On the powers of substitutions with prefunctions

Laurent Poinsot
(joint work with Gérard H. E. Duchamp)

LIPN - UMR CNRS 7030
Université Paris-Nord XIII

62nd Séminaire Lotharingien de Combinatoire

Main objective of the talk

The set of all pairs (μ, σ) of formal power series in a field $\mathbb{K}$ of characteristic zero such that

- $\mu = 1 + \mu_+$, $\omega(\mu_+) > 0$,
- $\sigma = \mathrm{x} + \sigma_+$, $\omega(\sigma_+) > 1$

Main objective of the talk

The set of all pairs (μ, σ) of formal power series in a field $\mathbb{K}$ of characteristic zero such that

- $\mu = 1 + \mu_+, \omega(\mu_+) > 0$,
- $\sigma = \mathrm{x} + \sigma_+, \omega(\sigma_+) > 1$

becomes a group, sometimes called *Riordan group*, under the product

$$(\mu_1, \sigma_1) \rtimes (\mu_2, \sigma_2) := ((\mu_1 \circ \sigma_2) \times \mu_2, \sigma_1 \circ \sigma_2)$$

with $(1, \mathrm{x})$ as identity.

Main objective of the talk

The set of all pairs (μ, σ) of formal power series in a field $\mathbb{K}$ of characteristic zero such that

- $\mu = 1 + \mu_+, \omega(\mu_+) > 0,$
- $\sigma = \mathrm{x} + \sigma_+, \omega(\sigma_+) > 1$

becomes a group, sometimes called *Riordan group*, under the product

$$(\mu_1, \sigma_1) \rtimes (\mu_2, \sigma_2) := ((\mu_1 \circ \sigma_2) \times \mu_2, \sigma_1 \circ \sigma_2)$$

with $(1, \mathrm{x})$ as identity. The Riordan group is the semi-direct product UP $\rtimes$ US of the group UP of *unipotent prefunctions* and the group US of *unipotent substitutions*.

In the Riordan group, we can define the usual *nth power*:

$$(\mu,\sigma)^{\rtimes n} := \begin{cases} (1,\mathrm{x}) & \text{if} \quad n=0\ , \\ \underbrace{(\mu,\sigma)\rtimes\cdots\rtimes(\mu,\sigma)}_{n\ \textit{times}} & \text{if} \quad n>0\ . \end{cases}$$

In the Riordan group, we can define the usual *nth power*:

$$(\mu,\sigma)^{\rtimes n} := \begin{cases} (1,\mathrm{x}) & \text{if} \quad n=0\ , \\ \underbrace{(\mu,\sigma)\rtimes\cdots\rtimes(\mu,\sigma)}_{n\ \textit{times}} & \text{if} \quad n>0\ . \end{cases}$$

Our main objective is to define the generalized powers $(\mu,\sigma)^{\rtimes\lambda}$ for arbitrary exponents $\lambda \in \mathbb{K}$.

In the Riordan group, we can define the usual *nth power*:

$$(\mu,\sigma)^{\rtimes n} := \begin{cases} (1,\mathrm{x}) & \text{if} \quad n=0\ , \\ \underbrace{(\mu,\sigma)\rtimes\cdots\rtimes(\mu,\sigma)}_{n\ \textit{times}} & \text{if} \quad n>0\ . \end{cases}$$

Our main objective is to define the generalized powers $(\mu,\sigma)^{\rtimes\lambda}$ for arbitrary exponents $\lambda \in \mathbb{K}$.

1. Why the Riordan group ?

In the Riordan group, we can define the usual *nth power*:

$$(\mu,\sigma)^{\rtimes n} := \begin{cases} (1,\mathrm{x}) & \text{if} \quad n=0\ , \\ \underbrace{(\mu,\sigma)\rtimes\cdots\rtimes(\mu,\sigma)}_{n\ \text{times}} & \text{if} \quad n>0\ . \end{cases}$$

Our main objective is to define the generalized powers $(\mu,\sigma)^{\rtimes\lambda}$ for arbitrary exponents $\lambda\in\mathbb{K}$.

1. Why the Riordan group ?
2. Why generalized powers ?

Why the Riordan group ?

Why the Riordan group ?

This group occurs in several fields of combinatorics:

1. Riordan matrices (Shapiro *et al* 1991, Roman 1984), Sheffer sequences, umbral calculus;

Why the Riordan group ?

This group occurs in several fields of combinatorics:

1. Riordan matrices (Shapiro *et al* 1991, Roman 1984), Sheffer sequences, umbral calculus;
2. Combinatorial physics: problem of normal ordering for boson strings.

Normal ordering for boson strings

Normal ordering for boson strings

Let a and $a^\dagger$ be two letters (they play the role of quantum operators),

Normal ordering for boson strings

Let a and $a^\dagger$ be two letters (they play the role of quantum operators), respectively called *annihilation* and *creation* operators.

Normal ordering for boson strings

Let a and $a^\dagger$ be two letters (they play the role of quantum operators), respectively called *annihilation* and *creation* operators. The Weyl algebra $\mathcal{W} := \mathbb{C}\{a, a^\dagger\}/\langle aa^\dagger - a^\dagger a - 1\rangle$

Normal ordering for boson strings

Let a and $a^\dagger$ be two letters (they play the role of quantum operators), respectively called *annihilation* and *creation* operators. The Weyl algebra $\mathcal{W} := \mathbb{C}\{a, a^\dagger\}/\langle aa^\dagger - a^\dagger a - 1\rangle$ has $((a^\dagger)^i a^j)_{i,j\in\mathbb{N}}$ as an algebraic basis.

Normal ordering for boson strings

Let a and $a^\dagger$ be two letters (they play the role of quantum operators), respectively called *annihilation* and *creation* operators. The Weyl algebra $\mathcal{W} := \mathbb{C}\{a, a^\dagger\}/\langle aa^\dagger - a^\dagger a - 1\rangle$ has $((a^\dagger)^i a^j)_{i,j\in\mathbb{N}}$ as an algebraic basis.

An element of $\mathcal{W}$ - called a *boson string* - is said to be in **normal form**

Normal ordering for boson strings

Let a and $a^\dagger$ be two letters (they play the role of quantum operators), respectively called *annihilation* and *creation* operators. The Weyl algebra $\mathcal{W} := \mathbb{C}\{a, a^\dagger\}/\langle aa^\dagger - a^\dagger a - 1\rangle$ has $((a^\dagger)^i a^j)_{i,j\in\mathbb{N}}$ as an algebraic basis.

An element of $\mathcal{W}$ - called a *boson string* - is said to be in **normal form** if, and only if, it written in the basis $((a^\dagger)^i a^j)_{i,j\in\mathbb{N}}$.

Normal ordering for boson strings (cont’d)

Normal ordering for boson strings (cont’d)

In a series of recent papers Duchamp *et al* showed that for a certain kind of boson strings Ω,

Normal ordering for boson strings (cont'd)

In a series of recent papers Duchamp *et al* showed that for a certain kind of boson strings Ω, when written in normal form as $\Omega = \sum_{i,j} m_{i,j}(a^{\dagger})^{i}a^{j}$,

Normal ordering for boson strings (cont’d)

In a series of recent papers Duchamp *et al* showed that for a certain kind of boson strings Ω, when written in normal form as $\Omega = \sum_{i,j} m_{i,j}(a^\dagger)^i a^j$, then the doubly-infinite matrix $M = (m_{i,j})_{i,j}$ defines a Riordan matrix.

Why generalized powers ?

Why generalized powers ?

The generalized powers could be used to define one-parameter subgroups $\lambda \mapsto (\mu, \sigma)^{\rtimes \lambda}$ with $(\mu, \sigma)^{\rtimes(\alpha+\beta)} = (\mu, \sigma)^{\rtimes \alpha}(\mu, \sigma)^{\rtimes \beta}$.

Why generalized powers ?

The generalized powers could be used to define one-parameter subgroups $\lambda \mapsto (\mu, \sigma)^{\rtimes \lambda}$ with $(\mu, \sigma)^{\rtimes(\alpha+\beta)} = (\mu, \sigma)^{\rtimes \alpha}(\mu, \sigma)^{\rtimes \beta}$. Moreover such one-parameter groups are relevant in the field of combinatorial physics because it is possible that at some time t_0 the coefficients of $(\mu, \sigma)^{\rtimes t_0}$ are integers and could count certain physical quantities.

Remarks

- In order to define $(\mu,\sigma)^{\rtimes\lambda}$, for any $\lambda\in\mathbb{K}$, we follow the usual way: we use the binomial series.

Remarks

- In order to define $(\mu, \sigma)^{\rtimes\lambda}$, for any $\lambda \in \mathbb{K}$, we follow the usual way: we use the binomial series. Therefore we need to embed the Riordan group into an algebraic-structure with an addition.

Remarks

- In order to define $(\mu, \sigma)^{\rtimes\lambda}$, for any $\lambda \in \mathbb{K}$, we follow the usual way: we use the binomial series. Therefore we need to embed the Riordan group into an algebraic-structure with an addition.
- In this talk, three successive attempts to define generalized powers will be presented (work in progress):

Remarks

- In order to define $(\mu,\sigma)^{\rtimes\lambda}$, for any $\lambda \in \mathbb{K}$, we follow the usual way: we use the binomial series. Therefore we need to embed the Riordan group into an algebraic-structure with an addition.
- In this talk, three successive attempts to define generalized powers will be presented (work in progress):
  1. In a skew-algebra $\mathbb{K}[[x]] \rtimes x\mathbb{K}[[x]]$;

Remarks

- In order to define $(\mu, \sigma)^{\rtimes\lambda}$, for any $\lambda \in \mathbb{K}$, we follow the usual way: we use the binomial series. Therefore we need to embed the Riordan group into an algebraic-structure with an addition.
- In this talk, three successive attempts to define generalized powers will be presented (work in progress):
  1. In a skew-algebra $\mathbb{K}[[x]] \rtimes x\mathbb{K}[[x]]$;
  2. In an algebra $\mathbb{K}[[\mu_+, \sigma_+]]$ isomorphic to $\mathbb{K}[[x]]$;

Remarks

- In order to define $(\mu, \sigma)^{\rtimes\lambda}$, for any $\lambda \in \mathbb{K}$, we follow the usual way: we use the binomial series. Therefore we need to embed the Riordan group into an algebraic-structure with an addition.
- In this talk, three successive attempts to define generalized powers will be presented (work in progress):
  1. In a skew-algebra $\mathbb{K}[[\mathrm{x}]] \rtimes \mathrm{x}\mathbb{K}[[\mathrm{x}]]$;
  2. In an algebra $\mathbb{K}[[\mu_+, \sigma_+]]$ isomorphic to $\mathbb{K}[[\mathrm{x}]]$;
  3. In a subalgebra of infinite matrices.

Remarks

- In order to define $(\mu, \sigma)^{\rtimes\lambda}$, for any $\lambda \in \mathbb{K}$, we follow the usual way: we use the binomial series. Therefore we need to embed the Riordan group into an algebraic-structure with an addition.
- In this talk, three successive attempts to define generalized powers will be presented (work in progress):
  1. In a skew-algebra $\mathbb{K}[[\mathrm{x}]] \rtimes \mathrm{x}\mathbb{K}[[\mathrm{x}]]$;
  2. In an algebra $\mathbb{K}[[\mu_+, \sigma_+]]$ isomorphic to $\mathbb{K}[[\mathrm{x}]]$;
  3. In a subalgebra of infinite matrices.
- Based on a preprint "*Generalized powers for the Riordan group*" (can be found on ArXiv).

Our goal is to define $(\mu, \sigma)^{\rtimes\lambda}$, where (μ, σ) belongs to the Riordan group.

Our goal is to define $(\mu, \sigma)^{\rtimes\lambda}$, where (μ, σ) belongs to the Riordan group. We will use binomial series, and therefore need to embed the Riordan group in an algebra-like structure.

Our goal is to define $(\mu, \sigma)^{\rtimes\lambda}$, where (μ, σ) belongs to the Riordan group. We will use binomial series, and therefore need to embed the Riordan group in an algebra-like structure. The first structure we define for is not a true algebra, but a *right-distributive algebra*.

- A *right-distributive algebra* is a kind of algebra for which only right-distributivity holds: $(u + v) * w = u * w + v * w$;

- A *right-distributive algebra* is a kind of algebra for which only right-distributivity holds: $(u+v)*w = u*w + v*w$;
- In a right-distributive algebra, in general, $u*(v+w) \neq u*w + v*w$ and $u*(\alpha v) \neq (\alpha u)*v$ for $\alpha \in \mathbb{K}$. The other axioms of an (associative) algebra (with unit) are true;

- A *right-distributive algebra* is a kind of algebra for which only right-distributivity holds: $(u+v)*w = u*w + v*w$;
- In a right-distributive algebra, in general, $u*(v+w) \neq u*w + v*w$ and $u*(\alpha v) \neq (\alpha u)*v$ for $\alpha \in \mathbb{K}$. The other axioms of an (associative) algebra (with unit) are true;
- Examples:
  1. Every (associative) algebra (with unit) is also a right-distributive algebra;

- A *right-distributive algebra* is a kind of algebra for which only right-distributivity holds: $(u+v)*w = u*w + v*w$;
- In a right-distributive algebra, in general, $u*(v+w) \neq u*w + v*w$ and $u*(\alpha v) \neq (\alpha u)*v$ for $\alpha \in \mathbb{K}$. The other axioms of an (associative) algebra (with unit) are true;
- Examples:
  1. Every (associative) algebra (with unit) is also a right-distributive algebra;
  2. $\mathrm{x}\mathbb{K}[[\mathrm{x}]]$ with composition as multiplication and usual addition is a right-distributive algebra which is not an algebra in the usual sense.

- A *right-distributive algebra* is a kind of algebra for which only right-distributivity holds: $(u+v)*w = u*w+v*w$;
- In a right-distributive algebra, in general, $u*(v+w) \neq u*w+v*w$ and $u*(\alpha v) \neq (\alpha u)*v$ for $\alpha \in \mathbb{K}$. The other axioms of an (associative) algebra (with unit) are true;
- Examples:
  1. Every (associative) algebra (with unit) is also a right-distributive algebra;
  2. $\mathrm{x}\mathbb{K}[[\mathrm{x}]]$ with composition as multiplication and usual addition is a right-distributive algebra which is not an algebra in the usual sense. For instance, $\mathrm{x}^2 \circ (2\mathrm{x}) = 4\mathrm{x}^2 \neq \mathrm{x}^2 \circ \mathrm{x} + \mathrm{x}^2 \circ \mathrm{x} = 2\mathrm{x}^2$ ($\mathbb{K}$ of characteristic zero).

- A *right-distributive algebra* is a kind of algebra for which only right-distributivity holds: $(u+v)*w = u*w + v*w$;
- In a right-distributive algebra, in general, $u*(v+w) \neq u*w + v*w$ and $u*(\alpha v) \neq (\alpha u)*v$ for $\alpha \in \mathbb{K}$. The other axioms of an (associative) algebra (with unit) are true;
- Examples:
  1. Every (associative) algebra (with unit) is also a right-distributive algebra;
  2. $\mathrm{x}\mathbb{K}[[\mathrm{x}]]$ with composition as multiplication and usual addition is a right-distributive algebra which is not an algebra in the usual sense. For instance, $\mathrm{x}^2 \circ (2\mathrm{x}) = 4\mathrm{x}^2 \neq \mathrm{x}^2 \circ \mathrm{x} + \mathrm{x}^2 \circ \mathrm{x} = 2\mathrm{x}^2$ ($\mathbb{K}$ of characteristic zero).
- Notions of (two-sided, left, right) ideals, units and group of units are extended in the obvious way.

Let $\mathfrak{M} := \mathrm{x}\mathbb{K}[[\mathrm{x}]]$.

Let $\mathfrak{M} := x\mathbb{K}[[x]]$.
$\mathbb{K}[[x]] \times \mathfrak{M}$ is a right-distributive algebra with

Let $\mathfrak{M} := x\mathbb{K}[[x]]$.
$\mathbb{K}[[x]] \times \mathfrak{M}$ is a right-distributive algebra with

1. the component-wise addition and scalar multiplication (for the vector space structure);

Let $\mathfrak{M} := x\mathbb{K}[[x]]$.
$\mathbb{K}[[x]] \times \mathfrak{M}$ is a right-distributive algebra with

1. the component-wise addition and scalar multiplication (for the vector space structure);
2. the (associative) multiplication

$$(\mu_1, \sigma_1) \rtimes (\mu_2, \sigma_2) := ((\mu_1 \circ \sigma_2) \times \mu_2, \sigma_1 \circ \sigma_2)$$

which extends the product of UP $\rtimes$ US.

Let $\mathfrak{M} := \mathrm{x}\mathbb{K}[[\mathrm{x}]]$.
$\mathbb{K}[[\mathrm{x}]] \times \mathfrak{M}$ is a right-distributive algebra with

1. the component-wise addition and scalar multiplication (for the vector space structure);
2. the (associative) multiplication

$$(\mu_1, \sigma_1) \rtimes (\mu_2, \sigma_2) := ((\mu_1 \circ \sigma_2) \times \mu_2, \sigma_1 \circ \sigma_2)$$

which extends the product of UP $\rtimes$ US.

This algebra is denoted $\mathbb{K}[[\mathrm{x}]] \rtimes \mathfrak{M}$ and called *skew Riordan algebra*.

Let $\mathfrak{M} := \mathrm{x}\mathbb{K}[[\mathrm{x}]]$.
$\mathbb{K}[[\mathrm{x}]] \times \mathfrak{M}$ is a right-distributive algebra with

1. the component-wise addition and scalar multiplication (for the vector space structure);
2. the (associative) multiplication

$$(\mu_1, \sigma_1) \rtimes (\mu_2, \sigma_2) := ((\mu_1 \circ \sigma_2) \times \mu_2, \sigma_1 \circ \sigma_2)$$

which extends the product of UP $\rtimes$ US.

This algebra is denoted $\mathbb{K}[[\mathrm{x}]] \rtimes \mathfrak{M}$ and called *skew Riordan algebra*.
The Riordan group UP $\rtimes$ US is a subgroup of the group of units of the skew Riordan algebra.

Extension of usual powers from UP $\rtimes$ US to $\mathbb{K}[[x]] \rtimes \mathfrak{M}$

Let $(\mu, \sigma) \in \mathbb{K}[[x]] \rtimes \mathfrak{M}$ and $n \in \mathbb{N}$. We denote the usual nth power by

Extension of usual powers from UP $\rtimes$ US to $\mathbb{K}[[\mathrm{x}]] \rtimes \mathfrak{M}$

Let $(\mu, \sigma) \in \mathbb{K}[[\mathrm{x}]] \rtimes \mathfrak{M}$ and $n \in \mathbb{N}$. We denote the usual nth power by

$$(\mu, \sigma)^{\rtimes n} := \begin{cases} (1, \mathrm{x}) & \text{if} \quad n = 0\ , \\ \underbrace{(\mu, \sigma) \rtimes \cdots \rtimes (\mu, \sigma)}_{n \text{ times}} & \text{if} \quad n > 0\ . \end{cases}$$

We denote

1. $\mathbb{K}[[x]]^+ := \mathfrak{M}$ is a semigroup under multiplication;
2. $\mathfrak{M}^+ := \{\sigma_+ \in \mathfrak{M} : \omega(\sigma_+) > 1\}$ is a semigroup under formal substitution.

We denote

1. $\mathbb{K}[[x]]^+ := \mathfrak{M}$ is a semigroup under multiplication;
2. $\mathfrak{M}^+ := \{\sigma_+ \in \mathfrak{M} : \omega(\sigma_+) > 1\}$ is a semigroup under formal substitution.

$\mathbb{K}[[x]]^+ \rtimes \mathfrak{M}^+$ is a two-sided ideal of the skew Riordan algebra.

The ideal $\mathbb{K}[[x]]^{+} \rtimes \mathfrak{M}^{+}$ as an interesting property: it allows us to define a formal calculus on the skew Riordan algebra.

The ideal $\mathbb{K}[[x]]^+ \rtimes \mathfrak{M}^+$ as an interesting property: it allows us to define a formal calculus on the skew Riordan algebra. That means that we can define an operation of formal power series on $\mathbb{K}[[x]] \rtimes \mathfrak{M}$ using converging series in powers of elements of $\mathbb{K}[[x]]^+ \rtimes \mathfrak{M}^+$.

Operation of formal power series

Proposition

For each formal power series $f = \sum_{n \geq 0} f_n \mathrm{x}^n$ and each $(\mu_+, \sigma_+) \in \mathbb{K}[[\mathrm{x}]]^+ \rtimes \mathfrak{M}^+$, the series $\sum_{n \geq 0} f_n (\mu_+, \sigma_+)^{\rtimes n}$ converges in the skew Riordan algebra.

In the particular case where f is a binomial series $\sum_{n\geq 0}\binom{\lambda}{n}\mathrm{x}^n$,
the last proposition gives us a ”natural” definition for generalized powers in the Riordan group.

In the particular case where f is a binomial series $\sum_{n\geq 0}\binom{\lambda}{n}\mathrm{x}^n$, the last proposition gives us a ”natural” definition for generalized powers in the Riordan group. However it is not as natural as it seems.

For $\lambda \in \mathbb{K}$ and $n \in \mathbb{N}$, $\binom{\lambda}{n} := \dfrac{\lambda(\lambda-1)\dots(\lambda-n+1)}{n!}$ is the generalized binomial coefficient.

For $\lambda \in \mathbb{K}$ and $n \in \mathbb{N}$, $\binom{\lambda}{n} := \dfrac{\lambda(\lambda-1)\ldots(\lambda-n+1)}{n!}$ is the generalized binomial coefficient.

Proposition

Let $(\mu, \sigma) = (1 + \mu_+, \mathrm{x} + \sigma_+) \in \mathrm{UP} \rtimes \mathrm{US}$ (with $(\mu_+, \sigma_+) \in \mathbb{K}[[\mathrm{x}]]^+ \rtimes \mathfrak{M}^+$).

For $\lambda \in \mathbb{K}$ and $n \in \mathbb{N}$, $\binom{\lambda}{n} := \dfrac{\lambda(\lambda-1)\dots(\lambda-n+1)}{n!}$ is the generalized binomial coefficient.

Proposition

Let $(\mu, \sigma) = (1 + \mu_+, \mathrm{x} + \sigma_+) \in \mathrm{UP} \rtimes \mathrm{US}$ (with $(\mu_+, \sigma_+) \in \mathbb{K}[[\mathrm{x}]]^+ \rtimes \mathfrak{M}^+$). Then for every $\lambda \in \mathbb{K}$, the binomial series

$$(\mu, \sigma)^{\rtimes\lambda} = ((1, \mathrm{x}) + (\mu_+, \sigma_+))^{\rtimes\lambda} := \sum_{n\geq 0} \binom{\lambda}{n} (\mu_+, \sigma_+)^{\rtimes n}$$

converges in the skew Riordan algebra to an element of $\mathrm{UP} \rtimes \mathrm{US}$.

A serious weakness

A serious weakness

The generalized power $((1, \mathrm{x}) + (\mu_+, \sigma_+))^{\rtimes\lambda}$ with $\lambda \in \mathbb{N}$ does not match with the usual *n*th power of $((1, \mathrm{x}) + (\mu_+, \sigma_+))$ in the Riordan group.

A serious weakness

The generalized power $((1, \mathrm{x}) + (\mu_+, \sigma_+))^{\rtimes\lambda}$ with $\lambda \in \mathbb{N}$ does not match with the usual *n*th power of $((1, \mathrm{x}) + (\mu_+, \sigma_+))$ in the Riordan group.
For instance, let $\mu = 1 + \mathrm{x}$ and $\sigma = \mathrm{x} + \mathrm{x}^2$.

A serious weakness

The generalized power $((1, \mathrm{x}) + (\mu_+, \sigma_+))^{\rtimes\lambda}$ with $\lambda \in \mathbb{N}$ does not match with the usual *n*th power of $((1, \mathrm{x}) + (\mu_+, \sigma_+))$ in the Riordan group.

For instance, let $\mu = 1 + \mathrm{x}$ and $\sigma = \mathrm{x} + \mathrm{x}^2$. If we compute $((1, \mathrm{x}) + (\mathrm{x}, \mathrm{x}^2))^{\rtimes 2}$ in the Riordan group as the usual square of (μ, σ), then it is equal to
$(1 + 2\mathrm{x} + 2\mathrm{x}^2 + \mathrm{x}^3, \mathrm{x} + 2\mathrm{x}^2 + 2\mathrm{x}^3 + \mathrm{x}^4)$.

A serious weakness

The generalized power $((1, \mathrm{x}) + (\mu_+, \sigma_+))^{\rtimes\lambda}$ with $\lambda \in \mathbb{N}$ does not match with the usual *n*th power of $((1, \mathrm{x}) + (\mu_+, \sigma_+))$ in the Riordan group.

For instance, let $\mu = 1 + \mathrm{x}$ and $\sigma = \mathrm{x} + \mathrm{x}^2$. If we compute $((1, \mathrm{x}) + (\mathrm{x}, \mathrm{x}^2))^{\rtimes 2}$ in the Riordan group as the usual square of (μ, σ), then it is equal to $(1 + 2\mathrm{x} + 2\mathrm{x}^2 + \mathrm{x}^3, \mathrm{x} + 2\mathrm{x}^2 + 2\mathrm{x}^3 + \mathrm{x}^4)$. If we compute the same quantity as a binomial series, the result obtained is $(1 + 2\mathrm{x} + \mathrm{x}^3, \mathrm{x} + 2\mathrm{x}^2 + \mathrm{x}^4)$.

A serious weakness

The generalized power $((1,\mathrm{x})+(\mu_+,\sigma_+))^{\rtimes\lambda}$ with $\lambda\in\mathbb{N}$ does not match with the usual *n*th power of $((1,\mathrm{x})+(\mu_+,\sigma_+))$ in the Riordan group.
For instance, let $\mu=1+\mathrm{x}$ and $\sigma=\mathrm{x}+\mathrm{x}^2$. If we compute $((1,\mathrm{x})+(\mathrm{x},\mathrm{x}^2))^{\rtimes 2}$ in the Riordan group as the usual square of (μ,σ), then it is equal to $(1+2\mathrm{x}+2\mathrm{x}^2+\mathrm{x}^3,\mathrm{x}+2\mathrm{x}^2+2\mathrm{x}^3+\mathrm{x}^4)$. If we compute the same quantity as a binomial series, the result obtained is $(1+2\mathrm{x}+\mathrm{x}^3,\mathrm{x}+2\mathrm{x}^2+\mathrm{x}^4)$. The difference comes essentially from the lack of left-distributivity in the skew Riordan algebra.

It is not a relevant definition for generalized powers in the Riordan group, but it may be accurate for another structure

It is not a relevant definition for generalized powers in the Riordan group, but it may be accurate for another structure $\Rightarrow$ Is there an algebraic structure for which these powers generalize the usual ones ?

The algebra $\mathbb{K}[[\mu_+, \sigma_+]]$

We define

$$\mathbb{K}[[\mu_+, \sigma_+]] := \{\sum_{n\geq 0} f_n(\mu_+, \sigma_+)^{\rtimes n} : f = \sum_{n\in\mathbb{N}} f_n \mathrm{x}^n \in \mathbb{K}[[\mathrm{x}]]\}.$$

The algebra $\mathbb{K}[[\mu_+, \sigma_+]]$

We define
$\mathbb{K}[[\mu_+, \sigma_+]] := \{\sum_{n\geq 0} f_n(\mu_+, \sigma_+)^{\rtimes n} : f = \sum_{n\in\mathbb{N}} f_n \mathrm{x}^n \in \mathbb{K}[[\mathrm{x}]]\}$.
Endowed with the usual Cauchy product

$$\left(\sum_{n\geq 0} f_n(\mu_+, \sigma_+)^{\rtimes n}\right) * \left(\sum_{n\geq 0} g_n(\mu_+, \sigma_+)^{\rtimes n}\right)$$
$$:= \sum_{n\geq 0} \left(\sum_{k=0}^{n} f_{n-k} g_k\right) (\mu_+, \sigma_+)^{\rtimes n}$$

$\mathbb{K}[[\mu_+, \sigma_+]]$ becomes an algebra isomorphic to $\mathbb{K}[[\mathrm{x}]]$.

Proposition

Let $(\mu, \sigma) = (1 + \mu_+, \mathrm{x} + \sigma_+) \in \mathrm{UP} \rtimes \mathrm{US}$, then the binomial series $(\mu, \sigma)^{*\lambda} := ((1, \mathrm{x}) + (\mu_+, \sigma_+))^{*\lambda} := \sum_{n \geq 0} \binom{\lambda}{n} (\mu_+, \sigma_+)^{\rtimes n} \in \mathbb{K}[[\mu_+, \sigma_+]]$ defines an element of $\mathrm{UP} \rtimes \mathrm{US}$.

Remarks

Let $(\mu, \sigma) = (1 + \mu_+, \mathrm{x} + \sigma_+) \in \mathrm{UP} \rtimes \mathrm{US}$.

Remarks

Let $(\mu, \sigma) = (1 + \mu_+, \mathrm{x} + \sigma_+) \in$ UP $\rtimes$ US.

1. If $\lambda \in \mathbb{N}$, then $(\mu, \sigma)^{*\lambda}$ is equal to the λth power of (μ, σ) in the algebra $\mathbb{K}[[\mu_+, \sigma_+]]$, not as an element of UP $\rtimes$ US;

Remarks

Let $(\mu, \sigma) = (1 + \mu_+, \mathrm{x} + \sigma_+) \in \text{UP} \rtimes \text{US}$.

1. If $\lambda \in \mathbb{N}$, then $(\mu, \sigma)^{*\lambda}$ is equal to the λth power of (μ, σ) in the algebra $\mathbb{K}[[\mu_+, \sigma_+]]$, not as an element of $\text{UP} \rtimes \text{US}$;
2. If $\lambda = -1$, then $(\mu, \sigma)^{*\lambda}$ is the inverse of (μ, σ) as an element of $\mathbb{K}[[\mu_+, \sigma_+]]$, not as an element of $\text{UP} \rtimes \text{US}$;

Remarks

Let $(\mu, \sigma) = (1 + \mu_+, \mathrm{x} + \sigma_+) \in \mathrm{UP} \rtimes \mathrm{US}$.

1. If $\lambda \in \mathbb{N}$, then $(\mu, \sigma)^{*\lambda}$ is equal to the λth power of (μ, σ) in the algebra $\mathbb{K}[[\mu_+, \sigma_+]]$, not as an element of $\mathrm{UP} \rtimes \mathrm{US}$;
2. If $\lambda = -1$, then $(\mu, \sigma)^{*\lambda}$ is the inverse of (μ, σ) as an element of $\mathbb{K}[[\mu_+, \sigma_+]]$, not as an element of $\mathrm{UP} \rtimes \mathrm{US}$;
3. For every $\alpha, \beta \in \mathbb{K}$, $(\mu, \sigma)^{*\alpha} * (\mu, \sigma)^{*\beta} = (\mu, \sigma)^{*(\alpha+\beta)}$ and $(\mu, \sigma)^{*0} = (1, \mathrm{x})$.

Remarks

Let $(\mu, \sigma) = (1 + \mu_+, \mathrm{x} + \sigma_+) \in \mathrm{UP} \rtimes \mathrm{US}$.

1. If $\lambda \in \mathbb{N}$, then $(\mu, \sigma)^{*\lambda}$ is equal to the λth power of (μ, σ) in the algebra $\mathbb{K}[[\mu_+, \sigma_+]]$, not as an element of $\mathrm{UP} \rtimes \mathrm{US}$;
2. If $\lambda = -1$, then $(\mu, \sigma)^{*\lambda}$ is the inverse of (μ, σ) as an element of $\mathbb{K}[[\mu_+, \sigma_+]]$, not as an element of $\mathrm{UP} \rtimes \mathrm{US}$;
3. For every $\alpha, \beta \in \mathbb{K}$, $(\mu, \sigma)^{*\alpha} * (\mu, \sigma)^{*\beta} = (\mu, \sigma)^{*(\alpha+\beta)}$ and $(\mu, \sigma)^{*0} = (1, \mathrm{x})$. So $\lambda \mapsto (\mu, \sigma)^{*\lambda}$ is a multiplicative one-parameter group.

This definition is ”better” than the former because it coincides with usual powers, but powers in the group of invertible elements of $\mathbb{K}[[\mu_+, \sigma_+]]$, not in UP $\rtimes$ US.

This definition is ”better” than the former because it coincides with usual powers, but powers in the group of invertible elements of $\mathbb{K}[[\mu_+, \sigma_+]]$, not in UP $\rtimes$ US.
$\Rightarrow$ we need to find another algebraic setting to define the generalized powers.

Elements of the Riordan group can be seen as operators acting on formal power series.

Elements of the Riordan group can be seen as operators acting on formal power series. As a group of operators, $\mathrm{UP} \rtimes \mathrm{US}$ is naturally embedded in the algebra of endomorphisms of $\mathbb{K}[[x]]$.

Elements of the Riordan group can be seen as operators acting on formal power series. As a group of operators, UP $\rtimes$ US is naturally embedded in the algebra of endomorphisms of $\mathbb{K}[[x]]$. This is the algebraic setting considered in this part.

- Every element $(\mu, \sigma) \in \mathrm{UP} \rtimes \mathrm{US}$ can be faithfully identified with a linear endomorphism of formal power series

$$f \in \mathbb{K}[[\mathrm{x}]] \mapsto \rho(\mu, \sigma)(f) = \mu \times (f \circ \sigma)$$

- Every element $(\mu, \sigma) \in \mathrm{UP} \rtimes \mathrm{US}$ can be faithfully identified with a linear endomorphism of formal power series

$$f \in \mathbb{K}[[\mathrm{x}]] \mapsto \rho(\mu, \sigma)(f) = \mu \times (f \circ \sigma)$$

such that $\rho((\mu_1, \sigma_1) \rtimes (\mu_2, \sigma_2)) = \rho(\mu_2, \sigma_2) \circ \rho(\mu_1, \sigma_1)$, $\rho((\mu, \sigma)^{\rtimes(-1)}) = \rho(\mu, \sigma)^{-1}$ and $\rho(1, \mathrm{x}) = \mathrm{Id}$.

- Every element $(\mu, \sigma) \in \mathrm{UP} \rtimes \mathrm{US}$ can be faithfully identified with a linear endomorphism of formal power series

$$f \in \mathbb{K}[[\mathrm{x}]] \mapsto \rho(\mu, \sigma)(f) = \mu \times (f \circ \sigma)$$

such that $\rho((\mu_1, \sigma_1) \rtimes (\mu_2, \sigma_2)) = \rho(\mu_2, \sigma_2) \circ \rho(\mu_1, \sigma_1)$, $\rho((\mu, \sigma)^{\rtimes(-1)}) = \rho(\mu, \sigma)^{-1}$ and $\rho(1, \mathrm{x}) = \mathrm{Id}$.

- Every such operator has a (bi-infinite) matrix representation: $M_{(\mu,\sigma)}(i, j) := [\mathrm{x}^i]\rho(\mu, \sigma)(\mathrm{x}^j) = [\mathrm{x}^i](\mu\sigma^j)$. Such matrices are called *Riordan matrices*.

UP $\rtimes$ US, seen as the group of Riordan matrices, is embedded into the algebra of infinite lower triangular matrices, and even in the group of unipotent matrices (because $M_{(\mu,\sigma)}(i,i) = 1$ for every $i \in \mathbb{N}$).

UP $\rtimes$ US, seen as the group of Riordan matrices, is embedded into the algebra of infinite lower triangular matrices, and even in the group of unipotent matrices (because $M_{(\mu,\sigma)}(i,i) = 1$ for every $i \in \mathbb{N}$). We will compute the generalized powers (in the third version) in this algebra.

It is possible to define a formal calculus on nilpotent matrices.

It is possible to define a formal calculus on nilpotent matrices. This calculus will be used to define generalized powers, just as in the first part of the presentation:

It is possible to define a formal calculus on nilpotent matrices. This calculus will be used to define generalized powers, just as in the first part of the presentation: $\mathbb{K}[[x]]^+ \rtimes \mathfrak{M}^+$ is replaced by the ideal of nilpotent matrices.

Formal calculus on nilpotent matrices

Lemma

Let N be a (topological) nilpotent matrix N, *i.e.*, $N(i,j) = 0$ for every $j \geq i$. Then for every $f = \sum_{n \in \mathbb{N}} f_n \mathrm{x}^n$, the sum

$$\sum_{n \geq 0} f_n N^n$$

is convergent and its sum is a lower triangular matrix.

Formal calculus on nilpotent matrices

Lemma

Let N be a (topological) nilpotent matrix N, *i.e.*, $N(i,j) = 0$ for every $j \geq i$. Then for every $f = \sum_{n \in \mathbb{N}} f_n \mathrm{x}^n$, the sum

$$\sum_{n \geq 0} f_n N^n$$

is convergent and its sum is a lower triangular matrix. If $f_0 = 0$, then the sum is a nilpotent matrix and if $f_0 = 1$, the sum is a unipotent matrix.

Taking the binomial series in place of the series f in the last lemma leads to a new definition for generalized powers.

Let $(\mu, \sigma) \in \mathrm{UP} \rtimes \mathrm{US}$. Then $M_{(\mu,\sigma)} = \mathrm{Id} + N_{(\mu,\sigma)}$ where $N_{(\mu,\sigma)}$ is a nilpotent matrix.

Let $(\mu, \sigma) \in \mathrm{UP} \rtimes \mathrm{US}$. Then $M_{(\mu,\sigma)} = \mathrm{Id} + N_{(\mu,\sigma)}$ where $N_{(\mu,\sigma)}$ is a nilpotent matrix. For every $\lambda \in \mathbb{K}$, we define

$$(\mu, \sigma)^{\rtimes \lambda} := \sum_{n \geq 0} \binom{\lambda}{n} N^n_{(\mu,\sigma)} \ .$$

Remarks

1. $(\mu, \sigma)^{\rtimes\lambda}$ is a unipotent matrix;

Remarks

1. $(\mu, \sigma)^{\rtimes\lambda}$ is a unipotent matrix;
2. When $\lambda \in \mathbb{N}$, then the generalized power reduces to the usual λth power in the group UP $\rtimes$ US;

Remarks

1. $(\mu, \sigma)^{\rtimes\lambda}$ is a unipotent matrix;
2. When $\lambda \in \mathbb{N}$, then the generalized power reduces to the usual λth power in the group UP $\rtimes$ US;
3. For $\lambda = -1$, it is the inverse in the group UP $\rtimes$ US;

Remarks

1. $(\mu, \sigma)^{\rtimes\lambda}$ is a unipotent matrix;
2. When $\lambda \in \mathbb{N}$, then the generalized power reduces to the usual λth power in the group UP $\rtimes$ US;
3. For $\lambda = -1$, it is the inverse in the group UP $\rtimes$ US;
4. In would be nice to prove that for an arbitrary $\lambda \in \mathbb{K}$, $(\mu, \sigma)^{\rtimes\lambda}$ belongs to UP $\rtimes$ US.

Remarks

1. $(\mu, \sigma)^{\rtimes\lambda}$ is a unipotent matrix;
2. When $\lambda \in \mathbb{N}$, then the generalized power reduces to the usual λth power in the group UP $\rtimes$ US;
3. For $\lambda = -1$, it is the inverse in the group UP $\rtimes$ US;
4. In would be nice to prove that for an arbitrary $\lambda \in \mathbb{K}$, $(\mu, \sigma)^{\rtimes\lambda}$ belongs to UP $\rtimes$ US. For this we need a little bit analysis.

- Suppose that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with their usual topology. We put on LT$(\mathbb{N}, \mathbb{C})$ the topology of *simple convergence on coefficients*:

- Suppose that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with their usual topology. We put on LT$(\mathbb{N}, \mathbb{C})$ the topology of *simple convergence on coefficients*: it becomes a **Fréchet algebra** (it is not a Banach algebra because its group of units is not open);

- Suppose that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with their usual topology. We put on $\mathrm{LT}(\mathbb{N}, \mathbb{C})$ the topology of *simple convergence on coefficients*: it becomes a **Fréchet algebra** (it is not a Banach algebra because its group of units is not open);
- A Fréchet algebra is similar to a Banach algebra except that the topology is given by a denumerable family of seminorms $(\| \cdot \|)_{n \in \mathbb{N}}$ rather than a unique norm.

- Suppose that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with their usual topology. We put on LT$(\mathbb{N}, \mathbb{C})$ the topology of *simple convergence on coefficients*: it becomes a **Fréchet algebra** (it is not a Banach algebra because its group of units is not open);
- A Fréchet algebra is similar to a Banach algebra except that the topology is given by a denumerable family of seminorms $(\| \cdot \|)_{n \in \mathbb{N}}$ rather than a unique norm.
- In any Fréchet algebra (with identity) $\mathcal{A}$, we can define an analytic calculus:

- Suppose that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with their usual topology. We put on $\mathrm{LT}(\mathbb{N}, \mathbb{C})$ the topology of *simple convergence on coefficients*: it becomes a **Fréchet algebra** (it is not a Banach algebra because its group of units is not open);
- A Fréchet algebra is similar to a Banach algebra except that the topology is given by a denumerable family of seminorms $(\| \cdot \|)_{n \in \mathbb{N}}$ rather than a unique norm.
- In any Fréchet algebra (with identity) $\mathcal{A}$, we can define an analytic calculus: for every $a \in \mathcal{A}$ and every analytic power series $f = \sum_{n \geq 0} f_n \mathrm{x}^n$ of radius R_f, the series $f(a) = \sum_{n \geq 0} f_n a^n$ is convergent in $\mathcal{A}$, if and only if, for every $n \in \mathbb{N}$, $\|a\|_n < R_f$.

- In any Fréchet algebra (with identity) $\mathcal{A}$, $\exp(a)$ or $\log(1+a)$ exist for $a \in \mathcal{A}$ (the later needs the condition that $\|a\|_n < 1$ for every $n \in \mathbb{N}$) and satisfy their usual properties;

- In any Fréchet algebra (with identity) $\mathcal{A}$, $\exp(a)$ or $\log(1+a)$ exist for $a \in \mathcal{A}$ (the later needs the condition that $\|a\|_n < 1$ for every $n \in \mathbb{N}$) and satisfy their usual properties;
- In particular $t \in \mathbb{K} \mapsto \exp(ta) \in \mathcal{A}$ is analytic and defines a one-parameter group;

- In any Fréchet algebra (with identity) $\mathcal{A}$, $\exp(a)$ or $\log(1+a)$ exist for $a \in \mathcal{A}$ (the later needs the condition that $\|a\|_n < 1$ for every $n \in \mathbb{N}$) and satisfy their usual properties;
- In particular $t \in \mathbb{K} \mapsto \exp(ta) \in \mathcal{A}$ is analytic and defines a one-parameter group;
- For every $b \in \mathcal{A}$ such that $\|b\|_n < 1$ for each n, we can define

$$a^{\lambda} := \exp(\lambda \log(a))$$

with $a := 1 + b$.

For every $(\mu,\sigma) \in \mathrm{US} \rtimes \mathrm{US}$, seen as an element of the Fréchet algebra $\mathrm{LT}(\mathbb{N},\mathbb{C})$, we may define $(\mu,\sigma)^{\lambda} := \exp(\lambda \log(\mu,\sigma))$, for every $\lambda \in \mathbb{C}$ (or $\lambda \in \mathbb{R}$). (For the moment, $(\mu,\sigma)^{\rtimes\lambda} \neq (\mu,\sigma)^{\lambda}$.)

For every $(\mu, \sigma) \in$ US $\rtimes$ US, seen as an element of the Fréchet algebra LT$(\mathbb{N}, \mathbb{C})$, we may define $(\mu, \sigma)^{\lambda} :=$ exp$(\lambda$ log$(\mu, \sigma))$, for every $\lambda \in \mathbb{C}$ (or $\lambda \in \mathbb{R}$). (For the moment, $(\mu, \sigma)^{\rtimes \lambda} \neq (\mu, \sigma)^{\lambda}$.)

Proposition

For every $\lambda \in \mathbb{C}$ and every $(\mu, \sigma) \in$ UP $\rtimes$ US, $(\mu, \sigma)^{\rtimes \lambda} \in$ UP $\rtimes$ US.

Idea of the proof

Idea of the proof

1. $\lambda \mapsto (\mu, \sigma)^{\lambda} = \exp(\lambda \log(\mu, \sigma))$ is continuous (actually it is an analytic function);

Idea of the proof

1. $\lambda \mapsto (\mu, \sigma)^{\lambda} = \exp(\lambda \log(\mu, \sigma))$ is continuous (actually it is an analytic function);
2. $(\mu, \sigma)^{\lambda} = \lim_{q \to \lambda, q \in \mathbb{Q}} (\mu, \sigma)^{q}$;

Idea of the proof

1. $\lambda \mapsto (\mu, \sigma)^\lambda = \exp(\lambda \log(\mu, \sigma))$ is continuous (actually it is an analytic function);
2. $(\mu, \sigma)^\lambda = \lim_{q \to \lambda, q \in \mathbb{Q}} (\mu, \sigma)^q$;
3. For every $q \in \mathbb{Q}$, $(\mu, \sigma)^q \in \mathrm{UP} \rtimes \mathrm{US}$, then $(\mu, \sigma)^\lambda \in \mathrm{UP} \rtimes \mathrm{US}$ (because $\mathrm{UP} \rtimes \mathrm{US}$ is closed in $\mathrm{LT}(\mathbb{N}, \mathbb{C})$);

Idea of the proof

1. $\lambda \mapsto (\mu, \sigma)^{\lambda} = \exp(\lambda \log(\mu, \sigma))$ is continuous (actually it is an analytic function);
2. $(\mu, \sigma)^{\lambda} = \lim_{q \to \lambda, q \in \mathbb{Q}} (\mu, \sigma)^{q}$;
3. For every $q \in \mathbb{Q}$, $(\mu, \sigma)^{q} \in \mathrm{UP} \rtimes \mathrm{US}$, then $(\mu, \sigma)^{\lambda} \in \mathrm{UP} \rtimes \mathrm{US}$ (because $\mathrm{UP} \rtimes \mathrm{US}$ is closed in $\mathrm{LT}(\mathbb{N}, \mathbb{C})$);
4. For every $q \in \mathbb{Q}$, $(\mu, \sigma)^{\rtimes q} = (\mu, \sigma)^{q}$ and $\lambda \mapsto (\mu, \sigma)^{\rtimes \lambda}$ is continuous, therefore $(\mu, \sigma)^{\rtimes \lambda} = (\mu, \sigma)^{\lambda} \in \mathrm{UP} \rtimes \mathrm{US}$.