonical inclusion. Let I be the differential ideal generated by j(x + y) - j(x) - j(y),  $j(x \cdot y) - j(x)j(y)$ , j(rx) - rj(x),  $x, y \in |M|$ ,  $r \in R$ .

Then, FDiffV(|M|)/I is the free differential algebra generated by  $(M, \cdot)$ .

### Forgetful functor (2/2) Universal property

Let  $(N, \cdot, e)$  be a differential algebra, and let  $\phi: (M, \cdot) \to (N, \cdot)$  be an algebra map.

Let  $\hat{\phi}$ : *FDiffV*(|M|)  $\rightarrow$  ( $N, \cdot, e$ ) be the unique differential algebra map such that  $\hat{\phi} \circ j = \phi$ .

Of course  $I \subseteq \ker \hat{\phi}$  (since  $\phi$  is an algebra map).

Hence there is a unique differential algebra map  $\tilde{\phi} \colon FDiffV(|M|)/I \to (N, \cdot, e)$  such that  $\tilde{\phi} \circ \pi \circ j = \phi$ .

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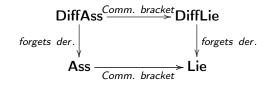


# Extension of the usual universal enveloping algebra to the differential setting

Let (A, d) be a differential (associative) algebra.

One has d([x, y]) = d(xy - yx) = d(x)y + xd(y) - d(y)x - yd(x) = [d(x), y] + [x, d(y)]. Hence, (A, [-, -], d) is a differential Lie algebra.

This gives rise to a functor **DiffAss**  $\rightarrow$  **DiffLie** which makes commute the following diagram (of forgetful functors).



All functors in this diagram admit a left adjoint.

## A construction

Let  $(\mathfrak{g}, [-, -], d)$  be a differential Lie algebra.

Let  $\partial$  be the unique derivation on T(g) that extends d. It satisfies  $\partial(xy - yx - [x, y]) = d(x)y + xd(y) - d(y)x - yd(x) - [d(x), y] - [x, d(y)] = d(x)y - yd(x) - [d(x), y] + xd(y) - d(y)x - [x, d(y)]$ , so it factors as a linear map  $\tilde{\partial}: \mathcal{U}(g) \to \mathcal{U}(g)$  which is easily seen to be a derivation.

 $(\mathcal{U}(\mathfrak{g}), \tilde{\partial})$  satisfies the following universal property:

Let (A, D) be a differential algebra, and let  $\phi: (\mathfrak{g}, [-, -], d) \rightarrow (A, [-, -], D)$  be a homomorphism of differential Lie algebras.

Then, there is a unique homomorphism of differential algebras  $\hat{\phi} : (\mathcal{U}(\mathfrak{g}), \tilde{\partial}) \to (A, D)$  such that  $\hat{\phi} \circ j = \phi$ .

Indeed,  $\phi$  is of course a homomorphism of Lie algebras from  $(\mathfrak{g}, [-, -])$  to (A, [-, -]), hence there is a unique algebra map  $\hat{\phi} \colon \mathcal{U}(\mathfrak{g}) \to A$  such that  $\hat{\phi} \circ j = \phi$ .

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### The Wronskian bracket

Let (A, d) be a commutative differential (associative and unital) *R*-algebra.

Let us define the Wronskian bracket

W(x,y) := xd(y) - d(x)y .

Of course it is alternating W(x, x) = xd(x) - d(x)x = 0 (since A is commutative).

Moreover it satisfies Jacobi identity.

Hence (A, W) turns to be a Lie algebra.

Moreover d(W(x, y)) = d(xd(y) - d(x)y) = $d(x)d(y) + xd^2(y) - d^2(x)y - d(x)d(y) = xd^2(y) - d^2(x)y.$ 

While

 $W(d(x), y) + W(x, d(y)) = d(x)d(y) - d^{2}(x)y + xd^{2}(y) - d(x)d(y).$ 

Hence (A, W, d) is a differential Lie algebra.

This defines a functor, say the Wronskian,  $(A, d) \mapsto (A, W, d)$  from DiffComAss to DiffLie.

## Wronskian enveloping algebra

One observes that the Wronskian functor preserves the obvious forgetful functors,

so it is an algebraic functor,

and it admits a left adjoint  $\ensuremath{\mathcal{W}}$  , the Wronskian enveloping algebra.

### Construction of the differential enveloping algebra (1/2)

Let  $(\mathfrak{g}, [-, -], d)$  be a differential Lie algebra.

Let  $S(\mathfrak{g})$  be the symmetric algebra of the module  $\mathfrak{g}$  which becomes a differential algebra with the unique derivation  $\partial$  that extends the map  $\partial(x) = d(x)$  on the generators  $x \in \mathfrak{g}$ 

#### Remark

Actually, one defines the derivation  $\partial$  on the tensor algebra  $T(\mathfrak{g})$ , and since it commutes to the permutation of variables, it factors through  $S(\mathfrak{g})$ .

## Construction of the Wronskian enveloping algebra (2/2)

Let us consider the (algebraic) ideal I generated by d(x)y - xd(y) - [x, y],  $x, y \in \mathfrak{g}$ .

One observes that  $\partial(I) \subseteq I$ . Hence I is actually a differential ideal.

Then, the Wronskian enveloping algebra  $\mathcal{W}(\mathfrak{g}, [-, -], d)$  is  $(S(\mathfrak{g})/I, \tilde{\partial})$ .

### Universal property of the Wronskian enveloping algebra

Let  $(A, \delta)$  be any commutative differential algebra, and let  $\phi : (g, [-, -], d) \mapsto (A, W, \delta)$  be a homomorphism of differential Lie algebras.

Then, there exists a unique differential algebra map  $\tilde{\phi} \colon (\mathsf{S}(\mathfrak{g})/I, \tilde{\partial}) \to (A, \delta)$  such that  $\tilde{\phi}(x + I) = \phi(x)$  for each  $x \in \mathfrak{g}$ .

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Moreover it satisfies  $\hat{\phi}(d(x)y - xd(y) - [x, y]) = \delta(\phi(x))\phi(y) - \phi(x)\delta(\phi(y)) - [\phi(x), \phi(y)]$ 

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Hence it factors through I and provides a unique homomorphism of differential algebras  $\tilde{\phi}$  from  $(S(\mathfrak{g})/I, \tilde{\partial})$  to  $(A, \delta)$  such that  $\tilde{\phi}(x + I) = \phi(x), x \in \mathfrak{g}$ .

### A special case: a Lie algebra with the zero derivation

Let  $(\mathfrak{g}, [-, -])$  be a Lie algebra. Then it may be faithfully identified with the differential Lie algebra  $(\mathfrak{g}, [-, -], \mathbf{0})$ .

The derivation on  $\mathsf{S}(\mathfrak{g})$  that extends the zero derivation is also just the zero derivation.

The differential ideal *I* is equal to the (algebraic) ideal generated by [x, y],  $x, y \in \mathfrak{g}$ .

Hence it follows that in case  $\mathfrak{g}$  is not commutative (i.e., [-, -] is not identically null),  $\mathfrak{g}$  does not embed into its universal enveloping differential (commutative) algebra  $\mathcal{W}(\mathfrak{g})$  even if R is a field!

Of course if  $\mathfrak{g}$  is a commutative Lie algebra (i.e., with a zero bracket), then it embeds into its Wronskian enveloping algebra which is just  $S(\mathfrak{g})$  (and the same as its universal enveloping algebra).

# $\mathfrak{sl}_2(\mathbb{K})$

Let  $\mathbb K$  be a field of characteristic zero.

The Lie algebra  $\mathfrak{sl}_2(\mathbb{K})$  embeds into the algebra of vector fields of  $\mathbb{K}[x]$  by the identification of the elements of its Chevalley basis e = -1, h = -2x, and  $f = x^2$  (the familiar commutation rules are satisfied [h, e] = 2e, [h, f] = -2f and [e, f] = h).

It is a differential Lie algebra when equipped with the usual derivation of polynomials.

Hence it embeds into the commutative differential algebra  $(\mathbb{K}[x], \partial)$  as a sub-Lie algebra under the Wronskian bracket, therefore it embeds into its Wronskian enveloping algebra.

### Conclusion

The problem of embeddability of a differential Lie algebra into its Wronskian enveloping algebra seems to be quite harder than the classical situation (e.g., the case of a non-commutative differential Lie algebra with a zero derivation).

It also seems to be connected to the (faithful) realization of a Lie algebra as a Lie algebra of vector fields. For instance, given two polynomials (seen as vector fields)  $P(x)\frac{d}{dx}$ ,  $Q(x)\frac{d}{dx}$ , one has  $[P(x)\frac{d}{dx}, Q(x)\frac{d}{dx}] = W(P(x), Q(x))\frac{d}{dx}$ .

But Lie algebras of vector fields satisfy some non-trivial identities.

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 $B(x) \cdot B(y) = B(B(x) \cdot y + x \cdot B(y)) .$ 

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The variety **V** embeds into the variety **RBV** since any algebra in **V** may be seen as a Rota-Baxter algebra with the trivial Rota-Baxter operator.

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Moreover, *B* becomes an algebra map from  $(A, *_B)$  to  $(A, \cdot)$ .

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Moreover, given a Rota-Baxter map  $\phi: (A_1, \cdot, B_1) \to (A_2, \cdot, B_2)$ , then  $\phi$  is also a Rota-Baxter map from  $(A_1, *_{B_1}, B_1)$  to  $(A_2, *_{B_2}, B_2)$ .

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Hence one gets a functor Dbl: **RBAss**  $\rightarrow$  **RBAss**.

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Let us once again define the double bracket

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Hence  $(\mathfrak{g}, [-, -]_B)$  is again a Lie algebra

Moreover  $B([x, y]_B) = B([B(x), y] + [x, B(y)]) = [B(x), B(y)]$  hence B is a Lie map from  $(\mathfrak{g}, [-, -]_B)$  to  $(\mathfrak{g}, [-, -])$ .

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Therefore one gets a functor  $\mathsf{Dbl}_{\mathsf{Lie}}$ :  $\mathsf{RBLie} \to \mathsf{RBLie}$ .

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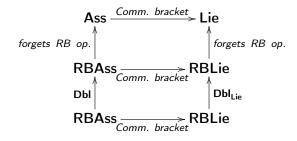
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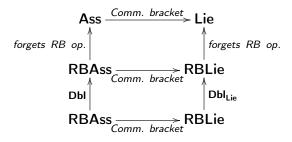
Therefore, one gets a functor  $RBAss \rightarrow RBLie$ .

## Commutative diagram of "forgetful" functors



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Each of these functors is algebraic, hence admits a left adjoint. In particular, one can form the universal enveloping Rota-Baxter algebra on a Rota-Baxter Lie algebra.