Communications in Algebra

ISSN: 0092-7872 (Print) 1532-4125 (Online) Journal homepage: https://www.tandfonline.com/loi/lagb20

Topological rigidity as a monoidal equivalence

Laurent Poinsot

To cite this article: Laurent Poinsot (2019) Topological rigidity as a monoidal equivalence, Communications in Algebra, 47:9, 3457-3480, DOI: 10.1080/00927872.2019.1566957

To link to this article: https://doi.org/10.1080/00927872.2019.1566957

Published online: 04 Mar 2019.

COMMUNICATIONS IN ALGEBRA®
2019, VOL. 47, NO. 9, 3457–3480
https://doi.org/10.1080/00927872.2019.1566957

Topological rigidity as a monoidal equivalence

Laurent Poinsot[a,b]

[a]LIPN-UMR CNRS 7030, University Paris 13, Sorbonne Paris Cité, Villetaneuse, France; [b]CREA, French Air Force Academy, Salon-de-Provence, France

ABSTRACT

A topological commutative ring is said to be *rigid* when for every set *X*, the topological dual of the *X*-fold topological product of the ring is isomorphic to the free module over *X*. Examples are fields with a ring topology, discrete rings, and normed algebras. Rigidity translates into a dual equivalence between categories of free modules and of "topologically free" modules and, with a suitable topological tensor product for the latter, one proves that it lifts to an equivalence between monoids in this category (some suitably generalized topological algebras) and some coalgebras. In particular, we provide a description of its relationship with the standard duality between algebras and coalgebras, namely finite duality.

ARTICLE HISTORY
Received 15 July 2018
Revised 3 December 2018
Communicated by Alberto Facchini

KEYWORDS
Coalgebras; finite duality; topological dual space; topological basis

MATHEMATICS SUBJECT CLASSIFICATION
13J99; 54H13; 46A20; 19D23

1. Introduction

The main result of [13] states that given a (Hausdorff) topological field $(\mathbb{k}, \tau)$, for every set X, the topological dual $((\mathbb{k}, \tau)^X)'$ of the X-fold topological product $(\mathbb{k}, \tau)^X$ is isomorphic to the vector space $\mathbb{k}^{(X)}$ of finitely supported $\mathbb{k}$-valued maps defined on X (i.e., those maps $X \xrightarrow{f} \mathbb{k}$ such that for all but finitely many members x of X, $f(x) = 0$).

Actually this topological property of *rigidity* is shared by more general topological (commutative unital) rings than only topological fields (a fact not noticed in [13]). For instance any discrete ring is rigid in the above sense (see Lemma 15). And even if not all topological rings are rigid (see Section 4.3 for a counterexample), many of them still are (e.g., every real or complex normed commutative algebra).

It is our intention to study in more details some consequences of the property of rigidity for arbitrary commutative rings in particular at the level of some of their topological algebras.[1] So far, for a topological ring (R, τ), rigidity reads as $((R, \tau)^X)' \simeq R^{(X)}$ for each set X. Suitably topologized (see Section 3.1), the algebraic dual $(R^{(X)})^*$ turns out to be isomorphic to $(R, \tau)^X$.

More appropriately the above correspondence may be upgraded into a dual equivalence of categories between free and topologically free modules, i.e., those topological modules isomorphic to some $(R, \tau)^X$ (Theorem 45) under the algebraic and topological dual functors. (This extends a similar interpretation from [13] to the realm of arbitrary commutative rigid rings.)

Under the rigidity assumption, the aforementioned dual equivalence enables to provide a topological tensor product $\circledast_{(\mathsf{R},\tau)}$ for topologically free (R, τ)-modules by transporting the algebraic tensor product $\otimes_{\mathsf{R}}$ along the dual equivalence. It turns out that $\circledast_{(\mathsf{R},\tau)}$ is (coherently) associative,

CONTACT Laurent Poinsot laurent.poinsot@lipn.univ-paris13.fr CREA, French Air Force Academy, Base aérienne 701, 13661 Salon-de-Provence, France.

[1]The results of the present contribution also serve in a subsequent paper under preparation about topological semisimplicity of commutative topological algebras.

commutative, and unital, i.e., makes monoidal the category of topologically free modules (Proposition 61). Not too surprisingly the above dual equivalence remains well-behaved, i.e., monoidal, with respect to the (algebraic and topological) tensor products (Theorem 64). According to the theory of monoidal categories, this in turn provides a dual equivalence between monoids in the tensor category of topologically free modules (some suitably generalized topological algebras) and coalgebras with a free underlying module (Corollary 66). So there are two constructions: a *topological dual coalgebra* of a monoid (in the tensor category of topologically free modules) and an *algebraic dual monoid* of a coalgebra, and these constructions are inverse one from the other (up to isomorphism).

There already exists a standard duality theory between algebras and coalgebras, over a field, known as *finite duality* but contrary to our "topological duality" it is merely an adjunction, not an equivalence. One discusses how these dualities interact (see Section 7) and in particular one proves that the algebraic dual monoid of a coalgebra essentially corresponds to its finite dual (Section 7.2), that over a discrete field, the topological dual coalgebra of a monoid is a subcoalgebra of the finite dual coalgebra of its underlying algebra and furthermore that they are equal exactly when finite duality provides an equivalence of categories (Theorem 77).

2. Conventions, notations, and basic definitions

2.1. Conventions and notations

One assumes that the reader is familiar with standard notions and notations from category theory ([12]), and some others will be introduced in the text.

Except as otherwise stipulated, all topologies are Hausdorff, and every ring is assumed unital and commutative. Algebras are only assumed associative and commutative.

For a ring R, R denotes both its underlying set and the canonical left R-module structure on its underlying additive group, and $m_{\mathsf{R}} : R \times R \to R$ is its bilinear multiplication. Likewise if A is an R-algebra, then A is both its underlying set and its underlying R-module. The unit of a ring R (resp., unital algebra A) is denoted by 1_{R} (resp. 1_{A}). A ring map (or morphism of rings) is assumed to preserve the units.

A product of topological spaces always has the product topology unless otherwise stated. When for each $x \in X$, all (E_x, τ_x)'s are equal to the same topological space (E, τ), then the X-fold topological product $\prod_{x \in X}(E_x, \tau_x)$ is canonically identified with the set E^X of all maps from X to E equipped with the topology of simple convergence, and is denoted by $(E, \tau)^X$. Under this identification, the canonical projections $(E, \tau)^X \xrightarrow{\pi_x} (E, \tau)$ are given by $\pi_x(f) = f(x), x \in X, f \in E^X$. The symbol d always represents the discrete topology.

2.2. Basic definitions

Definition 1. Let R be a ring. A (Hausdorff, following our conventions) topology τ of (the carrier set of) the ring is called a ring topology when addition, multiplication, and additive inversion of the ring are continuous. By topological ring (R, τ) is meant a ring together with a ring topology τ on it[2]. By a field with a ring topology, denoted $(\mathbb{k}, \tau)$, is meant a topological ring $(\mathbb{k}, \tau)$ with $\mathbb{k}$ a field.

Let (R, τ) be a topological ring. A pair (M, σ) consisting of a (left and unital[3]) R-module M and a topology σ on M which makes continuous the addition, additive inversion, and scalar multiplication $R \times M \to M$, is called a *topological* (R, τ)*-module*. Such a topology is referred to as a (R, τ)-module topology. In particular, when R is a field $\mathbb{k}$, then this provides *topological*

[2]In view of Section 2.1, the multiplication of a topological ring is jointly continuous.
[3]Unital means that the scalar action of the unit of R is the identity on the module.

$(\mathbb{k}, \tau)$-*vector spaces*. Given topological (R, τ)-modules $(M, \sigma), (N, \gamma)$, a *(continuous) homomorphism* $(M, \sigma) \xrightarrow{f} (N, \gamma)$ is a R-linear map $M \xrightarrow{f} N$ which is continuous. Topological (R, τ)-modules and these morphisms form a category $\mathbf{TopMod}_{(\mathsf{R},\tau)}$, which is denoted $\mathbf{TopVect}_{(\mathbb{k},\tau)}$, when $\mathbb{k}$ is a field.

A pair (A, σ), with A a unital R-algebra, and σ a topology on A, is a *topological* (R, τ)-*algebra*, when σ is a module topology for the underlying R-module A, and the multiplication of A is a bilinear (jointly) continuous map. Given topological (R, τ)-algebras $(\mathsf{A}, \sigma), (\mathsf{B}, \gamma)$, a *continuous* (R, τ)-*algebra map* $(\mathsf{A}, \sigma) \xrightarrow{f} (\mathsf{B}, \gamma)$ is a unit preserving R-algebra map $\mathsf{A} \xrightarrow{f} \mathsf{B}$ which is also continuous. Topological (R, τ)-algebras with these morphisms form a category ${}_1\mathbf{TopAlg}_{(\mathsf{R},\tau)}$. One also has the full subcategory ${}_{1,c}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$ of unital and commutative topological algebras.

2.3. X-fold product and finitely supported maps

Let R be a ring. $\mathbf{Mod}_{\mathsf{R}}$ is the category of unital left-R-modules with R-linear maps. When R is a field $\mathbb{k}$ one uses $\mathbf{Vect}_{\mathbb{k}}$ instead.

Let X be a set. The R-module R^X of all maps from X to R, equivalently defined as the X-fold power of R in the category $\mathbf{Mod}_{\mathsf{R}}$, is the object component of a functor P from the opposite $\mathbf{Set}^{\mathsf{op}}$ of the category of sets to $\mathbf{Mod}_{\mathsf{R}}$, whose action on maps is as follows: given $X \xrightarrow{f} Y$ and $g \in R^Y, P_{\mathsf{R}}(f)(g) = g \circ f$. R^X is merely not just a R-module but, under point-wise multiplication $R^X \times R^X \xrightarrow{M_X} R^X$, a commutative R-algebra, the usual *function algebra* on X, denoted $\mathsf{A}_{\mathsf{R}}(X)$, with unit $1_{\mathsf{A}_{\mathsf{R}}(X)} := \sum_{x\in X} \delta_x^{\mathsf{R}}$, where δ_x^{R}, or simply δ_x, is the member of R^X with $\delta_x^{\mathsf{R}}(x) = 1_{\mathsf{R}}, x \in R$, and for $y \in X, y \neq x, \delta_x^{\mathsf{R}}(y) = 0$. This actually provides a functor $\mathbf{Set}^{\mathsf{op}} \xrightarrow{\mathsf{A}_{\mathsf{R}}} {}_{1,c}\mathbf{Alg}_{\mathsf{R}}$, where ${}_{1,c}\mathbf{Alg}_{\mathsf{R}}$ is the full subcategory spanned by unital and commutative algebras of the category ${}_1\mathbf{Alg}_{\mathsf{R}}$ of (associative) unital R-algebras with unit preserving algebra maps. (The multiplication m_{A} of an algebra A thus is a R-bilinear map $\mathsf{A} \times \mathsf{A} \xrightarrow{m_{\mathsf{A}}} \mathsf{A}$.)

Let $f \in R^X$. The *support* of f is the set $supp(f) := \{ x \in X : f(x) \neq 0 \}$. Let $R^{(X)}$ be the sub-R-module of R^X consisting of all *finitely supported maps* (or maps with *finite support*), i.e., the maps f such that $supp(f)$ is finite.

$R^{(X)}$ is actually the free R-module over X, and a basis is given by $\{ \delta_x^{\mathsf{R}} : x \in X \}$. Observe that the map $X \xrightarrow{\delta_X^{\mathsf{R}}} R^X, x \mapsto \delta_x$, is one-to-one if, and only if, R is nontrivial and $|X|>1$, or $|X| \leq 1$ and R is arbitrary (even trivial).

Remark 2. There is the free module functor $\mathbf{Set} \xrightarrow{F_{\mathsf{R}}} \mathbf{Mod}_{\mathsf{R}}$ which is a left adjoint of the usual forgetful functor $\mathbf{Mod}_{\mathsf{R}} \xrightarrow{|\cdot|} \mathbf{Set}$; $F_R(X) := R^{(X)}$, and for $X \xrightarrow{f} Y, F_{\mathsf{R}}(f)(\delta_x^{\mathsf{R}}) := \delta_{f(x)}^{\mathsf{R}}, x \in X$. The map $X \xrightarrow{\delta_X^{\mathsf{R}}} |R^{(X)}|$ is the component at X of the unit of the adjunction $F_{\mathsf{R}} \dashv |\cdot| : \mathbf{Set} \to \mathbf{Mod}_{\mathsf{R}}$.

Let (R, τ) be a topological ring and let X be a set. Since for a map $X \xrightarrow{f} Y, \pi_x \circ P_{(\mathsf{R},\tau)}(f) = \pi_{f(x)}, x \in X, (R, \tau)^Y \xrightarrow{P_{\mathsf{R}}(f)} (R, \tau)^X$ is continuous, and thus one has a *topological power functor* $\mathbf{Set}^{\mathsf{op}} \xrightarrow{P_{(\mathsf{R},\tau)}} \mathbf{TopMod}_{(\mathsf{R},\tau)}$.

Lemma 3. *$(R, \tau)^X \times (R, \tau)^X \xrightarrow{M_X} (R, \tau)^X$ is continuous.*

Proof. M_X is of course separately continuous in both of its variables. Continuity at zero of m_{R} almost directly implies that of M_X, and thus its continuity by [19, Theorem 2.14, p. 16]. □

$\mathsf{A}_{(\mathsf{R},\tau)} : X \mapsto ((R, \tau)^X, M_X, 1_{\mathsf{A}_{\mathsf{R}}(X)})$ is a functor too and the diagram below commutes, with the forgetful functors unnamed.

$$\begin{array}{ccccc} {}_{1,c}\mathbf{TopAlg}_{(\mathsf{R},\tau)} & & \longrightarrow & & {}_{1,c}\mathbf{Alg}_{\mathsf{R}} \\ \downarrow & \overset{A_{(\mathsf{R},\tau)}}{\nwarrow} & \mathbf{Set}^{\mathsf{op}} & \overset{A_{\mathsf{R}}}{\nearrow} & \downarrow \\ & \underset{P_{(\mathsf{R},\tau)}}{\swarrow} & & \underset{P_{\mathsf{R}}}{\searrow} & \\ \mathbf{TopMod}_{(\mathsf{R},\tau)} & & \longrightarrow & & \mathbf{Mod}_{\mathsf{R}} \end{array} \tag{1}$$

Notation 4. The underlying topological ring of $\mathsf{A}_{(\mathsf{R},\tau)}(X)$ is denoted $(\mathsf{R},\tau)^X$ (and is the X-fold product of (R,τ) in the category of topological rings).

3. Recollection of results about algebraic and topological duals

3.1. Algebraic dual functor

Let R be a ring. Let M be a R-module. Let $M^* := \mathbf{Mod}_{\mathsf{R}}(M,R)$ be the *algebraic* (or *linear*) *dual* of M. This is readily a R-module on its own.

When (R,τ) is a topological ring, then M^* may be topologized with the initial topology ([5, p. 30] or [19, Theorem 2.17, p. 17]) $w^*_{(\mathsf{R},\tau)}$, called the *weak-$*$ topology*, induced by the family $(M^* \overset{\Lambda_M(v)}{\to} R)_{v\in M}$ of *evaluations* at some points, where $(\Lambda_M(v))(\ell) := \ell(v)$. A basis of neighborhoods of zero in this topology is given by the sets of the form $\cap_{v\in F}\Lambda_M(v)^{-1}(\{\ U_v\ \}) = \{\ \ell\in M^* : \forall v\in F, \ell(v)\in U_v\ \}$, where $F\subseteq M$ is a finite set and for each $v\in F, 0\in U_v\in\tau$. This provides a structure of topological (R,τ)-module on M^*, which is even Hausdorff by [5, Corollary 1, p. 78]. As an initial topology it is characterized by the following universal property: a linear map $N\overset{f}{\to}M^*$, where (N,σ) is a topological (R,τ)-module, is continuous from (M,σ) to $(M^*,w^*_{(\mathsf{R},\tau)})$ if, and only if, $\Lambda_M(v)\circ f : (N,\sigma)\to(R,\tau)$ is continuous for each $v\in M$.

Moreover given a linear map $M\overset{f}{\to}N, N^*\overset{f^*}{\to}M^*, \ell\mapsto f^*(\ell) := \ell\circ f$, is continuous for the above topologies. Consequently, this provides a functor $\mathbf{Mod}_{\mathsf{R}}^{\mathsf{op}} \overset{Alg_{(\mathsf{R},\tau)}}{\to} \mathbf{TopMod}_{(\mathsf{R},\tau)}$ called the *algebraic dual functor*.

Remark 5. M^* and f^* stand respectively for $Alg_{(\mathsf{R},\tau)}(M)$ and $Alg_{(\mathsf{R},\tau)}(f)$.

Up to isomorphism, one recovers the module of all R-valued maps on a set X, with its product topology, as the algebraic dual of the module of all finitely supported maps duly topologized as above.

Lemma 6. *For each set X, $(R,\tau)^X \simeq Alg_{(\mathsf{R},\tau)}(R^{(X)})$ (in $\mathbf{TopMod}_{(\mathsf{R},\tau)}$) under the map*

$$\rho_X : (R,\tau)^X \to \left(\left(R^{(X)}\right)^*, w^*_{(\mathsf{R},\tau)}\right)$$

given by

$$\left(\rho_X(f)(p)\right) := \sum_{x\in X} p(x)f(x), f\in R^X, p\in R^{(X)}.$$

Proof. Let $\ell\in(R^{(X)})^*$. Let us define $X\overset{\widehat{\ell}}{\to}\mathsf{R}$ by $\widehat{\ell}(x) := \ell(\delta_x), x\in X$. That the two constructions are linear and inverse one from the other is clear.

It remains to make sure that there are also continuous. Let $\ell\in(R^{(X)})^*$, and let $x\in X$. Then, $\pi_x(\widehat{\ell}) = \widehat{\ell}(x) = \ell(\delta_x) = (\Lambda_{R^{(X)}}(\delta_x))(\ell)$, which ensures continuity of $((R^{(X)})^*, w^*_{(\mathsf{R},\tau)}) \overset{\rho_X^{-1}}{\to} (R,\tau)^X$. Let $f\in R^X$, and $p\in R^{(X)}$. As $(\Lambda_{R^{(X)}}(p))(\rho_X(f)) = (\rho_X(f))(p) = \sum_{x\in X} p(x)f(x) = \sum_{x\in X}\pi_x(p)f(x) =$

$\sum_{x\in X}\pi_x(p)\pi_x(f), \Lambda_{R^{(X)}}(p)\circ\rho_X$ is a finite linear combination of projections, whence is continuous for the product topology, so is ρ_X. □

Let M be a free R-module. Let B be a basis of M. This defines a family of R-linear maps, the *coefficient maps* $(M\xrightarrow{b^*}R)_{b\in\mathsf{B}}$ such that each $v\in M$ is uniquely represented as a finite linear combination $v=\sum_{b\in\mathsf{B}}b^*(v)b$. One denotes $\mathbf{FreeMod}_{\mathsf{R}}$ the full subcategory of $\mathbf{Mod}_{\mathsf{R}}$ spanned by the free modules. When $\mathbb{k}$ is a field, $\mathbf{FreeMod}_{\mathbb{k}}$ is just $\mathbf{Vect}_{\mathbb{k}}$ itself.

Example 7. For each set X, $p_x=\delta_x^*$, where $p_x:=R^{(X)}\hookrightarrow R^X\xrightarrow{\pi_x}R$, $x\in X$.

Remark 8. $b^*(d)=\delta_b(d), b,d\in B$. So $(-)^*:B\to B^*:=\{\ b^*:b\in B\ \}$ is a bijection.

Given a free R-module, any choice of a basis B provides the initial topology Π_B^τ on M^* induced by $(\Lambda_M(b))_{b\in B}$. (Of course, $\Pi_B^\tau\subseteq w^*_{(\mathsf{R},\tau)}$.)

Lemma 9. *Let M be a free R-module. For any basis B of M, $\Pi_B^\tau=w^*_{(\mathsf{R},\tau)}$ and $(M^*,w^*_{(\mathsf{R},\tau)})\simeq(R,\tau)^B$ (in $\mathbf{TopMod}_{(\mathsf{R},\tau)}$).*

Proof. For $\ell\in M^*$, $(\Lambda_M(v))(\ell)=\sum_{b\in B}b^*(v)\ell(b)=\sum_{b\in B}b^*(v)(\Lambda_M(b))(\ell), v\in M$, thus $\Lambda_M(v)$ is a finite linear combination of some $\Lambda_M(b)$'s, whence is continuous for Π_B^τ, and so $w^*_{(\mathsf{R},\tau)}\subseteq\Pi_B^\tau$. The last assertion is clear. □

3.2. Topological dual functor

Let (R,τ) be a topological ring, and let (M,σ) be a topological (R,τ)-module. Let $(M,\sigma)':=\mathbf{TopMod}_{(\mathsf{R},\tau)}((M,\sigma),(R,\tau))$ be the *topological dual* of (M,σ), which is a R-submodule of M^*. Let $(M,\sigma)\xrightarrow{f}(N,\gamma)$ be a continuous homomorphism between topological modules. Let $(N,\gamma)'\xrightarrow{f'}(M,\sigma)'$ be the R-linear map given by $f'(\ell):=\ell\circ f$. All of this evidently forms a functor $\mathbf{TopMod}^{\mathsf{op}}_{(\mathsf{R},\tau)}\xrightarrow{Top_{(\mathsf{R},\tau)}}\mathbf{Mod}_{\mathsf{R}}$.

Let (R,τ) be a topological ring, and let X be a set. Let $R^{(X)}\xrightarrow{\lambda_X}(R^X)^*$ be given by $(\lambda_X(p))(f):=\sum_{x\in X}p(x)f(x), p\in R^{(X)}, f\in R^X$.

Let ρ_X be the map from Lemma 6. Then, for each $p\in R^{(X)}$, $\lambda_X(p)=\Lambda_{R^{(X)}}(p)\circ\rho_X$, which ensures continuity of $\lambda_X(p)$, i.e., $\lambda_X(p)\in((R,\tau)^X)'$. Next lemma follows from the equality $p(x)=(\lambda_X(p))(\delta_x), p\in R^{(X)}, x\in X$.

Lemma 10. *$\lambda_X:R^{(X)}\to((R,\tau)^X)'$ is one-to-one.*

4. Rigid rings: definitions and (counter-)examples

The notion of *rigidity*, recalled at the beginning of the Introduction, was originally but only implicitly introduced in [13, Theorem 5, p. 156] as the main result therein and the possibility that its conclusion could remain valid for more general topological rings than topological division rings was not noticed. Since a large part of this presentation is given for arbitrary rigid rings (Definition 12 below), one here provides a stock of basic examples.

As [13, Lemma 13, p. 158], one has the following fundamental lemma.

Lemma 11. *Let (R,τ) be a topological ring, and let X be a set. For each $f\in R^X$, $(f(x)\delta_x)_{x\in X}$ is summable in $(R,\tau)^X$ with sum f.*

Definition 12. Let (R,τ) be a topological ring. It is said to be rigid when for each set X, $R^{(X)}\xrightarrow{\lambda_X}((R,\tau)^X)'$ is an isomorphism in $\mathbf{Mod}_{\mathsf{R}}$, i.e., λ_X is onto. In this situation, one sometimes also called rigid a ring topology τ such that (R,τ) is rigid.

Lemma 13. *Let $\ell \in ((R,\tau)^X)'$. $\ell \in im(\lambda_X)$ if, and only if, $\widehat{\ell} : X \to R$ given by $\widehat{\ell}(x) := \ell(\delta_x)$, belongs to $R^{(X)}$. Moreover, $(\widehat{-}) : im(\lambda_X) \to R^{(X)}, \ell \mapsto \widehat{\ell} = \sum_{x\in X} \ell(\delta_x)\delta_x$, is the inverse of λ_X.*

4.1. Basic stock of examples

Of course, the trivial ring is rigid (under the (in)discrete topology!). The first result below is a slight generalization of [13, Theorem 5, p. 156] since its proof does not use continuity of the inversion.

Lemma 14. *Let $(\Bbbk,\tau)$ be a field with a ring topology. Then, $(\Bbbk,\tau)$ is rigid.*

Lemma 15. *For each ring R, the discretely topologized ring (R,d) is rigid.*

Proof. Let $\ell \in ((R,\mathrm{d})^X)'$. As a consequence of Lemma 11, $(\ell(\delta_x))_x$ is summable in (R,d), with sum $\ell(1_{\mathsf{A}_R(X)})$. Since $\{\ 0\ \}$ is an open neighborhood of zero in $(R,\mathrm{d}), \ell(\delta_x) = 0$ for all but finitely many $x \in X$ ([19, Theorem 10.5, p. 73]). The conclusion follows by Lemma 13. □

Every normed, complex or real, commutative, and unital algebra (e.g., Banach or C^*-algebra) is rigid.

Lemma 16. *Let $\Bbbk = \mathbb{R}, \mathbb{C}$. Let $(\mathsf{A}, ||\cdot||)$ be a commutative normed $\Bbbk$-algebra[4] with a unit. Then, as a topological ring under the topology induced by the norm, it is rigid.*

Proof. Let $\tau_{||-||}$ be the topology on A induced by the norm of A, where A is the underlying $\Bbbk$-vector space of A. Let X be a set. Let $\ell \in ((A,\tau_{||-||})^X)'$. Let $f \in A^X$ be given by $f(x) = \frac{1}{||\ell(\delta_x)||} 1_{\mathsf{A}}$ if $x \in supp(\widehat{\ell})$ and $f(x) = 0$ for $x \notin supp(\widehat{\ell})$. Since by Lemma 11, $(f(x)\delta_x)_{x\in X}$ is summable with sum f, $(f(x)\ell(\delta_x))_{x\in X}$ is summable in $(\mathsf{A},\tau_{||-||})$ with sum $\ell(f)$. So according to [19, Theorem 10.5, p. 73], for $1>\epsilon>0$, there exists a finite set $F_\epsilon \subseteq X$ such that $||f(x)\ell(\delta_x)||<\epsilon$ for all $x \in X\setminus F_\epsilon$. But $1 = ||f(x)\ell(\delta_x)||$ for all $x \in supp(\widehat{\ell})$ so that $supp(\widehat{\ell})$ is finite, and λ_X is onto by Lemma 13. □

4.2. A supplementary example: von Neumann regular rings

A (commutative and unital) ring is said to be *von Neumann regular* if for each $x \in R$, there exists $y \in R$ such that $x = xyx$ [9, Theorem 4.23, p. 65].

Let us assume that R is a (commutative) von Neumann regular ring. For each $x \in R$, there is a unique $x^\dagger \in R$, called the *weak inverse* of x, such that $x = xx^\dagger x$ and $x^\dagger = x^\dagger x x^\dagger$.

Example 17. A field is von Neumann regular with $x^\dagger := x^{-1}, x \neq 0$, *and* $0^\dagger = 0$. More generally, let $(\Bbbk_i)_{i\in I}$ be a family of fields. Let R be a ring, and let $\jmath\colon \mathsf{R} \to \prod_{i\in I}\Bbbk_i$ be a one-to-one ring map. Assume that for each $x \in R, \jmath(x)^\dagger \in im(\jmath)$, where for $(x_i)_{i\in I} \in \prod_{i\in I}\Bbbk_i, (x_i)^\dagger_{i\in I} := (x_i^\dagger)_{i\in I}$. Then, R is von Neumann regular.

[4] In a normed algebra $(\mathsf{A}, ||\cdot||)$, unital or not, commutative or not, the norm is assumed *sub-multiplicative*, i.e., $||xy|| \leq ||x||||y||$, which ensures that the multiplication of A is jointly continuous with respect to the topology induced by the norm.

Remark 18. Let R be a von Neumann regular ring. For each $x \in R, x \neq 0$ if, and only if, $xx^\dagger \neq 0$. Also $xx^\dagger$ belongs to the set $E(\mathsf{R})$ of all idempotents $(e^2 = e)$ of R.

Proposition 19. *Let* (R, τ) *be a topological ring with* R *von Neumann regular. If* $0 \notin \overline{E(\mathsf{R}) \setminus \{ 0 \}}$, *then* (R, τ) *is rigid. In particular, if* $E(\mathsf{R})$ *is finite, then* (R, τ) *is rigid.*

Proof. That the second assertion follows from the first is immediate. Let X be a set. Let us assume that $0 \notin \overline{E(\mathsf{R}) \setminus \{ 0 \}}$. Let $V \in \mathfrak{V}_{(\mathsf{R},\tau)}(0)$[5] such that $V \cap (E(\mathsf{R}) \setminus \{ 0 \}) = \emptyset$. Let $\ell \in ((R, \tau)^X)'$. Let $f \in R^X$ be given by $f(x) := \ell(\delta_x)^\dagger$ for each $x \in X$. Since $(f(x)\ell(\delta_x))_{x \in X}$ is summable in (R, τ) with sum $\ell(f)$, by Cauchy's condition [19, Definition 10.3, p. 72], there exists a finite set $A_{f,V} \subseteq X$ such that for all $x \notin A_{f,V}, f(x)\ell(\delta_x) \in V$. But for $x \in X, f(x)\ell(\delta_x) = \ell(\delta_x)^\dagger \ell(\delta_x) \in E(\mathsf{R})$. Whence, in view of Remark 18, for all but finitely many x's, $f(x)\ell(\delta_x) = 0$, i.e., $\ell(\delta_x) = 0$. □

Remark 20.

1. Let us point out that a von Neumann regular ring with only finitely many idempotents is classically semisimple so is (by commutativity) a finite product of fields.
2. Lemma 14 becomes a consequence of Proposition 19.

Now, let $(E_i, \tau_i)_{i \in I}$ be a family of topological spaces. On $\prod_{i \in I} E_i$ is defined the *box topology* [8, p. 107] a basis of open sets of which is given by the "box" $\prod_{i \in I} V_i$, where each $V_i \in \tau_i, i \in I$. The product $\prod_{i \in I} E_i$ together with the box topology is denoted by $\sqcap_{i \in I}(E_i, \tau_i)$. (This topology is Hausdorff as soon as all the (E_i, τ_i)'s are.)

It is not difficult to see that given a family $(\mathsf{R}_i, \tau_i)_{i \in I}$ of topological rings, then $\sqcap_{i \in I}(\mathsf{R}_i, \tau_i)$ still is a topological ring (under component wise operations).

Proposition 21. *Let* $(\mathbb{k}_i)_{i \in I}$ *be a family of fields, and for each* $i \in I$, *let* τ_i *be a ring topology on* $\mathbb{k}_i$. *Let* R *be a ring with a one-to-one ring map* $\jmath : \mathsf{R} \hookrightarrow \prod_{i \in I} \mathbb{k}_i$. *Let us assume that for each* $x \in R, \jmath(x)^\dagger \in im(\jmath)$ $((x_i)_i^\dagger$ *as in Example 17). Let* R *be topologized with the subspace topology* $\tau_\jmath$ *inherited from* $\sqcap_{i \in I}(\mathbb{k}_i, \tau_i)$. *Then,* $(\mathsf{R}, \tau_\jmath)$ *is rigid.*

Proof. Naturally $(x_i)_{i \in I} \in E(\prod_{i \in I} \mathbb{k}_i)$ if, and only if, $x_i \in \{ 0, 1_{\mathbb{k}_i} \}$ for each $i \in I$. Now, for each $i \in I$, let U_i be an open neighborhood of zero in $(\mathbb{k}_i, \tau_i)$ such that $1_{\mathbb{k}_i} \notin U_i$. Then, $\prod_i U_i$ is an open neighborhood of zero in $\sqcap_{i \in I}(\mathbb{k}_i, \tau_i)$ whose only idempotent member is 0. Therefore, $0 \notin \overline{E(\prod_{i \in I} \mathbb{k}_i) \setminus \{ 0 \}}$.

Under the assumptions of the statement, an application of Example 17 states that R is a (commutative) von Neumann regular ring. It is also of course a topological ring under $\tau_\jmath$ (since $\jmath$ is a one-to-one ring map). It is also clear that $E(\mathsf{R}) \simeq E(\jmath(\mathsf{R})) \subseteq E(\prod_i \mathbb{k}_i)$. Furthermore, $\jmath(\overline{E(\mathsf{R}) \setminus \{ 0 \}}) = \overline{E(\jmath(\mathsf{R})) \setminus \{ 0 \}} \cap \jmath(\mathsf{R}) \subseteq \overline{E(\prod_i \mathbb{k}_i) \setminus \{ 0 \}}$, and thus $0 \notin \overline{E(\mathsf{R}) \setminus \{ 0 \}}$ according to the above discussion. Therefore, by Proposition 19, $(\mathsf{R}, \tau_\jmath)$ is rigid. □

4.3. A counterexample

Let (R, τ) be a topological ring, and let us consider the topological $(\mathsf{R}, \tau)^X$-module $((R, \tau)^X)^X$ for a given set X. To avoid confusion one denotes by $(R^X)^X \xrightarrow{\Pi_x} R^X$ the canonical projection, $x \in X$.

Let us define a linear map $(R^X)^X \xrightarrow{d} (R, \tau)^X$ by setting $d(f) : x \mapsto (f(x))(x), f \in (R^X)^X$. d is continuous, and thus belongs to $(((R, \tau)^X)^X)'$, since for each $x \in X, \pi_x \circ d = \pi_x \circ \Pi_x$. Now, for each

[5]Given a topological space (E, τ) and $x \in E$, $\mathfrak{V}_{(E,\tau)}(x)$ is the set of all neighborhoods of x.

$x\in X, (d(\delta_x^{\mathsf{R}^X}))(x) = (\delta_x^{\mathsf{R}^X}(x))(x) = 1_{\mathsf{R}^X}(x) = 1_{\mathsf{R}}$, i.e., $d(\delta_x^{R^X}) = \delta_x^{\mathsf{R}}$, so that $supp(\widehat{d}) = X$, when $\mathsf{R}\neq(0)$.

Proposition 22. *Let (R,τ) be a nontrivial topological ring, and let X be a set. If X is infinite, then $(\mathsf{R},\tau)^X$ is not rigid.*

The above negative result may be balanced by the following.

Proposition 23. *Let (R,τ) be a rigid ring. If I is finite, then $(\mathsf{R},\tau)^I$ is rigid too.*

Proof. Let (R,τ) be a topological ring. For a set I, recall from Notation 4 that $(\mathsf{R},\tau)^I$ is the underlying ring of $\mathsf{A}_{(\mathsf{R},\tau)}(I)$. Any topological $(\mathsf{R},\tau)^I$-module is also a topological (R,τ)-module under restriction of scalars along the unit map $(\mathsf{R},\tau)\xrightarrow{\eta_I}(\mathsf{R},\tau)^I, \eta_I(1_{\mathsf{R}}) = 1_{\mathsf{R}^I}$, which of course is a ring map, and is continuous (because $\eta_I(\alpha) = m_{\mathsf{R}^I}(\eta_I(\alpha), 1_{\mathsf{R}^I}), \alpha\in R$.)

Let X be a set, and let $\ell\in(((R,\tau)^I)^X)'$, i.e., $((R,\tau)^I)^X\xrightarrow{\ell}(R,\tau)^I$ is continuous and R^I-linear, and by restriction of scalar along η_I it is also a continuous homomorphism of topological (R,τ)-modules. Therefore for each $i\in I, ((R,\tau)^I)^X\xrightarrow{\ell}(R,\tau)^I\xrightarrow{\pi_i}(R,\tau)$ belongs to the topological dual space of $((R,\tau)^I)^X$ seen as a (R,τ)-module.

Let us assume that (R,τ) is rigid. Then, by Lemma 13, $supp(\widehat{\pi_i\circ\ell})$ is finite for each $i\in I$. One also has $supp(\widehat{\ell}) = \cup_{i\in I} supp(\widehat{\pi_i\circ\ell})$, with $X\xrightarrow{\widehat{\ell}}R^I, \widehat{\ell}(x) := \ell(\delta_x^{\mathsf{R}^I}), x\in X$. Whence if I is finite, then $supp(\widehat{\ell})$ is finite too. □

5. Rigidity as an equivalence of categories

The main result of this section is Theorem 45 which provides a translation of the rigidity condition on a topological ring into a dual equivalence between the category of free modules and that of topologically free modules (see below), provided by the topological dual functor with equivalence inverse the (opposite of the) algebraic dual functor, with both functors conveniently co-restricted. The purpose of this section thus is to prove this result.

Topologically free modules. Let (R,τ) be a topological ring. Let (M,σ) be a topological (R,τ)-module. It is said to be a *topologically free* (R,τ)*-module* if $(M,\sigma)\simeq(R,\tau)^X$, in $\mathbf{TopMod}_{(\mathsf{R},\tau)}$, for some set X. Such topological modules span the full subcategory $\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$ of $\mathbf{TopMod}_{(\mathsf{R},\tau)}$. For a field $(\mathbb{k},\tau)$ with a ring topology, one defines correspondingly the category $\mathbf{TopFreeVect}_{(\mathbb{k},\tau)}$ of *topologically free* $(\mathbb{k},\tau)$*-vector spaces.*

Remark 24. The topological power functor $\mathbf{Set}^{\mathsf{op}}\xrightarrow{P_{(\mathsf{R},\tau)}}\mathbf{TopMod}_{(\mathsf{R},\tau)}$ factors as indicated below (the co-restriction obtained is still called $P_{(\mathsf{R},\tau)}$).

$$\begin{array}{ccc}\mathbf{Set}^{\mathsf{op}} & \xrightarrow{P_{(\mathsf{R},\tau)}} & \mathbf{TopMod}_{(\mathsf{R},\tau)} \\ & \searrow^{P_{(\mathsf{R},\tau)}} & \uparrow \\ & & \mathbf{TopFreeMod}_{(\mathsf{R},\tau)}\end{array} \quad (2)$$

Topologically free modules are characterized by having "topological bases" (see Corollary 28 below) which makes easier a number of calculations and proofs, once such a basis is chosen.

Definition 25. Let (M,σ) be a topological (R,τ)-module. Let $B\subseteq M$. It is said to be a topological basis of (M,σ) if the following hold.

1. For each $v \in M$, there exists a unique family $(b'(v))_{b\in B}$, with $b'(v) \in R$ for each $b \in B$, such that $(b'(v)b)_{b\in B}$ is summable in (M,σ) with sum v. $b'(v)$ is referred to as the coefficient of v at $b \in B$.
2. For each family $(\alpha_b)_{b\in B}$ of elements of R, there is a member v of M such that $b'(v) = \alpha_b, b \in B$. (By the above point such v is unique.)
3. σ is equal to the initial topology induced by the (topological) coefficient maps $(M \xrightarrow{b'} (R,\tau))_{b\in B}$. (According to the two above points, each b' is R-linear.)

Remark 26. It is an immediate consequence of the definition that for a topological basis B of some topological module, $0 \notin B$ and $b'(d) = \delta_b(d), b, d \in B$ (since $\sum_{b\in B}\delta_b(d)b = d = \sum_{b\in B} b'(d)b$). In particular, $(-)' : B \to B' := \{\ b' : b \in B\ \}$ is a bijection.

Lemma 27. *Let (M,σ) and (N,γ) be isomorphic topological (R,τ)-modules. Let $\Theta : (M,\sigma) \simeq (N,\gamma)$ be an isomorphism (in $\mathbf{TopMod}_{(\mathsf{R},\tau)}$). Let B be a topological basis of (M,σ). Then, $\Theta(B) = \{\ \Theta(b) : b \in B\ \}$ is a topological basis of (N,γ).*

Corollary 28. *Let (M,σ) be a (Hausdorff) topological (R,τ)-module. It admits a topological basis if, and only if, it is topologically free.*

Example 29. Let (R,τ) be a topological ring. For each set X, $\{\ \delta_x : x \in X\ \}$ is a topological basis of $(R,\tau)^X$. Moreover $\pi_x = \delta'_x, x \in X$.

Let us now take the time to establish a certain number of quite useful properties of topological bases.

Lemma 30. *Let (M,σ) be a topologically free (R,τ)-module with topological basis B. Then, B is R-linearly independent and the linear span $\langle B\rangle$ of B is dense in (M,σ).*

Proof. Concerning the assertion of independence, it suffices to note that 0 may be written as $\sum_{b\in B} 0b$, and conclude by the uniqueness of the decomposition in a topological basis. Let $u \in M$ and let $V := \{\ v \in M : b'(v) \in U_b, b \in A\ \} \in \mathfrak{V}_{(M,\sigma)}(0)$, where A is a finite subset of B and $U_b \in \mathfrak{V}_{(\mathsf{R},\tau)}(0), b \in A$. Let $\alpha_b \in U_b, b \in A$, and $v := \sum_{b\in A}\alpha_b b - \sum_{b\in B\setminus A} b'(u)b \in V$. So $u + v \in \langle B\rangle$. Thus, $u + V$ meets $\langle B\rangle$ and $\langle B\rangle$ is dense in (M,σ). □

Corollary 31. *Let (M,σ) be a topologically free (R,τ)-module, and let (N,γ) be a topological (R,τ)-module. Let $(M,\sigma) \overset{f,g}{\rightarrow} (N,\gamma)$ be two continuous homomorphisms of topological (R,τ)-modules. $f = g$ if, and only if, for any topological basis B of (M,σ), $f(b) = g(b)$ for each $b \in B$.*

Topologically free modules allow for the definition of changes of topological bases (see Proposition 48 for a related construction).

Lemma 32. *Let (M,σ) and (N,γ) be topologically free (R,τ)-modules, with respective topological bases B, D. Let $f : B \to D$ be a bijection. Then, there is a unique isomorphism g in $\mathbf{TopMod}_{(\mathsf{R},\tau)}$ such that $g(b) = f(b), b \in B$.*

Proof. The question of uniqueness is settled by Corollary 31, and an isomorphism is given by $g(v) = \sum_{d\in D}(f^{-1}(d))'(v)d, v \in M$. □

Lemma 33. *Let M be a free module with basis B. Then, $(M^*, w^*_{(\mathsf{R},\tau)})$ is a topologically free module with topological basis $B^* := \{\ b^* : b \in B\ \}$.*

Proof. According to Lemma 9, $(M^*, w^*_{(\mathsf{R},\tau)})$ is a topologically free module. Let $\theta_B : M \to R^{(B)}$ be the isomorphism given by $\theta_B(b) = \delta_b, b \in B$. Thus, $\theta^*_B : (R^{(B)})^* \simeq M^*$, and $\theta^*_B \circ \rho_B : R^B \simeq$

$(R^{(B)})^* \simeq M^*$ is given by $\theta_B^*(\rho_B(\delta_b^{\mathsf{R}})) = \rho_B(\delta_b^{\mathsf{R}}) \circ \theta_B = p_b \circ \theta_B = b^*$ for $b \in B$ (see Example 7 for the definition of ρ_B). Now, $\{ \delta_b : b \in B \}$ being a topological basis of R^B, by Lemma 27 this shows that B^* is a topological basis of $(M^*, w^*_{(\mathsf{R},\tau)})$. □

Example 34. $\{ p_x : x \in X \}$ is a topological basis of $(R^{(X)})^*$ (Example 7).

Remark 35. If B is a basis of a free module M, then $B \simeq B^*$ under $b \mapsto b^*$, because for each $b, d \in B, b^*(d) = \delta_b(d)$.

Corollary 36. *Let (R, τ) be a topological ring. The functor $Alg_{(\mathsf{R},\tau)} : \mathbf{Mod}_{\mathsf{R}}^{\mathsf{op}} \to \mathbf{TopMod}_{(\mathsf{R},\tau)}$ factors as illustrated in the diagram below.*[6] *Moreover the resulting co-restriction of $Alg_{(\mathsf{R},\tau)}$ (the bottom arrow of the diagram) is essentially surjective on objects.*

$$\begin{array}{ccc} \mathbf{Mod}_{\mathsf{R}}^{\mathsf{op}} & \xrightarrow{Alg_{(\mathsf{R},\tau)}} & \mathbf{TopMod}_{(\mathsf{R},\tau)} \\ \uparrow & & \uparrow \\ \mathbf{FreeMod}_{\mathsf{R}}^{\mathsf{op}} & \longrightarrow & \mathbf{TopFreeMod}_{(\mathsf{R},\tau)} \end{array} \tag{4}$$

Proof. The first assertion is merely Lemma 33. Regarding the second assertion, let (M, σ) be a topologically free module. So, for some set X, $(M, \sigma) \simeq (R, \tau)^X$. By Lemma 6, $(R, \tau)^X \simeq Alg_{(\mathsf{R},\tau)}(R^{(X)})$. □

Lemma 37. *Let (R, τ) be a rigid ring. Let (M, σ) be a topologically free (R, τ)-module with topological basis B. Then, $(M, \sigma)'$ is free with basis $B' := \{ b' : b \in B \}$.*

Proof. Let $\Theta_B : (M, \sigma) \simeq (R, \tau)^B$ be given by $\Theta_B(b) = \delta_b$. Thus $\Theta_B' : ((R, \tau)^B)' \simeq (M, \sigma)'$, and thus one has an isomorphism $\Theta_B' \circ \lambda_B : R^{(B)} \simeq (M, \sigma)'$. Since a module isomorphic to a free module is free, $(M, \sigma)'$ is free. The previous isomorphism acts as: $\Theta_B'(\lambda_B(\delta_b)) = \pi_b \circ \Theta_B = b'$ for $b \in B$. It follows from Lemma 27 that B' is a basis of $(M, \sigma)'$. □

Example 38. Let (R, τ) be a rigid ring. Let $(M, \sigma) = (R, \tau)^X$. By Example 29, $\{ \delta_x' : x \in X \} = \{ \pi_x : x \in X \}$ is a linear basis of $((R, \tau)^X)'$.

Corollary 39. *Let (R, τ) be a rigid ring. The functor $\mathbf{TopMod}^{\mathsf{op}}_{(\mathsf{R},\tau)} \xrightarrow{Top_{(\mathsf{R},\tau)}} \mathbf{Mod}_{\mathsf{R}}$ factors*[7] *as indicated by the diagram below.*

[6]When $\Bbbk$ is a field with a ring topology τ, then one has the corresponding factorization of $Alg_{(\Bbbk,\tau)} : \mathbf{Vect}_{\Bbbk}^{\mathsf{op}} \to \mathbf{TopVect}_{(\Bbbk,\tau)}$.

$$\begin{array}{ccc} \mathbf{Vect}_{\Bbbk}^{\mathsf{op}} & \xrightarrow{Alg_{(\Bbbk,\tau)}} & \mathbf{TopVect}_{(\Bbbk,\tau)} \\ & \searrow & \uparrow \\ & & \mathbf{TopFreeVect}_{(\Bbbk,\tau)} \end{array} \tag{3}$$

[7]Correspondingly for a field $(\Bbbk, \tau)$ with a ring topology,

$$\begin{array}{ccc} \mathbf{TopVect}^{\mathsf{op}}_{(\Bbbk,\tau)} & \xrightarrow{Top_{(\Bbbk,\tau)}} & \mathbf{Vect}_{\Bbbk} \\ \uparrow & \nearrow & \\ \mathbf{TopFreeVect}^{\mathsf{op}}_{(\Bbbk,\tau)} & & \end{array} \tag{6}$$

$$\begin{array}{ccc} \mathbf{TopMod}^{op}_{(\mathsf{R},\tau)} & \xrightarrow{Top_{(\mathsf{R},\tau)}} & \mathbf{Mod}_{\mathsf{R}} \\ \uparrow & & \uparrow \\ \mathbf{TopFreeMod}^{op}_{(\mathsf{R},\tau)} & \longrightarrow & \mathbf{FreeMod}_{\mathsf{R}} \end{array} \tag{6}$$

The topological dual of the algebraic dual of a free module. Let (R,τ) be a topological ring. Let M be a R-module, and let us consider as in Section 3.1, the R-linear map $M\xrightarrow{\Lambda_M}(M^*, w^*_{(\mathsf{R},\tau)})'$ $(\Lambda_M(v))(\ell) = \ell(v), v \in M, \ell \in M^*$.

Lemma 40. *Let M be a projective R-module. Then, Λ_M is one-to-one. This holds in particular when M is a free R-module.*

Proof. Let us consider a dual basis for M, i.e., sets $B \subseteq M$ and $\{\ \ell_e : e \in B\ \} \subseteq M^*$, such that for all $v \in M, \ell_e(v) = 0$ for all but finitely many $\ell_e \in B^*$ and $v = \sum_{e\in B} \ell_e(v)e$ ([9, p. 23]). Let $v \in \ker\Lambda_M$, i.e., $(\Lambda_M(v))(\ell) = \ell(v) = 0$ for each $\ell \in M^*$. Then, in particular, $\Lambda_M(v)(\ell_e) = \ell_e(v) = 0$ for all $e \in B$, and thus $v = 0$. □

Let (R,τ) (resp. $(\mathbb{k},\tau)$) be a rigid ring (resp. field). The functors provided by Corollaries 36 and 39 are still denoted by $Alg_{(\mathsf{R},\tau)} : \mathbf{FreeMod}^{op}_{\mathsf{R}} \to \mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$ (resp. $Alg_{(\mathbb{k},\tau)} : \mathbf{Vect}^{op}_{\mathbb{k}} \to \mathbf{TopFreeVect}_{(\mathbb{k},\tau)}$) and by $Top_{(\mathsf{R},\tau)} : \mathbf{TopFreeMod}^{op}_{(\mathsf{R},\tau)} \to \mathbf{FreeMod}_{\mathsf{R}}$ (resp. $Top_{(\mathbb{k},\tau)} : \mathbf{TopFreeVect}^{op}_{(\mathbb{k},\tau)} \to \mathbf{Vect}_{\mathbb{k}}$).

Proposition 41. *Let us assume that (R,τ) is rigid. $\Lambda := (\Lambda_M)_M : id \Rightarrow Top_{(\mathsf{R},\tau)} \circ Alg^{op}_{(\mathsf{R},\tau)} : \mathbf{FreeMod}_{\mathsf{R}} \to \mathbf{FreeMod}_{\mathsf{R}}$ is a natural isomorphism.*

Proof. Naturality is clear. Let (R,τ) be a topological ring. Let M be a free R-module. For each free basis X of M, the following diagram commutes in $\mathbf{Mod}_{\mathsf{R}}$, where $M\xrightarrow{\theta_X}R^{(X)}$ is the canonical isomorphism given by $\theta_X(x) = \delta^{\mathsf{R}}_x, x \in X$. Consequently, when (R,τ) is rigid Λ_M is an isomorphism.

$$\begin{array}{ccccc} M & \xrightarrow{\Lambda_M} & (M^*, w^*_{(\mathsf{R},\tau)})' & & \\ \theta_X \Big\| & & & \overset{(\theta^*_X)'}{\simeq} & ((R^{(X)})^*, w^*_{(\mathsf{R},\tau)})' \\ R^{(X)} & \xrightarrow[\lambda_X]{} & ((R,\tau)^X)' & \overset{\rho'_X}{\simeq} & \end{array} \tag{7}$$

□

Corollary 42. *Let us assume that $(\mathbb{k},\tau)$ is a field with a ring topology. Then, $\Lambda = (\Lambda_M)_M : id \Rightarrow Top_{(\mathbb{k},\tau)} \circ Alg^{op}_{(\mathbb{k},\tau)} : \mathbf{Vect}_{\mathbb{k}} \to \mathbf{Vect}_{\mathbb{k}}$ is a natural isomorphism.*

The algebraic dual of the topological dual of a topologically-free module. Let (M,σ) be a topological (R,τ)-module. Let us consider the R-linear map $\Gamma_{(M,\sigma)} : M \to ((M,\sigma)\prime)^*$ by setting $(\Gamma_{(M,\sigma)}(v))(\ell) := \ell(v)$.

Proposition 43. *Let us assume that (R,τ) is a rigid ring. Then, $\Gamma : id \Rightarrow Alg_{(\mathsf{R},\tau)} \circ Top^{op}_{(\mathsf{R},\tau)} : \mathbf{TopFreeMod}_{(\mathsf{R},\tau)} \to \mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$ is a natural isomorphism, with $\Gamma := (\Gamma_{(M,\sigma)})_{(M,\sigma)}$.*

Proof. Naturality is clear. Let $\Theta : (M,\sigma) \simeq (R,\tau)^X$ be an isomorphism (in $\mathbf{TopMod}_{(\mathsf{R},\tau)}$). Since (R,τ) is rigid, $\lambda_X : R^{(X)} \simeq ((R,\tau)^X)'$ is an isomorphism. Therefore $R^{(X)}\xrightarrow{\lambda_X}((R,\tau)^X)'\xrightarrow{\Theta'}(M,\sigma)'$ is an isomorphism too in $\mathbf{Mod}_{\mathsf{R}}$. In particular, $(M,\sigma)'$ is free with basis $\{\ \Theta'(\lambda_X(\delta^{\mathsf{R}}_x)) : x \in X\ \}$. By

Lemma 9, the weak-$*$ topology on $((M,\sigma)\prime)^*$ is the same as the initial topology given by the maps $((M,\sigma)\prime)^* \overset{\Lambda_{(M,\sigma)'}(\pi_x\circ\Theta)}{\to} (R,\tau), x \in X$, because $\Theta'(\lambda_X(\delta_x^{\mathsf{R}})) = \Theta'(\pi_x) = \pi_x \circ \Theta$. Therefore, $\Gamma_{(M,\sigma)}$ is continuous if, and only if, for each $x \in X, \Lambda_{(M,\sigma)'}(\pi_x \circ \Theta) \circ \Gamma_{(M,\sigma)} = \pi_x \circ \Theta$ is continuous. Continuity of $\Gamma_{(M,\sigma)}$ thus is proved.

That $\Gamma_{(M,\sigma)}$ is an isomorphism in $\mathbf{TopMod}_{(\mathsf{R},\tau)}$ follows from the commutativity of the diagram (in $\mathbf{TopMod}_{(\mathsf{R},\tau)}$) below.

$$\begin{array}{ccccc} & & (M,\sigma) & \xrightarrow{\Gamma_{(M,\sigma)}} & (((M,\sigma)')^*, w^*_{(\mathsf{R},\tau)}) \\ & \overset{\Theta}{\swarrow} & & & \uparrow ((\Theta')^*)^{-1} \\ (R,\tau)^X & & & & \\ & \underset{\rho_X}{\searrow} & & & \\ & & ((R^{(X)})^*, w^*_{(\mathsf{R},\tau)}) & \xrightarrow[(\lambda_X^{-1})^*]{} & ((((R,\tau)^X)')^*, w^*_{(\mathsf{R},\tau)}) \end{array} \qquad (8)$$

□

Corollary 44. *Let us assume that* $(\mathbb{k},\tau)$ *is a field with a ring topology.* $\Gamma : id \Rightarrow Alg_{(\mathbb{k},\tau)} \circ Top^{\mathsf{op}}_{(\mathbb{k},\tau)} : \mathbf{TopFreeVect}_{(\mathbb{k},\tau)} \to \mathbf{TopFreeVect}_{(\mathbb{k},\tau)}$ *with* $\Gamma := (\Gamma_{(M,\sigma)})_{(M,\sigma)}$, *is a natural isomorphism.*

The equivalence and some of its immediate consequences. Collecting Propositions 41 and 43, one immediately gets the following.

Theorem 45. *Let us assume that* (R,τ) *is rigid.* $Top_{(\mathsf{R},\tau)} : \mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)} \to \mathbf{FreeMod}_{\mathsf{R}}$ *is an equivalence of categories, with equivalence inverse the functor* $Alg^{\mathsf{op}}_{(\mathsf{R},\tau)} : \mathbf{FreeMod}_{\mathsf{R}} \to \mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)}$.

Corollary 46. $Top_{(\mathbb{k},\tau)} : \mathbf{TopFreeVect}^{\mathsf{op}}_{(\mathbb{k},\tau)} \to \mathbf{Vect}_{\mathbb{k}}$ *is an equivalence of categories, and* $Alg^{\mathsf{op}}_{(\mathbb{k},\tau)} : \mathbf{Vect}_{\mathbb{k}} \to \mathbf{TopFreeVect}^{\mathsf{op}}_{(\mathbb{k},\tau)}$ *is its equivalence inverse, whenever* $(\mathbb{k},\tau)$ *is a field with a ring topology.*

Finite-dimensional vector spaces. Let $\mathbb{k}$ be a field. Let (M,σ) be a topologically free $(\mathbb{k},\mathsf{d})$-vector space with M finite dimensional. Then, σ is the discrete topology d on M. It follows that $(M,\sigma)' = M^*$, and the equivalence established in Corollary 46 coincides with the classical dual equivalence $\mathbf{FinDimVect}_{\mathbb{k}} \simeq \mathbf{FinDimVect}^{\mathsf{op}}_{\mathbb{k}}$ under the algebraic dual functor, where $\mathbf{FinDimVect}_{\mathbb{k}}$ is the category of finite dimensional $\mathbb{k}$-vector spaces.
Linearly compact vector spaces. Let $\mathbb{k}$ be a field. A topological $(\mathbb{k},\mathsf{d})$-vector space (M,σ) is said to be a *linearly compact* $\mathbb{k}$-vector space when $(M,\sigma) \simeq (\mathbb{k},\mathsf{d})^X$ for some set X (see [4, Proposition 24.4, p. 105]). The full subcategory $\mathbf{LCpVect}_{\mathbb{k}}$ of $\mathbf{TopVect}_{(\mathbb{k},\mathsf{d})}$ spanned by these spaces is equal to $\mathbf{TopFreeVect}_{(\mathbb{k},\mathsf{d})}$.

Corollary 47. (of Theorem 45) *Let* R *be a ring. For each rigid topologies* τ,σ *on* R, *the categories* $\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$ *and* $\mathbf{TopFreeMod}_{(\mathsf{R},\sigma)}$ *are equivalent. Moreover, for each field* $(\mathbb{k},\tau)$ *with a ring topology,* $\mathbf{TopFreeVect}_{(\mathbb{k},\tau)}$ *is equivalent to* $\mathbf{LCpVect}_{\mathbb{k}}$.

In particular, one recovers the result from [7] that $\mathbf{Vect}^{\mathsf{op}}_{\mathbb{k}} \simeq \mathbf{LCpVect}_{\mathbb{k}}$.
The universal property of $(R,\tau)^X$. For a ring R, the functor $|\cdot| : \mathbf{Mod}_{\mathsf{R}} \to \mathbf{Set}$ (see Remark 2) may be restricted as indicated in the following commutative diagram, and the restriction still is denoted $|\cdot| : \mathbf{FreeMod}_{\mathsf{R}} \to \mathbf{Set}$.

$$\begin{array}{ccc} \mathbf{Mod}_{\mathsf{R}} & \xrightarrow{|\cdot|} & \mathbf{Set} \\ \uparrow & \nearrow & \\ \mathbf{FreeMod}_{\mathsf{R}} & & \end{array} \qquad (9)$$

Likewise $F_{\mathsf{R}} : \mathbf{Set} \to \mathbf{Mod}_{\mathsf{R}}$ (see again Remark 2) may be co-restricted as indicated by the commutative diagram below, and the co-restriction is given the same name $F_{\mathsf{R}} : \mathbf{Set} \to \mathbf{FreeMod}_{\mathsf{R}}$.

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{F_{\mathsf{R}}} & \mathbf{Mod}_{\mathsf{R}} \\ & \searrow & \uparrow \\ & & \mathbf{FreeMod}_{\mathsf{R}} \end{array} \tag{10}$$

(When R is a field $\mathbb{k}$, there is no need to consider the corresponding co-restrictions.)

The adjunction $F_{\mathsf{R}} \dashv |\cdot| : \mathbf{Set} \to \mathbf{Mod}_{\mathsf{R}}$ gives rise to a new one $F_{\mathsf{R}} \dashv |\cdot| : \mathbf{Set} \to \mathbf{FreeMod}_{\mathsf{R}}$ [12, p. 147], and by composition, for each rigid ring (R, τ), there is also the adjunction $Alg^{\mathrm{op}}_{(\mathsf{R},\tau)} \circ F_{\mathsf{R}} \dashv |\cdot| \circ Top_{(\mathsf{R},\tau)} : \mathbf{Set} \to \mathbf{TopFreeMod}^{\mathrm{op}}_{(\mathsf{R},\tau)}$. Since $(R,\tau)^X \simeq ((R^{(X)})^*, w^*_{(\mathsf{R},\tau)})$ (Lemma 6), this may be translated into a *universal property* of $(R,\tau)^X$, as explained below, which somehow legitimates the terminology *topologically free*.

Proposition 48. *Let us assume that (R,τ) is rigid. Let X be a set. For each topologically free module (M,σ) and any map $f : X \to |(M,\sigma)'|$, there is a unique continuous homomorphism $f^\sharp : (M,\sigma) \to (R,\tau)^X$ such that $|(f^\sharp)'| \circ |\lambda_X| \circ \delta^{\mathsf{R}}_X = f$ (recall that $\delta^{\mathsf{R}}_X(x) = \delta^{\mathsf{R}}_x, x \in X$).*

Proof. There is a unique R-linear map $\widetilde{f} : R^{(X)} \to (M,\sigma)'$ such that $|\widetilde{f}| \circ \delta^{\mathsf{R}}_X = f$. Let us define the continuous linear map $(M,\sigma) \xrightarrow{f^\sharp} (R,\tau)^X := (M,\sigma) \xrightarrow{\Gamma_{(M,\sigma)}} (((M,\sigma)\prime)^*, w^*_{(\mathsf{R},\tau)}) \xrightarrow{(\widetilde{f})^*} ((R^{(X)})^*, w^*_{(\mathsf{R},\tau)}) \xrightarrow{\rho_X^{-1}} (R,\tau)^X$. One has

$$\begin{aligned}
|(f^\sharp)'| \circ |\lambda_X| \circ \delta^{\mathsf{R}}_X &= |\Gamma'_{(M,\sigma)}| \circ \left|\left(\left(\widetilde{f}\right)^*\right)'\right| \circ |(\rho_X^{-1})'| \circ |\lambda_X| \circ \delta^{\mathsf{R}}_X \\
&= |\Gamma'_{(M,\sigma)}| \circ \left|\left(\left(\widetilde{f}\right)^*\right)'\right| \circ |\Lambda_{R^{(X)}}| \circ \delta^{\mathsf{R}}_X \\
&\quad \left(\text{because } (\rho_X^{-1})' \circ \lambda_X = \Lambda_{R^{(X)}}\right) \\
&= |\Gamma'_{(M,\sigma)}| \circ |\Lambda_{(M,\sigma)'}| \circ |\widetilde{f}| \circ \delta^{\mathsf{R}}_X \\
&\quad (\text{by naturality of } \Lambda) \\
&= |\widetilde{f}| \circ \delta^{\mathsf{R}}_X \\
&\quad \left(\text{triangular identities for an adjunction } [12, \text{p. } 85]\right) \\
&= f.
\end{aligned} \tag{11}$$

It remains to check uniqueness of $f^\sharp$. Let $(M,\sigma) \xrightarrow{g} (R,\tau)^X$ be a continuous linear map such that $|g'| \circ |\lambda_X| \circ \delta^{\mathsf{R}}_X = f$. Then, $g' \circ \lambda_X = \widetilde{f}$. Thus, $\lambda_X^* \circ (g\prime)^* = \widetilde{f}^* = \rho_X \circ f^\sharp \circ \Gamma^{-1}_{(M,\sigma)}$. So $\rho_X \circ \Gamma^{-1}_{(R,\tau)^X} \circ (g\prime)^* = \rho_X \circ f^\sharp \circ \Gamma^{-1}_{(M,\sigma)}$ because $\Gamma_{(R,\tau)^X} = (\lambda_X^{-1})^* \circ \rho_X$ (by direct inspection), and thus $\Gamma^{-1}_{(R,\tau)^X} \circ (g\prime)^* = f^\sharp \circ \Gamma^{-1}_{(M,\sigma)}$. Then, by naturality of $\Gamma^{-1}, g \circ \Gamma^{-1}_{(M,\sigma)} = f^\sharp \circ \Gamma^{-1}_{(M,\sigma)}$. □

Corollary 49. *Let (R,τ) be a rigid ring. $P^{\mathrm{op}}_{(\mathsf{R},\tau)} : \mathbf{Set} \to \mathbf{TopFreeMod}^{\mathrm{op}}_{(R,\tau)}$ is a left adjoint of $\mathbf{TopFreeMod}^{\mathrm{op}}_{(R,\tau)} \xrightarrow{Top_{(\mathsf{R},\tau)}} \mathbf{FreeMod}_R \xrightarrow{|-|} \mathbf{Set}$, and thus is naturally equivalent to $\mathbf{Set} \xrightarrow{Alg^{\mathrm{op}}_{(\mathsf{R},\tau)} \circ F_{\mathsf{R}}} \mathbf{TopFreeMod}^{\mathrm{op}}_{(R,\tau)}$.*

Proof. A quick calculation shows that $P_{(\mathsf{R},\tau)}(f) = (|\lambda_Y| \circ \delta^{\mathsf{R}}_Y \circ f)^\sharp$ for a set theoretic map $f : X \to Y$. The relation $f \mapsto (\lambda_Y \circ \delta^{\mathsf{R}}_Y \circ f)^\sharp$ provides a functor from $\mathbf{Set}^{\mathrm{op}}$ to $\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$

whose opposite is, by construction, a left adjoint of $|-|\circ Top_{(\mathsf{R},\tau)}$ (this is basically the content of Proposition 48). □

6. Tensor product of topologically free modules

In this section most of the ingredients so far introduced and developed fit together to lift the equivalence $\mathbf{FreeMod}\simeq\mathbf{TopFreeMod}^{\mathrm{op}}_{(\mathsf{R},\tau)}$ to a duality between some (suitably generalized) topological algebras and some coalgebras.

6.1. Monoidal categories, monoidal functors, and (co-)monoids

Most of the notations and notions from the theory of monoidal categories needed hereafter are taken from [14, Section 2, pp. 4874–4876] and only a few more are introduced below, since they are indispensable.

Monoidal categories. Recall that each natural transformation $\alpha : F\Rightarrow G:\mathbf{C}\to\mathbf{D}$ has an *opposite natural transformation* $\alpha^{\mathrm{op}}: G^{\mathrm{op}}\Rightarrow F^{\mathrm{op}}:\mathbf{C}^{\mathrm{op}}\to\mathbf{D}^{\mathrm{op}}$ with $(\alpha^{\mathrm{op}})_C=(\alpha_C)^{\mathrm{op}}$ for each $\mathbf{C}$-object C, where for $f\in\mathbf{C}(C,D), f^{\mathrm{op}}\in\mathbf{C}^{\mathrm{op}}(D,C)=\mathbf{C}(C,D)$ is the corresponding $\mathbf{C}^{\mathrm{op}}$-morphism. Recall also that for every categories $\mathbf{C},\mathbf{D}$, $(\mathbf{C}\times\mathbf{D})^{\mathrm{op}}=\mathbf{C}^{\mathrm{op}}\times\mathbf{D}^{\mathrm{op}}$.

If $\mathbb{C}=(\mathbf{C},-\otimes-,I,\alpha,\lambda,\rho)=(\mathbf{C},-\otimes-,I)$ is a (symmetric) monoidal category, then so is its *dual* $\mathbb{C}^{\mathrm{op}}:=(\mathbf{C}^{\mathrm{op}},-\otimes^{\mathrm{op}}-,I,(\alpha^{-1})^{\mathrm{op}},(\varrho^{-1})^{\mathrm{op}},(\lambda^{-1})^{\mathrm{op}})$. (In [14] it is denoted by $\mathbb{C}^{\mathrm{op}}$ instead of $\mathbb{C}^{\mathrm{op}}$.)

Example 50. Let R be a ring. For each R-modules M, N, $M\otimes_{\mathsf{R}}N$ stands for their (algebraic) tensor product, and $\otimes: M\times N\to M\otimes_{\mathsf{R}}N$ is the universal R-bilinear map. $\mathbb{M}\mathrm{od}_{\mathsf{R}}:=(\mathbf{Mod}_{\mathsf{R}},\otimes_{\mathsf{R}},R)$, with the ordinary coherence constraints of associativity, of left and right units and of symmetry, is a symmetric monoidal category ([18, Example 11.2, p. 70]).

1. $\mathbf{Mon}(\mathbb{M}\mathrm{od}_{\mathsf{R}})$ is isomorphic to the category ${}_1\mathbf{Alg}_{\mathsf{R}}$ of "ordinary" unital R-algebras under the functor O, concrete over $\mathbf{Mod}_{\mathsf{R}}$, such that $O(\mathsf{A}):=(A,m_{\mathsf{A}},1_{\mathsf{A}})$, with $m_{\mathsf{A}}(x,y):=\mu(x\otimes y), x,y\in A$, and $1_{\mathsf{A}}:=\eta(1_{\mathsf{R}})$, where $\mathsf{A}=(A,\mu_{\mathsf{A}},\eta_{\mathsf{A}})$ is a monoid in $\mathbb{M}\mathrm{od}_{\mathsf{R}}$.
2. Likewise ${}_c\mathbf{Mon}(\mathbb{M}\mathrm{od}_{\mathsf{R}})\simeq{}_{1,c}\mathbf{Alg}_{\mathsf{R}}$ under the (co-)restriction of the above functor O.
3. $\mathbf{Comon}(\mathbb{M}\mathrm{od}_{\mathsf{R}})$ is the category ${}_\epsilon\mathbf{Coalg}_{\mathsf{R}}$ of counital R-coalgebras ([1,6]), and the category of cocommutative coalgebras ${}_{\epsilon,coc}\mathbf{Coalg}_{\mathsf{R}}$ is ${}_{coc}\mathbf{Comon}(\mathbb{M}\mathrm{od}_{\mathsf{R}})$.

By a *(symmetric) monoidal subcategory* of a (symmetric) monoidal category $\mathbb{C}=(\mathbf{C},-\otimes-,I)$ we mean a subcategory $\mathbf{C}'$ of $\mathbf{C}$, closed under tensor products, containing I, and the coherence constraints of $\mathbb{C}$ between $\mathbf{C}'$-objects. (The last condition is automatically fulfilled when $\mathbf{C}'$ is a full subcategory.) The embedding $E_{\mathbf{C}'}$ of $\mathbf{C}'$ into $\mathbf{C}$ then is a strict monoidal functor $\mathbb{E}_{\mathbb{C}'}$ (see e.g., [14, Definition 2, p. 4876]).

For instance, since given free modules M, N over a ring $\mathsf{R}, M\otimes_{\mathsf{R}}N$ is free too, $\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{\mathsf{R}}=(\mathbf{FreeMod}_{\mathsf{R}},\otimes_{\mathsf{R}},\mathsf{R})$ is a symmetric monoidal subcategory of $\mathbb{M}\mathrm{od}_{\mathsf{R}}$. $\mathbf{Comon}(\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{\mathsf{R}})$ (resp., ${}_{coc}\mathbf{Comon}(\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{\mathsf{R}})$) thus corresponds to the full subcategory of ${}_\epsilon\mathbf{Coalg}_{\mathsf{R}}$ (resp., ${}_{\epsilon,coc}\mathbf{Coalg}_{\mathsf{R}}$) spanned by the (resp. cocommutative) coalgebras whose underlying module is free.

Monoidal functors and their induced functors. For a (symmetric) monoidal category $\mathbb{C}, \mathbb{id}_{\mathbb{C}}:=(id_{\mathsf{C}},id_{-\otimes-},id_I)$, or simply $\mathbb{id}$, is a strict (symmetric, [18, p. 86]) monoidal functor from $\mathbb{C}$ to itself, which acts as a unit for the usual composition of monoidal functors (see [3, Chapter 3, p. 72]). By direct inspection one observes that the composite of strong (resp. symmetric) monoidal functors is strong (resp. symmetric) too.

Remark 51. Recall that for a monoidal functor $\mathbb{F} : \mathbb{C} \to \mathbb{C}'$, one let $\widetilde{\mathbb{F}} : \mathbf{Mon}\mathbb{C} \to \mathbf{Mon}\mathbb{C}'$ be the induced functor as described in [14, Proposition 3, p. 4876].

1. $\widetilde{\mathrm{id}_{\mathbb{C}}} = id_{\mathbf{Mon}(\mathbb{C})}$ and $\widetilde{\mathbb{G} \circ \mathbb{F}} = \widetilde{\mathbb{G}} \circ \widetilde{\mathbb{F}}$.
2. When $\mathbb{F}$ is symmetric (and $\mathbb{C}, \mathbb{C}'$ also are symmetric), then $\widetilde{\mathbb{F}}$ also provides an induced functor ${}_c\mathbf{Mon}\mathbb{C}$ *to* ${}_c\mathbf{Mon}\mathbb{C}'$ with similar properties as above.

A strong monoidal functor $\mathbb{F} = (F, \Phi, \phi) : \mathbb{C} \to \mathbb{C}'$ may be considered as a strong monoidal functor $\mathbb{F}^d := (F^{\mathrm{op}}, (\Phi^{-1})^{\mathrm{op}}, (\phi^{-1})^{\mathrm{op}}) : \mathbb{C}^{\mathrm{op}} \to (\mathbb{C}\prime)^{\mathrm{op}}$, the *dual of* $\mathbb{F}$ ([16, Proposition 17, p. 639]).

Monoidal transformations and equivalences.

Remark 52.
1. Let $\alpha : \mathbb{F} \Rightarrow \mathbb{G} : \mathbb{C} \to \mathbb{C}'$ be a monoidal transformation (see [3, pp. 64–65]). It induces a natural transformation $\widetilde{\alpha} : \widetilde{\mathbb{F}} \Rightarrow \widetilde{\mathbb{G}} : \mathbf{Mon}\mathbb{C} \to \mathbf{Mon}\mathbb{C}'$ with $\widetilde{\alpha}_{(C,m,e)} := \alpha_C$ ([3, Proposition 3.30, p. 78]).
2. When the monoidal functors and categories are symmetric, then α also induces $\widetilde{\alpha} : \widetilde{\mathbb{F}} \Rightarrow \widetilde{\mathbb{G}} : {}_c\mathbf{Mon}\mathbb{C} \to {}_c\mathbf{Mon}\mathbb{C}'$ ([3, Proposition 3.38]).

A *monoidal equivalence* of monoidal categories is given by a monoidal functor $\mathbb{F} : \mathbb{C} \to \mathbb{C}'$ such that there are a monoidal functor $\mathbb{G} : \mathbb{C}' \to \mathbb{C}$ and monoidal isomorphisms ([15, p. 948]) $\eta : \mathrm{id} \Rightarrow \mathbb{G} \circ \mathbb{F}$ and $\epsilon : \mathbb{F} \circ \mathbb{G} \Rightarrow \mathrm{id}$. In this situation $\mathbb{C}, \mathbb{C}'$ are said *monoidally equivalent*.

Remark 53. If $\mathbb{F}$ is a monoidal equivalence, then $\widetilde{\mathbb{F}}$ is an equivalence between the corresponding categories of monoids.

6.2. Topological tensor product of topologically free modules

We now wish to take advantage of the equivalence of categories $\mathbf{FreeMod}_{\mathsf{R}} \simeq \mathbf{TopFreeMod}^{\mathrm{op}}_{(\mathsf{R},\tau)}$ (Theorem 45) for a rigid ring (R, τ), to introduce a topological tensor product of topologically free modules.

From here to the end of Section 6.2, (R, τ) denotes a rigid ring.

The bifunctor $\circledast_{(\mathsf{R},\tau)}$. Let $(M, \sigma), (N, \gamma)$ be two topologically free (R, τ)-modules. One defines their *topological tensor product over* (R, τ) as

$$(M, \sigma)\circledast_{(\mathsf{R},\tau)}(N, \gamma) := Alg_{(\mathsf{R},\tau)}\big((M, \sigma)'\otimes_{\mathsf{R}}(N, \gamma)'\big). \tag{12}$$

One immediately observes that $(M, \sigma)\circledast_{(\mathsf{R},\tau)}(N, \gamma)$ still is a topologically free (R, τ)-module as $(M, \sigma)'$ and $(N, \gamma)'$ are free R-modules (Lemma 37), so is $(M, \sigma)'\otimes_{\mathsf{R}}(N, \gamma)'$, and the algebraic dual of a free module is topologically free (Lemma 33).

Actually, this definition is just the object component of a bifunctor

$$\mathbf{TopFreeMod}_{(\mathsf{R},\tau)} \times \mathbf{TopFreeMod}_{(\mathsf{R},\tau)} \xrightarrow{-\circledast_{(\mathsf{R},\tau)}-} \mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$$

that is $\mathbf{TopFreeMod}_{(\mathsf{R},\tau)} \times \mathbf{TopFreeMod}_{(\mathsf{R},\tau)} \xrightarrow{Top^{\mathrm{op}}_{(\mathsf{R},\tau)} \times Top^{\mathrm{op}}_{(\mathsf{R},\tau)}} \mathbf{Mod}^{\mathrm{op}}_{\mathsf{R}} \times \mathbf{Mod}^{\mathrm{op}}_{\mathsf{R}} \xrightarrow{\otimes^{\mathrm{op}}_{\mathsf{R}}} \mathbf{Mod}^{\mathrm{op}}_{\mathsf{R}} \xrightarrow{Alg_{(\mathsf{R},\tau)}} \mathbf{TopMod}_{(\mathsf{R},\tau)}$.

Given $f_i \in \mathbf{TopFreeMod}_{(\mathsf{R},\tau)}((M_i, \sigma_i), (N_i, \gamma_i))$, $i = 1, 2$, then $f_1\circledast_{(\mathsf{R},\tau)}f_2 := (M_1, \sigma_1)\circledast_{(\mathsf{R},\tau)}(M_2, \sigma_2) \xrightarrow{(f_1'\otimes_{\mathsf{R}} f_2')^*} (N_1, \gamma_1)\circledast_{(\mathsf{R},\tau)}(N_2, \gamma_2)$. Let $L \in ((M_1, \sigma_1)\prime\otimes_{\mathsf{R}}(M_2, \sigma_2)\prime)^*, \ell_1 \in (N_1, \gamma_1)'$ and $\ell_2 \in (N_2, \gamma_2)'$. Then,

$$\Big(\big(f_1 \circledast_{(\mathsf{R},\tau)} f_2\big)(L)\Big)(\ell_1 \otimes \ell_2) = L((\ell_1 \circ f_1) \otimes (\ell_2 \circ f_2)). \tag{13}$$

Remark 54. For every sets X, Y, $(R,\tau)^X \circledast_{(\mathsf{R},\tau)} (R,\tau)^Y \simeq (R,\tau)^{X\times Y}$ under $\rho^{-1}_{X\times Y} \circ \Upsilon^*_{X,Y} \circ (\lambda_X \otimes_{\mathsf{R}} \lambda_Y)^*$, where $\Upsilon_{X,Y} : R^{(X\times Y)} \simeq R^{(X)} \otimes_{\mathsf{R}} R^{(Y)}$ is the unique isomorphism such that $\Upsilon_{X,Y}(\delta^{\mathsf{R}}_{(x,y)}) = \delta^{\mathsf{R}}_x \otimes \delta^{\mathsf{R}}_y, x \in X, y \in Y$.

A topological basis of $(M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma)$. Our next goal will be to explicitly describe a topological basis (Definition 25) of $(M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma)$ in terms of topological bases of (M,σ) and (N,γ).

Definition 55. Given a ring S, for every S-modules M, N, one has a natural S-linear map $M^* \otimes_{\mathsf{S}} N^* \overset{\Theta_{M,N}}{\to} (M \otimes_{\mathsf{S}} M)^*$ given by $(\Theta_{M,N}(\ell_1 \otimes \ell_2))(u \otimes v) = \ell_1(u)\ell_2(v), \ell_1 \in M^*, \ell_2 \in N^*, u \in M$ and $v \in N$.

Let (M,σ), (N,γ) be topologically free (R,τ)-modules. Let $u \in M$ and $v \in N$. Let us define

$$u \circledast v := \Theta_{(M,\sigma)',(N,\gamma)'}\left(\Gamma_{(M,\sigma)}(u) \otimes_{\mathsf{R}} \Gamma_{(N,\gamma)}(v)\right) \in (M,\sigma) \otimes_{(\mathsf{R},\tau)} (N,\gamma). \tag{14}$$

In details, given $\ell_1 \in (M,\sigma)'$ and $\ell_2 \in (N,\gamma)'$, $(u \circledast v)(\ell_1 \otimes \ell_2) = \ell_1(u)\ell_2(v)$.

Lemma 56. *Let (M,σ) and (N,γ) be both topologically free (R,τ)-modules, with respective topological bases B, D. The map $B \times D \overset{-\circledast-}{\to} (M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma)$ given by $(b,d) \mapsto b \circledast d$, is one-to-one.*

Lemma 57. *Let (M,σ) and (N,γ) be both topologically free (R,τ)-modules. The map $M \times N \overset{-\circledast-}{\to} (M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma)$ is R-bilinear and separately continuous in both variable. Moreover, if $\tau = \mathsf{d}$, then $\circledast$ is even jointly continuous.*

Proof. R-bilinearity is clear. Since $(M,\sigma)' \otimes_{\mathsf{R}} (N,\gamma)'$ is free on $\{ x \otimes y : x \in X, y \in Y \}$, where X (resp. Y) is a basis of $(M,\sigma)'$ (resp. $(N,\gamma)'$), by Lemma 9, the topology $w^*_{(\mathsf{R},\tau)}$ on $(M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma)$ is the initial topology induced by the maps $((M,\sigma)' \otimes_{\mathsf{R}} ((N,\gamma)\prime)^* \overset{\Lambda_{(M,\sigma)' \otimes_{\mathsf{R}} (N,\gamma)'}(x \otimes y)}{\to} (R,\tau), x \in X, y \in Y$.

Let $x \in X, y \in Y, u \in M$ and $v \in N$. Then, $\Lambda_{(M,\sigma)' \otimes_{\mathsf{R}} (N,\gamma)'}(x \otimes y)(u \circledast v) = (u \circledast v)(x \otimes y) = x(u)y(v) = m_{\mathsf{R}}(x(u), x(v))$, and this automatically guarantees separate continuity in each variable of $\circledast$.

Let us assume that $\tau = \mathsf{d}$. According to the above general case, to see that $\circledast$ is continuous, by [19, Theorem 2.14, p. 17], it suffices to prove continuity at zero of $\circledast$. Let $A \subseteq X \times Y$ be a finite set, and for each $(x,y) \in A$, let $U_{(x,y)}$ be an open neighborhood of zero in (R,d). Let $A_1 := \{ x \in X : \exists y \in Y, (x,y) \in A \}$ and $A_2 := \{ y \in Y : \exists x \in X, (x,y) \in A \}$. A_1, A_2 are both finite and $A \subseteq A_1 \times A_2$. Let $u \in M$ such that $x(u) = 0$ for all $x \in A_1$, and $v \in N$ such that $y(v) = 0$ for all $y \in A_2$. Then, $(u \circledast v)(x \otimes y) = 0 \in U_{(x,y)}$ for all $(x,y) \in A_1 \times A_2$. □

Remark 58. Let X, Y be sets, and let $f \in R^X, g \in R^Y$. Since by Lemma 57, $\circledast$ is separately continuous[8]

[8]The second equality in Eq. (15) follows from the proof of [19, Theorem 10.15, p. 78] which, by inspection, shows that the cited result still is valid more generally after the replacement of a jointly continuous bilinear map by a separately continuous bilinear map.

$$f \circledast g = \left(\sum_{x\in X} f(x)\delta_x^{\mathsf{R}}\right) \circledast \left(\sum_{y\in Y} g(y)\delta_y^{\mathsf{R}}\right) = \sum_{(x,y)\in X\times Y} f(x)g(y)\delta_x^{\mathsf{R}} \circledast \delta_y^{\mathsf{R}}. \quad \text{(as a sum of a summable family)} \tag{15}$$

For the same reason as above, if B is a topological basis of (M,σ) and D is a topological basis of (N,γ), then $u\circledast v = \sum_{(b,d)\in B\times D} b'(u)d'(v)b\circledast d, u\in M, v\in N$. In particular, one observes that $(b\circledast d)'(u\circledast v) = b'(u)d'(v), b\in B, u\in M, d\in D$, and $v\in N$.

Proposition 59. *Let (M,σ) and (N,γ) be topologically free (R,τ)-modules, with respective topological bases B, D. Then, $(b\circledast d)_{(b,d)\in B\times D}$ is a topological basis of $(M,\sigma)\circledast_{(\mathsf{R},\tau)}(N,\gamma)$.*

Proof. By virtue of Lemma 27 and Remark 54, $\{ (\delta_x^{\mathsf{R}}\circledast\delta_y^{\mathsf{R}})_{(x,y)\in X\times Y} : (x,y)\in X\times Y \}$ is a topological basis of $(R,\tau)^X\circledast_{(\mathsf{R},\tau)}(R,\tau)^Y$ since one has $((\lambda_X^{-1}\otimes_{\mathsf{R}}\lambda_Y^{-1})^* ((\Upsilon_{X,Y}^{-1})^*(\rho_{X\times Y}(\delta_{(x,y)}^{\mathsf{R}})))) = \delta_x^{\mathsf{R}}\circledast\delta_y^{\mathsf{R}}$. Let $\Theta_B : (M,\sigma)\simeq(R,\tau)^B, \Theta_B(b) := \delta_b, b\in B$ (resp. $\Theta_D : (N,\gamma)\simeq(R,\tau)^D$). By functoriality, $\Theta_B\circledast_{(\mathsf{R},\tau)}\Theta_D : (M,\sigma)\circledast_{(\mathsf{R},\tau)}(N,\gamma)\simeq(R,\tau)^B\circledast_{(\mathsf{R},\tau)}(R,\tau)^D$, and since $(\theta_B\circledast_{(\mathsf{R},\tau)}\theta_D)(b\circledast d) = \delta_b\circledast\delta_d, (b,d)\in B\times D, \{ b\circledast d : (b,d)\in B\times D \}$ is a topological basis of $(M,\sigma)\circledast_{(\mathsf{R},\tau)}(N,\gamma)$. □

Corollary 60. *Let (M,σ) and (N,γ) be topologically free (R,τ)-modules. Then, $\{ u\circledast v : u\in M, v\in N \}$ spans a dense subspace in $(M,\sigma)\circledast_{(\mathsf{R},\tau)}(N,\gamma)$.*

Let (M_i,σ_i) and (N_i,γ_i), $i=1, 2$, be topologically free (R,τ)-modules. Let $(M_i,\sigma_i)\xrightarrow{f_i}(N_i,\gamma_i)$, $i=1, 2$, be continuous homomorphisms. Let $(u,v)\in M_1\times M_2$. By Eq. (13) it is clear that

$$(f_1\circledast_{(\mathsf{R},\tau)}f_2)(u\circledast v) = f_1(u)\circledast f_2(v). \tag{16}$$

If B and D are topological bases of (M_1,σ_1) and (M_2,σ_2) respectively, $(f_1\circledast_{(\mathsf{R},\tau)}f_2)(u\circledast v) = \sum_{(b,d)\in B\times D} b'(u)d'(v)f_1(b)\circledast f_2(d)$ (see Remark 58).

6.3. Monoidality of $\circledast_{(\mathsf{R},\tau)}$ and its direct consequences

Since most of the proofs from this section mainly consist in rather tedious, but simple, inspections of commutativity of some diagrams, essentially by working with given topological or linear bases,[9] and because they did not provide much understanding, they are not included in the presentation.

Proposition 61. *Let (R,τ) be a rigid ring.*

$$\mathbb{TopFreeMod}_{(\mathsf{R},\tau)} := \left(\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}, \circledast_{(\mathsf{R},\tau)}, (R,\tau)\right)$$

is a symmetric monoidal category.

Corollary 62. *For each field $(\mathbb{k},\tau)$ with a ring topology,*

$$\mathbb{TopFreeVect}_{(\mathbb{k},\tau)} := \left(\mathbf{TopFreeVect}_{(\mathbb{k},\tau)}, \circledast_{(\mathbb{k},\tau)}, (\mathbb{k},\tau)\right)$$

is a symmetric monoidal category.

Example 63. Let (R,τ) be a rigid ring. Let X be a set. Let us define a commutative monoid $M_{(\mathsf{R},\tau)}(X) := ((R,\tau)^X, \mu_X, \eta_X)$ in $\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$ by $\mu_X(f\circledast g) = \sum_{x\in X} f(x)g(x)\delta_x^{\mathsf{R}}, f, g\in R^X$

[9]E.g., associativity of $\circledast_{(\mathsf{R},\tau)}$ is given by the isomorphism $(b\circledast d)\circledast e\mapsto b\circledast(d\circledast e)$ on basis elements (Lemma 32).

(under $(R,\tau)^X \circledast_{(\mathsf{R},\tau)} (R,\tau)^X \simeq (R,\tau)^{X\times X}$ from Remark 54) and $\eta_X(1_\mathsf{R}) = \sum_{x\in X}\delta_x^\mathsf{R}$. This actually defines a functor $M_{(\mathsf{R},\tau)} : \mathbf{Set}^{\mathsf{op}} \to {}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)})$.

Let (R,τ) be a rigid ring. For each free R-modules M, N, let us define $\Phi_{M,N} := (M^*, w^*_{(\mathsf{R},\tau)}) \circledast_{(\mathsf{R},\tau)} (N^*, w^*_{(\mathsf{R},\tau)}) = Alg_{(\mathsf{R},\tau)}((M^*, w^*_{(\mathsf{R},\tau)})' \otimes_\mathsf{R} (N^*, w^*_{(\mathsf{R},\tau)})') \overset{(\Lambda_M \otimes_\mathsf{R} \Lambda_N)^*}{\to} ((M \otimes_\mathsf{R} N)^*, w^*_{(\mathsf{R},\tau)})$. According to Proposition 41, $\Phi_{M,N}$ is an isomorphism in $\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$. Naturality in M, N is clear, so this provides a natural isomorphism

$$\begin{aligned} \Phi : Alg_{(\mathsf{R},\tau)}(-)\circledast_{(\mathsf{R},\tau)} Alg_{(\mathsf{R},\tau)}(-) \Rightarrow Alg_{(\mathsf{R},\tau)}(-\otimes_\mathsf{R} -) : \\ \mathbf{FreeMod}_\mathsf{R}^{\mathsf{op}} \times \mathbf{FreeMod}_\mathsf{R}^{\mathsf{op}} \to \mathbf{TopFreeMod}_{(\mathsf{R},\tau)}. \end{aligned} \tag{17}$$

Furthermore, let us consider the isomorphism $(R,\tau) \overset{\phi}{\to} (R^*, w^*_{(\mathsf{R},\tau)})$ given by $\phi(1_\mathsf{R}) := id_\mathsf{R}$, with inverse $\phi^{-1}(\ell) = \ell(1_\mathsf{R})$.

Let $(M,\sigma), (N,\gamma)$ be topologically free (R,τ)-modules. One defines the map $\Psi_{(M,\sigma),(N,\gamma)} := (M,\sigma)' \otimes_\mathsf{R} (N,\gamma)' \overset{\Lambda_{(M,\sigma)'\otimes_\mathsf{R}(N,\gamma)'}}{\to} (Alg_{(\mathsf{R},\tau)}((M,\sigma)' \otimes_\mathsf{R} (N,\gamma)'))' = ((M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma))'$. This gives rise to a natural isomorphism

$$\begin{aligned} \Psi : Top_{(\mathsf{R},\tau)}(-) \otimes_\mathsf{R} Top_{(\mathsf{R},\tau)}(-) \Rightarrow Top_{(\mathsf{R},\tau)}\left(-\circledast_{(\mathsf{R},\tau)}-\right) : \\ \mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)} \times \mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)} \to \mathbf{FreeMod}_\mathsf{R}. \end{aligned} \tag{18}$$

Let also $R \overset{\psi}{\to} (R,\tau)'$ be given by $\psi(1_\mathsf{R}) = id_R$ and $\psi^{-1}(\ell) = \ell(1_\mathsf{R})$.

Theorem 64. *Let (R,τ) be a rigid ring.*

1. $\mathbb{Alg}_{(\mathsf{R},\tau)} := (Alg_{(\mathsf{R},\tau)}, \Phi, \phi) : \mathbb{FreeMod}_\mathsf{R}^{\mathsf{op}} \to \mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$ *is a strong symmetric monoidal functor.*
2. $\mathbb{Top}_{(\mathsf{R},\tau)} := (Top_{(\mathsf{R},\tau)}, \Psi, \psi) : \mathbb{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)} \to \mathbb{FreeMod}_\mathsf{R}$ *is a strong symmetric monoidal functor, so is its dual* $\mathbb{Top}^{\mathsf{d}}_{(\mathsf{R},\tau)}$ *(Remark 51) from* $\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$ *to* $\mathbb{FreeMod}_\mathsf{R}^{\mathsf{op}}$.
3. $\Lambda^{\mathsf{op}} : \mathbb{Top}^{\mathsf{d}}_{(\mathsf{R},\tau)} \circ \mathbb{Alg}_{(\mathsf{R},\tau)} \Rightarrow \mathbb{id} : \mathbb{FreeMod}_\mathsf{R}^{\mathsf{op}} \to \mathbb{FreeMod}_\mathsf{R}^{\mathsf{op}}$ *is a monoidal isomorphism.*
4. $\Gamma : \mathbb{id} \Rightarrow \mathbb{Alg}_{(\mathsf{R},\tau)} \circ \mathbb{Top}^{\mathsf{d}}_{(\mathsf{R},\tau)} : \mathbb{TopFreeMod}_{(\mathsf{R},\tau)} \to \mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$ *is a monoidal isomorphism.*

In particular, $\mathbb{FreeMod}_\mathsf{R}^{\mathsf{op}}$ and $\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$ are monoidally equivalent.

Corollary 65. *For each field $(\mathbb{k},\tau)$ with a ring topology, the monoidal categories $\mathbb{Vect}_\mathbb{k}^{\mathsf{op}}$ and $\mathbb{TopFreeVect}_{(\mathbb{k},\tau)}$ are monoidally equivalent.*

Corollary 66. *For each rigid ring (R,τ), the natural transformations $\widetilde{\Lambda^{\mathsf{op}}} : (\widetilde{\mathbb{Top}^{\mathsf{d}}_{(\mathsf{R},\tau)}}) \circ \widetilde{\mathbb{Alg}_{(\mathsf{R},\tau)}} \Rightarrow id_{\mathbf{Comon}(\mathbb{FreeMod}_\mathsf{R})^{\mathsf{op}}}$ and $\widetilde{\Gamma} : id_{\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)})} \Rightarrow \widetilde{\mathbb{Alg}_{(\mathsf{R},\tau)}} \circ (\widetilde{\mathbb{Top}^{\mathsf{d}}_{(\mathsf{R},\tau)}})$, induced as in Remark 52 by Λ^{op} and Γ, are natural isomorphisms. So are also the corresponding induced natural transformations at the level of the respective categories of (co)commutative (co)monoids (Remark 52).*

Corollary 67. *The equivalence from Corollary 66 restricts to an equivalence between the category ${}_\epsilon\mathbf{FinDimCoalg}_\mathbb{k}$ (resp. ${}_{\epsilon,coc}\mathbf{FinDimCoalg}_\mathbb{k}$) of finite dimensional (resp. cocommutative) coalgebras and the category of monoids $\mathbf{Mon}(\mathbb{FinDimVect}_\mathbb{k})$ (resp. ${}_c\mathbf{Mon}(\mathbb{FinDimVect}_\mathbb{k})$), where $\mathbb{FinDimVect}_\mathbb{k} = (\mathbf{FinDimVect}_\mathbb{k}, \otimes_\mathbb{k}, \mathbb{k})$.*

7. Relationship with finite duality

Over a field, there is a standard and well-known notion of duality between algebras and coalgebras, known as the finite duality [1,6] and we have the intention to understand the relations if any, between the equivalence of categories from Corollary 66 and this finite duality.

Let $(-)^* : \mathbf{Mod}_{\mathsf{R}}^{\mathsf{op}} \to \mathbf{Mod}_{\mathsf{R}}$ be the usual algebraic dual functor. Then, $\mathbb{D}_* := ((-)^*, \Theta, \theta)$ is a lax symmetric monoidal functor from $\mathbb{M}\mathrm{od}_{\mathsf{R}}^{\mathrm{op}}$ to $\mathbb{M}\mathrm{od}_{\mathsf{R}}$ (where Θ is as in Definition 55, and $\theta : R \to R^*$ is the isomorphism $\theta(1_{\mathsf{R}}) = id_R$ (and $\theta^{-1}(\ell) = \ell(1_{\mathsf{R}})$)). When $\mathbb{k}$ is a field, there is the *finite dual coalgebra functor* $D_{fin} : \mathbf{Mon}(\mathbb{V}\mathrm{ect}_{\mathbb{k}})^{\mathsf{op}} \to_{\epsilon} \mathbf{Coalg}_{\mathbb{k}}$ (denoted by $(-)^0$ in [1,6]). The aforementioned finite duality is the adjunction $D_{fin}^{\mathsf{op}} \dashv \widetilde{\mathbb{D}_*} : \mathbf{Mon}(\mathbb{V}\mathrm{ect}_{\mathbb{k}}) \to_{\epsilon} \mathbf{Coalg}_{\mathbb{k}}^{\mathsf{op}}$ (see e.g., [6, Theorem 1.5.22, p. 44], where $\widetilde{\mathbb{D}_*}$ is denoted by $(-)^*$).

7.1. The underlying algebra

Let (R, τ) be a rigid ring. Let $(M, \sigma), (N, \gamma)$ be topologically free (R, τ)-modules. According to Lemma 57, $M \times N \overset{-\circledast-}{\to} (M, \sigma) \circledast_{(\mathsf{R},\tau)} (N, \gamma)$ is R-bilinear. Denoting by $\mathbf{TopFreeMod}_{(\mathsf{R},\tau)} \overset{||\circledast||}{\to} \mathbf{Mod}_{\mathsf{R}}$ the canonical forgetful functor, this means that there is a unique R-linear map $||(M, \sigma)|| \otimes_{\mathsf{R}} ||(N, \gamma)|| \overset{\Xi_{(M,\sigma),(N,\gamma)}}{\to} ||(M, \sigma) \circledast_{(\mathsf{R},\tau)} (N, \gamma)||$ such that for each $u \in M, v \in N, \Xi_{(M,\sigma),(N,\gamma)}(u \otimes v) = u \circledast v$.

Lemma 68. *$\mathbb{A} := (||-||, (\Xi_{(M,\sigma),(N,\gamma)})_{(M,\sigma),(N,\gamma)}, id_R)$ is a lax symmetric monoidal functor from $\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}$ to $\mathbb{M}\mathrm{od}_{\mathsf{R}}$.*

Let $\widetilde{\mathbb{A}} : \mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}) \to \mathbf{Mon}(\mathbb{M}\mathrm{od}_{\mathsf{R}})$ be the functor induced by $\mathbb{A}$. Using the functorial isomorphism $O : \mathbf{Mon}(\mathbb{M}\mathrm{od}_{\mathsf{R}}) \simeq_1 \mathbf{Alg}_{\mathsf{R}}$ (Example 50), to any monoid in $\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}$ is associated an ordinary algebra.

Definition 69. Let us define $UA := \mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}) \overset{O \circ \widetilde{\mathbb{A}}}{\to}{}_1 \mathbf{Alg}_{\mathsf{R}}$. Given a monoid $((M, \sigma), \mu, \eta)$ in $\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}$, $UA((M, \sigma), \mu, \eta) = O(\widetilde{\mathbb{A}}((M, \sigma), \mu, \eta))$ is referred to as the underlying (ordinary) algebra of the monoid $((M, \sigma), \mu, \eta)$. In details, $UA((M, \sigma), \mu, \eta) = (M, \mu_{bil}, \eta(1_{\mathsf{R}}))$ with $\mu_{bil} : M \times M \to M$ given by $\mu_{bil}(u, v) := \mu(u \circledast v)$.

Remark 70. Since by Lemma 68, $\mathbb{A}$ is symmetric, it also induces a functor (see Remark 51) $\widetilde{\mathbb{A}} : {}_c\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}) \to {}_c\mathbf{Mon}(\mathbb{M}\mathrm{od}_{\mathsf{R}})$. Because one has the co-restriction $O : {}_c\mathbf{Mon}(\mathbb{M}\mathrm{od}_{\mathsf{R}}) \to_{1,c} \mathbf{Alg}_{\mathsf{R}}$, one may consider the underlying algebra functor $UA = {}_c\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}) \overset{O \circ \widetilde{\mathbb{A}}}{\to}{}_{1,c} \mathbf{Alg}_{\mathsf{R}}$.

Example 71. (Continuation of Example 63) $UA(M_{(\mathsf{R},\tau)}(X)) = \mathsf{A}_{\mathsf{R}} X$.

7.2. Relations with the algebraic dual algebra functor $\widetilde{\mathbb{D}_*}$

Let (R, τ) be a rigid ring. Let $\mathbf{FreeMod}_{\mathsf{R}} \overset{E}{\to} \mathbf{Mod}_{\mathsf{R}}$ be the canonical embedding functor. Since $\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{\mathsf{R}}$ is a symmetric monoidal subcategory of $\mathbb{M}\mathrm{od}_{\mathsf{R}}$ it follows that $\mathbb{E} = (E, id, id)$ is a strict monoidal functor from $\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{\mathsf{R}}$ to $\mathbb{M}\mathrm{od}_{\mathsf{R}}$.

One claims that $\mathbb{D}_* \circ \mathbb{E}^{\mathrm{d}} = \mathbb{A} \circ \mathbb{A}\mathrm{lg}_{(\mathsf{R},\tau)}$. In particular, if $\mathbb{k}$ is a field (and τ is a ring topology on $\mathbb{k}$), then this reduces to $\mathbb{D}_* = \mathbb{A} \circ \mathbb{A}\mathrm{lg}_{(\mathbb{k},\tau)}$.

That $||-|| \circ Alg_{(\mathsf{R},\tau)} = (-)^* \circ E^{\mathsf{op}}$ is due to the very definition of $Alg_{(\mathsf{R},\tau)}$. Of course, $||\phi|| = \theta$. That for each free modules M, N, $||(\Lambda_M \otimes \Lambda_N)^*|| \circ \Xi_{M^*,N^*} = \Theta_{M,N}$ is easy to check. So $((-)^* \circ E^{\mathsf{op}}, \Theta, \theta) = (||-||, \Xi, id_{\mathsf{R}}) \circ (Alg_{(\mathsf{R},\tau)}, \Phi, \phi) = (||-|| \circ Alg_{(\mathsf{R},\tau)}, (||\Phi_{M,N}|| \circ \Xi_{M^*,N^*})_{M,N}, ||\phi||)$.

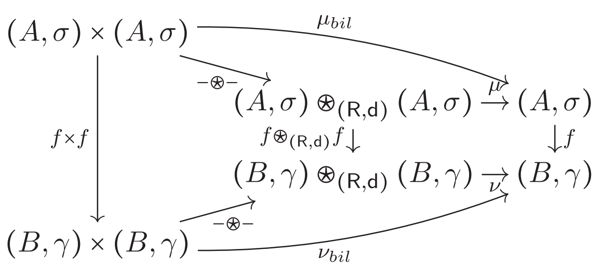
It follows that the algebraic dual monoid (in $\mathbb{V}\mathrm{ect}_{\mathbb{k}}$) $\widetilde{\mathbb{D}_*}(C)$ of a $\mathbb{k}$-coalgebra C, is equal to $\widetilde{\mathbb{A}}(\widetilde{\mathbb{A}\mathrm{lg}_{(\mathbb{k},\tau)}}(C))$ whatever is the ring topology τ on the field $\mathbb{k}$, and thus as ordinary algebras, $O(\widetilde{\mathbb{D}_*}(C)) = UA(\widetilde{\mathbb{A}\mathrm{lg}_{(\mathbb{k},\tau)}}(C))$.

Proposition 72. *Let* $(\mathbb{k}, \tau)$ *be a field with a ring topology. Then the functor* $O \circ \widetilde{\mathbb{A}}$: $\mathbf{Mon}(\mathbb{FreeTopVect}_{(\mathbb{k},\tau)}) \to {}_1\mathbf{Alg}_{\mathbb{k}}$ *has a left adjoint, namely* $\mathbb{Alg}_{(\mathbb{k},\tau)} \circ D_{fin}^{\mathsf{op}} \circ O^{-1}$.

Proof. One has $\widetilde{\mathbb{D}_*} = \widetilde{\mathbb{A}} \circ \widetilde{\mathbb{Alg}_{(\mathbb{k},\tau)}}$, whence $\widetilde{\mathbb{D}_*} \circ (\widetilde{\mathbb{Top}^{\mathsf{d}}_{(\mathbb{k},\tau)}}) = \widetilde{\mathbb{A}} \circ \widetilde{\mathbb{Alg}_{(\mathbb{k},\tau)}} \circ (\widetilde{\mathbb{Top}^{\mathsf{d}}_{(\mathbb{k},\tau)}})^{\mathsf{op}} \simeq \widetilde{\mathbb{A}}$ (natural isomorphism) by Theorem 64. Since $\mathbb{Alg}_{(\mathbb{k},\tau)} \circ D_{fin}^{\mathsf{op}}$ is a left adjoint of $\widetilde{\mathbb{D}_*} \circ (\widetilde{\mathbb{Top}^{\mathsf{d}}_{(\mathbb{k},\tau)}})$, it follows that it is also the left adjoint of $\widetilde{\mathbb{A}}$. □

7.3. The underlying topological algebra

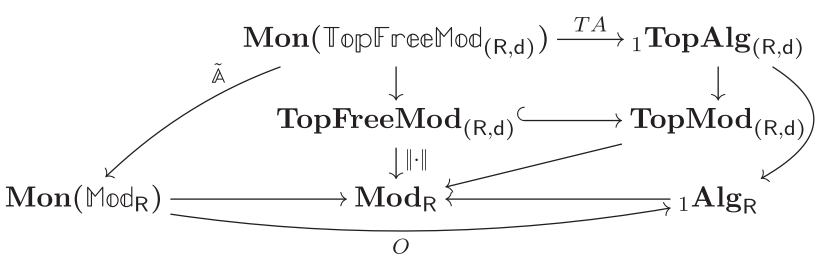
Let R be a ring. Let $\mathsf{A} = ((A, \sigma), \mu, \eta)$ be an object of $\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})})$. One knows that (A, σ) is an object of $\mathbf{TopMod}_{(\mathsf{R},\mathsf{d})}$ and $UA(\mathsf{A})$ is an object of ${}_1\mathbf{Alg}_{\mathsf{R}}$. Moreover, $(A, \sigma) \times (A, \sigma) \xrightarrow{\mu_{bil}} (A, \sigma)$ is continuous, since it is equal to the composition $(A, \sigma) \times (A, \sigma) \xrightarrow{-\circledast-} (A, \sigma) \circledast_{(\mathsf{R},\mathsf{d})} (A, \sigma) \xrightarrow{\mu} (A, \sigma)$ of continuous maps (see Lemma 57). Now, let $((A, \sigma), \mu, \eta) \xrightarrow{f} ((B, \gamma), \nu, \zeta)$ be a morphism in $\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathbb{R},\mathsf{d})})$. In particular, $(A, \sigma) \xrightarrow{f} (B, \gamma)$ is linear and continuous, and the following diagram commutes.



(19)

Since by assumption, one also has $f \circ \eta = \zeta$, it follows that $f(\eta(1_{\mathsf{R}})) = \zeta(1_{\mathsf{R}})$, and thus f is a continuous algebra map from $((A, \sigma), \mu_{bil}, \eta(1_{\mathsf{R}}))$ to $((B, \gamma), \nu_{bil}, \zeta(1_{\mathsf{R}}))$.

So is obtained a functor $\mathbf{Mon}(\mathbb{TopFreeMod}_{(R,\mathsf{d})}) \xrightarrow{TA} {}_1\mathbf{TopAlg}_{(\mathsf{R},\mathsf{d})}$, referred to as the *topological algebra functor*, and the following diagram commutes (the unnamed arrows are either the obvious forgetful functors or the evident embedding functor), so that TA is concrete over $\mathbf{TopMod}_{(\mathsf{R},\mathsf{d})}$, whence faithful.



(20)

Remark 73. *When* A *is a commutative monoid in* $\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})}$, *then* $TA(\mathsf{A})$ *is a commutative topological algebra.*

Example 74. For each set X, $TA(M_{(\mathsf{R},\mathsf{d})}(X)) = \mathsf{A}_{(\mathsf{R},\mathsf{d})}(X)$.

Proposition 75. *TA is a full embedding functor.*

Proof. Let $\mathsf{A} = ((A,\sigma),\mu,\eta)$ and $\mathsf{B} = ((B,\gamma),\nu,\zeta)$ be monoids in $\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})}$. Let $TA(\mathsf{A})\xrightarrow{g}TA(\mathsf{B})$ be a morphism in ${}_1\mathbf{TopAlg}_{(\mathsf{R},\mathsf{d})}$. Therefore in particular, $g\in{}_1\mathbf{Alg}_{\mathsf{R}}(UA(\mathsf{A}),UA(\mathsf{B}))\cap\mathbf{Top}((|A|,\sigma),(|B|,\gamma))$. (Recall from Remark 2 that $\mathbf{Mod}_{\mathsf{R}}\xrightarrow{|\cdot|}\mathbf{Set}$ is the usual forgetful functor, and **Top** is the category of Hausdorff topological spaces.)

By assumption, for each $u,v\in A, g(\mu(u\circledast v)) = g(\mu_{bil}(u,v)) = \nu_{bil}(g(u),g(v)) = \nu(g(u)\circledast g(v))$. Thus, $g\circ\mu = \nu\circ(g\circledast_{(\mathsf{R},\mathsf{d})}g)$ on $\{\ u\circledast v : u,v\in A\ \}$. Since this set spans a dense subset of $(A,\sigma)\otimes_{(R,\tau)}(A,\sigma)$ (according to Corollary 60), by linearity and continuity, $g\circ\mu = \nu\circ(g\circledast_{(\mathsf{R},\mathsf{d})}g)$ on the whole of $(A,\sigma)\circledast_{(\mathsf{R},\mathsf{d})}(A,\sigma)$.

Moreover, $g(\eta(1_{\mathsf{R}})) = \zeta(1_{\mathsf{R}})$, then $g\circ\eta = \zeta$. Therefore, g may be seen as a morphism $\mathsf{A}\xrightarrow{f}\mathsf{B}$ in $\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})})$ with $TA(f) = g$, i.e., TA is full.

Let $\mathsf{A} = ((A,\sigma),\mu,\eta), \mathsf{B} = ((B,\gamma),\nu,\zeta)$ be monoids in $\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})}$ such that $TA(\mathsf{A}) = TA(\mathsf{B})$. In particular, $(A,\sigma) = (B,\gamma)$, and $\eta = \zeta$. By assumption $\mu_{bil} = \nu_{bil}$. Whence $\mu = \nu$ on $\{\ u\circledast v : u\in A, v\in B\ \}$, and by continuity they are equal on $(A,\sigma)\circledast_{(\mathsf{R},\mathsf{d})}(B,\gamma)$. So $\mathsf{A} = \mathsf{B}$, i.e., TA is injective on objects. □

As a consequence of Proposition 75, $\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})})$ is isomorphic to a full subcategory of ${}_1\mathbf{TopAlg}_{(\mathsf{R},\mathsf{d})}$ ([2, Proposition 4.5, p. 49]).

It is clear that ${}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})})\xrightarrow{TA}{}_{1,c}\mathbf{TopAlg}_{(\mathsf{R},\mathsf{d})}$ of TA (see Remark 73) also is a full embedding functor.

7.4. Relations with the finite dual coalgebra functor $\mathsf{d}_{\mathsf{fin}}$

Let V be any vector space on a field $\mathbb{k}$. Then, V^* has a somewhat natural topology called the *V-topology* ([1]) or the *finite topology* ([6]), with a fundamental system of neighborhoods of zero consisting of subspaces

$$W^\dagger := \{\ \ell\in V^* : \forall w\in W, \ell(w) = 0\ \} \tag{21}$$

where W runs over the finite dimensional subspaces of V. This is manifestely the same topology as our $w^*_{(\mathbb{k},\mathsf{d})}$ (see Section 3.1). Accordingly this turns V^* into a linearly compact $\mathbb{k}$-vector space (p. §). The closed subspace of $(V^*, w^*_{(\mathbb{k},\mathsf{d})})$ are exactly the subspaces of the form $W^\dagger$, where W is any subspace of V ([4, Proposition 24.4, p. 105]).

Lemma 76. *Let W be a subspace of V.* $\operatorname{codim}(W^\dagger)$ *is finite if, and only if,* $\dim(W)$ *is finite. In this case,* $\operatorname{codim}(W^\dagger) = \dim(W)$.

Proof. One observes that $V^*/W^\dagger \simeq W^*$ because the map $incl_W^* : V^*\to W^*$ is onto, where $incl_W : W\to V$ is the canonical inclusion, and $\ker incl_W^* = W^\dagger$. Since $V^*/W^\dagger\simeq W^*$, it follows that $\operatorname{codim}(W^\dagger) = \dim W^*$. □

Theorem 77. *Let $\mathbb{k}$ be a field. For each monoid A in $\mathbb{TopFreeVect}_{(\mathbb{k},\mathsf{d})}$, the topological dual coalgebra $(\widetilde{\mathbb{Top}^{\mathsf{d}}_{(\mathbb{k},\mathsf{d})}})(\mathsf{A})$ of A is a subcoalgebra of $D^{\mathsf{op}}_{fin}(\widetilde{\mathbb{A}}(\mathsf{A}))$. Furthermore, the assertions below are equivalent.*

1. In $TA(\mathsf{A})$ every finite codimensional ideal is closed.
2. $\widetilde{\mathbb{A}}(\mathsf{A})$ is reflexive[10].

[10] A monoid A in $\mathbb{Vect}_{\mathbb{k}}$ is *reflexive* when $\mathsf{A}\simeq\widetilde{\mathbb{D}_*}(D^{\mathsf{op}}_{fin}(\mathsf{A}))$ under the linear map $u\mapsto(\ell\mapsto\ell(u))$, which is the unit of the adjunction $\widetilde{D^{\mathsf{op}}_{fin}}\dashv\mathbb{D}_*$.

3. The coalgebra $(\widetilde{\mathbb{T}\mathrm{op}^{\mathrm{d}}_{(\mathbb{k},\mathrm{d})}})(\mathsf{A})$ is coreflexive[11].
4. $(\widetilde{\mathbb{T}op^{\mathsf{d}}_{(\mathbb{k},\mathsf{d})}})(\mathsf{A}) = D^{\mathsf{op}}_{fin}(\widetilde{\mathbb{A}}(\mathsf{A}))$.

Proof. Let $\mathsf{A} = ((A,\sigma),\mu,\eta)$ be a monoid in $\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathbb{k},\mathrm{d})}$. Whence its underlying topological vector space is a linearly compact vector space (p. 15). Let $C := (\widetilde{\mathbb{T}\mathrm{op}^{\mathrm{d}}_{(\mathbb{k},\mathrm{d})}})(\mathsf{A})$. Since $\widetilde{\mathbb{A}} \circ \widetilde{\mathbb{A}\mathrm{lg}_{(\mathbb{k},\mathrm{d})}} = \widetilde{\mathbb{D}_*}$ it follows that $\widetilde{\mathbb{A}} \simeq \widetilde{\mathbb{A}} \circ \widetilde{\mathbb{A}\mathrm{lg}_{(\mathbb{k},\mathrm{d})}} \circ (\widetilde{\mathbb{T}\mathrm{op}^{\mathrm{d}}_{(\mathbb{k},\mathrm{d})}}) \simeq \widetilde{\mathbb{D}_*} \circ (\widetilde{\mathbb{T}\mathrm{op}^{\mathrm{d}}_{(\mathbb{k},\mathrm{d})}})$ (naturally isomorphic). In particular, $\widetilde{\mathbb{A}}(\mathsf{A}) \simeq \widetilde{\mathbb{D}_*}(C)$. By construction, the underlying topological vector space of A, namely (A,σ), is also the underlying topological vector space of $TA(\mathsf{A})$. Also $\mathsf{A}, TA(\mathsf{A})$ and $\widetilde{\mathbb{A}}(\mathsf{A})$ share the same underlying vector space A, which is isomorphic to C^*, where C is the underlying vector space of the coalgebra C. Of course, $(A,\sigma) \simeq Alg_{(\mathbb{k},\mathrm{d})}(C) = (C^*, w^*_{(\mathbb{k},\mathrm{d})})$. Therefore, up to such an isomorphism, (A,σ) has a fundamental system of neighborhoods of zero consisting of $V^\dagger = \{\ \ell \in A : \forall v \in V, \ell(v) = 0\ \}$ where V is a finite dimensional subspace of C (see Eq. (21)).

Let $\ell \in (A,\sigma)'$. By continuity of ℓ, there exists a finite dimensional subspace V of C such that $V^\dagger \subseteq \ker\ell$. Let B be a (finite) basis of V, and let D be the (necessarily finite dimensional, by [10, Theorem 1.3.2, p. 21]) subcoalgebra of C it generates. Then, $V \subseteq D$, which implies that $D^\dagger \subseteq V^\dagger \subseteq \ker\ell$. But $D^\dagger$ is a finite codimensional ideal of $\widetilde{\mathbb{A}}(\mathsf{A})$ (by Lemma 76 and [1, Theorem 2.3.1, p. 78]), whence $\ell \in A^0$.[12]

It remains to check that the above inclusion $incl_{(A,\sigma)'}$ is a coalgebra map from $(\widetilde{\mathbb{T}\mathrm{op}^{\mathrm{d}}_{(\mathbb{k},\mathrm{d})}})(\mathsf{A})$ to $D^{\mathsf{op}}_{fin}(\widetilde{\mathbb{A}}(\mathsf{A}))$, which would equivalently mean that $(A,\sigma)'$ is a subcoalgebra of $D^{\mathsf{op}}_{fin}(\widetilde{\mathbb{A}}(\mathsf{A}))$. One thus needs to make explicit the two coalgebra structures so as to make possible a comparison. By construction the comultiplication of $(\widetilde{\mathbb{T}\mathrm{op}^{\mathrm{d}}_{(\mathbb{k},\mathrm{d})}})(\mathsf{A})$ is given by the composition $\Lambda^{-1}_{(A,\sigma)'\otimes_{\mathbb{k}}(A,\sigma)'} \circ \mu'$. So for $\ell \in (A,\sigma)'$, $(\Lambda^{-1}_{(A,\sigma)'\otimes_{\mathbb{k}}(A,\sigma)'} \circ \mu')(\ell) = \sum_{i=1}^n \ell_i \otimes r_i$, for some $\ell_i, r_i \in (A,\sigma)'$. Therefore, given $\ell \in (A,\sigma)', u, v \in A, \ell(\mu(u \star v)) = \sum_{i=1}^n \ell_i(u) r_i(v)$. The counit of $(\widetilde{\mathbb{T}\mathrm{op}^{\mathrm{d}}_{(\mathbb{k},\mathrm{d})}})(\mathsf{A})$ is $(A,\sigma)' \xrightarrow{\eta'} (\mathbb{k},\mathsf{d})' \xrightarrow{\psi^{-1}} \mathbb{k}$, i.e., $\ell \mapsto \ell(\eta(1_{\mathbb{k}}))$. It follows easily, from the explicit description of $D_{fin}(\mathsf{B})$ provided in [6, p. 35], for a monoid B in $\mathbb{V}\mathrm{ect}_{\mathbb{k}}$, that the above comultiplication coincides with that of $D_{fin}(\widetilde{\mathbb{A}}(A))$, and because it is patent that the counit of $(\widetilde{\mathbb{T}\mathrm{op}^{\mathrm{d}}_{(\mathbb{k},\mathrm{d})}})(\mathsf{A})$ is the restriction of that of $D^{\mathsf{op}}_{fin}(\widetilde{\mathbb{A}}(\mathsf{A}))$, $(A,\sigma)'$ is a subcoalgebra of $D^{\mathsf{op}}_{fin}(\widetilde{\mathbb{A}}(\mathsf{A}))$.

It remains to prove the equivalence of the four assertions given in the statement. $2 \iff 3$ since finite duality restricts to an equivalence of categories between the full categories of reflexive algebras and of coreflexive coalgebras (in a standard way; see e.g., [11, Proposition 4.2, p. 16]), and $\widetilde{\mathbb{A}}(\mathsf{A}) \simeq \widetilde{\mathbb{D}_*}((\widetilde{\mathbb{T}\mathrm{op}^{\mathrm{d}}_{(\mathbb{k},\mathrm{d})}})(\mathsf{A}))$.

The coalgebra $C := (\widetilde{\mathbb{T}\mathrm{op}^{\mathrm{d}}_{(\mathbb{k},\mathrm{d})}})(\mathsf{A})$ is coreflexive if, and only if, every finite codimensional ideal of $\widetilde{\mathbb{D}_*}(C) \simeq \widetilde{\mathbb{A}}(\mathsf{A})$ is closed in the finite topology of C^* ([1, Lemma 2.2.15, p. 76]), which coincides with our topology $w^*_{(\mathbb{k},\mathrm{d})}$, and thus it turns out that $(\widetilde{\mathbb{D}_*}(C), w^*_{(\mathbb{k},\mathrm{d})}) \simeq TA(\mathsf{A})$ (since C^* under the finite topology is equal to $Alg_{(\mathbb{k},\mathrm{d})}(C) \simeq (A,\sigma)$ by functoriality). Whence $3 \iff 1$.

Let us assume that in $TA(\mathsf{A})$ every finite codimensional ideal is closed. Let $\ell \in D_{fin}(\widetilde{\mathbb{A}}(\mathsf{A}))$. By definition $\ker\ell$ contains a finite codimensional ideal say I of $\widetilde{\mathbb{A}}(\mathsf{A})$. Since I is closed, there exists a finite dimensional subspace D of C such that $D^\dagger = I$ (since the closed subspaces are of the

[11] A coalgebra C is *coreflexive*, when $C \simeq D^{\mathsf{op}}_{fin}(\widetilde{\mathbb{D}_*}(C))$ under the natural inclusion $u \mapsto (\ell \mapsto \ell(u))$, which is the counit of $D^{\mathsf{op}}_{fin} \dashv \widetilde{\mathbb{D}_*}$.

[12] $A^0 := \{\ \ell \in A^* : \ker\ell$ contains a finite-codimensional (two-sided) ideal of $UA(\mathsf{A})\ \}$ is the underlying vector space of the finite dual coalgebra $D_{fin}(\widetilde{\mathbb{A}}(\mathsf{A}))$.

form $D^\dagger$ for a subspace D of C and by Lemma 76, $\mathrm{codim}(I) = \mathrm{codim}(D^\dagger) = \dim(D)$), whence I is open, which shows that $\underset{\sim}{(A,\sigma)\overset{\ell}{\to}(\Bbbk,\mathrm{d})}$ is continuous so $1 \Rightarrow 4$.

Let $C := (\widetilde{\mathbb{T}\mathrm{op}^{\mathrm{d}}_{(\Bbbk,\mathrm{d})}})(\mathsf{A}) = D^{\mathsf{op}}_{fin}(\widetilde{\mathbb{A}}(\mathsf{A}))$, so that $\widetilde{\mathbb{A}}(\mathsf{A}) \simeq \widetilde{\mathbb{D}_*}(C)$, as above. Whence $C = D^{\mathsf{op}}_{fin}(\widetilde{\mathbb{A}}(\mathsf{A})) \simeq D^{\mathsf{op}}_{fin}(\widetilde{\mathbb{D}_*}(C))$. This is not sufficient to ensure coreflexivity of C, since there is at this stage no guaranty that the above isomorphism corresponds to the counit of the adjunction $D^{\mathsf{op}}_{fin} \dashv \widetilde{\mathbb{D}_*}$ (see Footnote 3). One knows from the beginning of the proof that $\widetilde{\mathbb{A}}(\widetilde{\Gamma}_{\mathsf{A}}) : \mathbb{A}(\mathsf{A}) \simeq \widetilde{\mathbb{A}}(\widetilde{\mathbb{A}\mathrm{lg}_{(\Bbbk,\mathrm{d})}}((\widetilde{\mathbb{T}\mathrm{op}^{\mathrm{d}}_{(\Bbbk,\mathrm{d})}})(\mathsf{A})))$ which, in this case where $(\widetilde{\mathbb{T}\mathrm{op}^{\mathrm{d}}_{(\Bbbk,\mathrm{d})}})(\mathsf{A}) = D^{\mathsf{op}}_{fin}(\widetilde{\mathbb{A}}(\mathsf{A}))$, is the isomorphism $||\Gamma_{(A,\sigma)}|| : A \simeq ((A,\sigma)\prime)^* = (A^0)^*, u \mapsto (\ell \mapsto \ell(u))$. So $\widetilde{\mathbb{A}}(\mathsf{A})$ is reflexive, and thus its finite dual coalgebra C is coreflexive. Thus $4 \Rightarrow 3$. □

Example 78. Let R be a ring. Let $C_{\mathsf{R}}X = (R^{(X)}, d_X, e_X)$ be the group-like coalgebra on X, i.e., $d_X(\delta_x) = \delta_x \otimes \delta_x$, and $e_X(\delta_x) = 1_{\mathsf{R}}, x \in X$. The following diagram commutes for a rigid ring (R, τ).

$$\begin{array}{ccc} ((R,\tau)^X)' & \xrightarrow{\mu'_X} & ((R,\tau)^X \otimes_{(\mathsf{R},\tau)} (R,\tau)^X)' \\ \lambda_X \downarrow & & \uparrow \Lambda_{((R,\tau)^X)' \otimes_{\mathsf{R}} ((R,\tau)^X)'} \\ R^{(X)} \xrightarrow[d_X]{} R^{(X)} \otimes_{\mathsf{R}} R^{(X)} & \xrightarrow[\lambda_X^{-1} \otimes_{\mathsf{R}} \lambda_X^{-1}]{} & ((R,\tau)^X)' \otimes_{\mathsf{R}} ((R,\tau)^X)' \end{array} \tag{22}$$

Moreover $\eta'_X(\ell) = \psi(e_X(\lambda_X(\ell)))$ for each $\ell \in ((R,\tau)^X)'$. All of this shows that $\lambda_X : (\widetilde{\mathbb{T}\mathrm{op}^{\mathrm{d}}_{(\mathsf{R},\tau)}})(M_{(\mathsf{R},\tau)}(X)) \simeq C_{\mathsf{R}}X$ is an isomorphism of coalgebras.

Let $\Bbbk$ be a field. It follows from Theorem 77 and Example 74 that in $\mathsf{A}_{(\Bbbk,\mathrm{d})}(X)$ every finite codimensional ideal is closed if, and only if $C_{\Bbbk}X$ is coreflexive if, and only if, $\{\ \pi_x : x \in X\ \} =_1 \mathbf{Alg}_{\Bbbk}(\mathsf{A}_{\Bbbk}(X), \Bbbk)$ ([17, Corollary 3.2, p. 528]). This holds in particular if $|X| \leq |\Bbbk|$ (see [17, Corollary 3.6, p. 529]). If $\Bbbk$ is a finite field, then $C_{\Bbbk}(X)$ is coreflexive if, and only if, X is finite (see [17, Remark 3.7, p. 530]).

Acknowledgements

We are grateful to the referee's thorough work and constructive comments which led to a considerable improvement of the presentation.

References

[1] Abe, E. (2004). *Hopf Algebras*, Vol. 74. Cambridge, UK: CUP.
[2] Adámek, J., Herrlich, H., Strecker, G. E. (2004). *Abstract and Concrete Categories. The Joy of Cats*. New York: Dover Publication.
[3] Aguiar, M., Mahajan, S. A. (2010). *Monoidal Functors, Species and Hopf Algebras*, Vol. 29. Providence: Amer. Math. Soc.
[4] Bergman, G. M., Hausknecht, A. O. (1996). *Cogroups and Co-rings in Categories of Associative Rings*, Vol. 45. Providence: Amer. Math. Soc.
[5] Bourbaki, N. (1995). *General Topology: Chapters 1–4*, Vol. 18. Berlin, Germany: Springer-Verlag.
[6] Dăscălescu, S., Năstăsescu, C., Raianu, Ş. (2001). *Hopf Algebras. Pure and Applied Mathematics*, Vol. 401. New York: Marcel Dekker.
[7] Dieudonné, J. (1951). Linearly compact spaces and double vector spaces over sfields. *Amer. J. Math.* 73(1): 13–19.
[8] Kelley, J. L. (1955). *General Topology*, Vol. 27. Berlin, Germany: Springer-Verlag.
[9] Lam, T. Y. (1999). *Lectures on Modules and Rings*, Vol. 189. Berlin, Germany: Springer-Verlag.

[10] Lambe, L. A., Radford, D. E. (2013). *Introduction to the Quantum Yang-Baxter Equation and Quantum Groups: An Algebraic Approach*, Vol. 423. Berlin, Germany: Springer-Verlag.
[11] Lambek, J., Scott, P. J. (1988). *Introduction to Higher-order Categorical Logic*, Vol. 7. Cambridge, UK: CUP.
[12] MacLane, S. (1998). *Categories for the Working Mathematician*, Vol. 5. Berlin, Germany: Springer-Verlag.
[13] Poinsot, L. (2015). Rigidity of topological duals of spaces of formal series with respect to product topologies. *Topol. Appl.* 189:147–175.
[14] Poinsot, L., Porst, H.-E. (2015). Free monoids over semigroups in a monoidal category: Construction and applications. *Comm. Algebra* 43(11):4873–4899.
[15] Poinsot, L., Porst, H.-E. (2016). The dual rings of an *R*-coring revisited. *Comm. Algebra* 44(3):944–964.
[16] Porst, H. E. (2015). The formal theory of Hopf algebras part I: Hopf monoids in a monoidal category. *Quaest. Math.* 38(5):631–682.
[17] Radford, D. E. (1973). Coreflexive coalgebras. *J. Algebra* 26(3):512–535.
[18] Street, R. (2007). *Quantum Groups: A Path to Current Algebra*. Cambridge, UK: CUP.
[19] Warner, S. (1993). *Topological Rings*, Vol. 178. Amsterdam, Netherlands: Elsevier.